Consistent equations for open-channel flows in the smooth turbulent regime with shearing effects

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Online supplementary material

Calculation of the asymptotic expansions at order 1

This online supplementary document details the calculation of the asymptotic expansions at order 1. It gives some intermediate results that are needless for most readers but that could be useful for those who want to follow closely the method or to check the results. The section §3.3 is rewritten as in the paper, adding the various stages of the long calculations. The references are the same as those of the paper.

3.3. *Order 1*

At order 1, the momentum equation in the Ox direction (2.24) is

$$\frac{\partial u_0'}{\partial t'} + \frac{\partial u_0'^2}{\partial x'} + \frac{\partial u_0' w_0'}{\partial z'} = \kappa^2 \frac{\partial \tau_1'}{\partial z'} - \frac{1}{F^2}\frac{\partial p_0'}{\partial x'}. \tag{3.31}$$

Making use of the boundary condition at the free surface, integration of (3.31) gives:

$$\kappa^2 \tau_1' = \frac{\partial}{\partial t'}\int_{h'}^{z'} u_0' \,\mathrm{d}z' + \frac{\partial}{\partial x'}\int_{h'}^{z'} u_0'^2 \,\mathrm{d}z' + u_0' w_0' + \frac{1}{F^2}\frac{\partial}{\partial x'}\int_{h'}^{z'} p_0' \,\mathrm{d}z'. \tag{3.32}$$

The valued of $\partial h'/\partial t'$ is estimated with (3.27). The first term can be written

$$\frac{\partial}{\partial t'}\int_{h'}^{z'} u'_0 \,\mathrm{d}z' = \frac{\partial}{\partial t'}\left[h' \int_1^s u'_0 \,\mathrm{d}s\right], \tag{3.33}$$

where $s = z'/h'$. The calculation of this integral yields

$$\frac{\partial}{\partial t'}\int_{h'}^{z'} u'_0 \,\mathrm{d}z' = \sqrt{\lambda_0 h'}\frac{\partial h'}{\partial t'}\Bigg\{-3 + s + \mu\Bigg[3 - \frac{3}{2}R + \sqrt{1-s} + \frac{R}{2}s - \frac{9}{2}\ln 2 + \frac{1}{2}sj(s) + \frac{3}{2}s\ln 2 - \frac{3}{2}\ln M + \frac{1}{2}s\ln M\Bigg]\Bigg\}. \tag{3.34}$$

The derivative $\partial h'/\partial t'$ can be written

$$\frac{\partial h'}{\partial t'} = -\frac{\partial h' U'_0}{\partial x'} + O(\varepsilon) \tag{3.35}$$

$$= -\frac{3}{2}\sqrt{\lambda_0 h'}\frac{\partial h'}{\partial x'}\left[2 + \mu\left(R + 3\ln 2 - \frac{8}{3} + \ln M\right)\right] + O(\varepsilon). \tag{3.36}$$

This gives the final expression of the first term

$$\frac{\partial}{\partial t'}\int_{h'}^{z'} u'_0\,\mathrm{d}z' = \lambda_0 h'\frac{\partial h'}{\partial x'}\Bigg\{3\left(3-s\right)+\mu\Bigg[-21+9R-3\sqrt{1-s}+4s-3Rs$$
$$+27\ln 2-9s\ln 2+9\ln M-3s\ln M-\frac{3}{2}sj\left(s\right)\Bigg]+\mu^2\Bigg[12-\frac{21}{2}R+\frac{9}{4}R^2+4\sqrt{1-s}$$
$$-\frac{3}{2}R\sqrt{1-s}+2Rs-\frac{3}{4}R^2s-\frac{63}{2}\ln 2+\frac{27}{2}R\ln 2-\frac{9}{2}\ln 2\sqrt{1-s}+6s\ln 2$$
$$-\frac{9}{2}Rs\ln 2+\frac{81}{4}\ln^2 2-\frac{27}{4}s\ln^2 2-\frac{21}{2}\ln M+\frac{9}{2}R\ln M-\frac{3}{2}\sqrt{1-s}\ln M+2s\ln M$$
$$-\frac{3}{2}Rs\ln M+\frac{27}{2}\ln 2\ln M-\frac{9}{2}s\ln 2\ln M+\frac{9}{4}\ln^2 M-\frac{3}{4}s\ln^2 M+2sj\left(s\right)$$
$$-\frac{3}{4}Rsj\left(s\right)-\frac{9}{4}sj\left(s\right)\ln 2-\frac{3}{4}sj\left(s\right)\ln M\Bigg]\Bigg\}. \quad (3.37)$$

The second term of (3.32) can be calculated with

$$\frac{\partial}{\partial x'}\left[\int_{h'}^{z'} u_0'^2\,\mathrm{d}z'\right] = \frac{\partial}{\partial x'}\left[h'\int_1^s u_0'^2\,\mathrm{d}s\right]. \quad (3.38)$$

This leads to

$$\frac{\partial}{\partial x'}\left[\int_{h'}^{z'} u_0'^2\,\mathrm{d}z'\right] = \lambda_0 h'\frac{\partial h'}{\partial x'}\Bigg\{4\left(s-2\right)+\mu\Bigg[18-8R+\frac{16}{3}\sqrt{1-s}-6s+4Rs$$
$$+\frac{8}{3}s\sqrt{1-s}-24\ln 2+12s\ln 2-8\ln M+4s\ln M-8s\ln\left(1+\sqrt{1-s}\right)+4s\ln s\Bigg]$$
$$+\mu^2\Bigg[-\frac{23}{3}+9R-2R^2-\frac{4}{3}s-3Rs+R^2s+27\ln 2-12R\ln 2-9s\ln 2+6Rs\ln 2$$
$$-18\ln^2 2+9s\ln^2 2+9\ln M-4R\ln M-3s\ln M+2Rs\ln M-12\ln 2\ln M$$
$$+6s\ln 2\ln M-2\ln^2 M+s\ln^2 M-\frac{8}{3}\ln s-3sj\left(s\right)+2Rsj\left(s\right)+6sj\left(s\right)\ln 2$$
$$+2sj\left(s\right)\ln M+sj\left(s\right)^2-6\sqrt{1-s}+\frac{8}{3}R\sqrt{1-s}+\frac{4}{3}Rs\sqrt{1-s}+8\ln 2\sqrt{1-s}$$
$$+4s\sqrt{1-s}\ln 2+\frac{8}{3}\sqrt{1-s}\ln M+\frac{4}{3}s\sqrt{1-s}\ln M$$
$$+\frac{8}{3}\sqrt{1-s}j\left(s\right)+\frac{4}{3}s\sqrt{1-s}j\left(s\right)\Bigg]\Bigg\}. \quad (3.39)$$

The expression of the third term of (3.32) is

$$u_0'w_0' = \lambda_0 h'\frac{\partial h'}{\partial x'}\Bigg\{-2s+\mu\Bigg[2-2\sqrt{1-s}+3s-2Rs-2s\sqrt{1-s}-6s\ln 2-2s\ln M-2sj\left(s\right)\Bigg]$$
$$+\mu^2\Bigg[-5+R+5\sqrt{1-s}-R\sqrt{1-s}+2s+\frac{3}{2}Rs-\frac{R^2}{2}s-Rs\sqrt{1-s}+3\ln 2-3\sqrt{1-s}\ln 2$$
$$+\frac{9}{2}s\ln 2-3Rs\ln 2-3s\sqrt{1-s}\ln 2-\frac{9}{2}s\ln^2 2+\ln M-\sqrt{1-s}\ln M+\frac{3}{2}s\ln M$$
$$-Rs\ln M-s\sqrt{1-s}\ln M-3s\ln 2\ln M-\frac{1}{2}s\ln^2 M+j\left(s\right)-\sqrt{1-s}j\left(s\right)+\frac{3}{2}sj\left(s\right)$$
$$-Rsj\left(s\right)-s\sqrt{1-s}j\left(s\right)-3sj\left(s\right)\ln 2-sj\left(s\right)\ln M-\frac{1}{2}sj\left(s\right)^2\Bigg]\Bigg\} \quad (3.40)$$

The expression of the fourth term of (3.32) is

$$\frac{1}{F^2}\frac{\partial}{\partial x'}\int_{h'}^{z'} p_0'\,\mathrm{d}z' = -\lambda_0 h'\frac{\partial h'}{\partial x'}\frac{\cos\theta}{\lambda_0 F^2}\left(1-s\right). \tag{3.41}$$

We find finally

$$\tau_1' = \frac{\lambda_0 h'}{\kappa^2}\frac{\partial h'}{\partial x'}\left[T_0\left(s\right)+\mu T_1\left(s\right)+\mu^2 T_2\left(s\right)\right], \tag{3.42}$$

where $s = z'/h'$ and:

$$\begin{aligned}
T_0\left(s\right) &= \left(1-\frac{\cos\theta}{\lambda_0 F^2}\right)\left(1-s\right),\\
T_1\left(s\right) &= \left(R-1+3\ln 2+\ln M\right)\left(1-s\right)+\frac{\sqrt{1-s}}{3}+\frac{2}{3}s\sqrt{1-s}+\frac{1}{2}sj\left(s\right),\\
T_2\left(s\right) &= \Big[\frac{R^2}{4}-\frac{R}{2}-\frac{2}{3}+\frac{3}{2}\left(R-1+\frac{3}{2}\ln 2\right)\ln 2\\
&\quad +\frac{1}{2}\left(R-1+3\ln 2+\frac{1}{2}\ln M\right)\ln M\Big]\left(1-s\right)\\
&\quad +\left(3+\frac{R}{6}+\frac{\ln 2}{2}+\frac{1}{6}\ln M\right)\sqrt{1-s}+\frac{1}{3}\left(R+3\ln 2+\ln M\right)s\sqrt{1-s}\\
&\quad -2\ln\left(1+\sqrt{1-s}\right)+\frac{5}{3}\sqrt{1-s}j\left(s\right)+\frac{1}{4}\left(2+R+3\ln 2+\ln M\right)sj\left(s\right)\\
&\quad +\frac{1}{3}s\sqrt{1-s}j\left(s\right)+\frac{1}{2}sj(s)^2-\frac{5}{3}\ln s.
\end{aligned} \tag{3.43}$$

It appears that τ_1' possesses a finite limit when $z'\to 0$ (or $s\to 0$), namely

$$\tau_1'\left(0\right) = \frac{\lambda_0 h'}{\kappa^2}\frac{\partial h'}{\partial x'}\left[T_0\left(0\right)+\mu T_1\left(0\right)+\mu^2 T_2\left(0\right)\right], \tag{3.44}$$

with

$$\begin{aligned}
T_0\left(0\right) &= 1-\frac{\cos\theta}{\lambda_0 F^2},\\
T_1\left(0\right) &= R-\frac{2}{3}+3\ln 2+\ln M,\\
T_2\left(0\right) &= \frac{7}{3}-\frac{19}{3}\ln 2+\frac{9}{4}\ln^2 2-\frac{R}{3}+\frac{3}{2}R\ln 2+\frac{R^2}{4}\\
&\quad +\left(\frac{R}{2}-\frac{1}{3}+\frac{3}{2}\ln 2+\frac{1}{4}\ln M\right)\ln M.
\end{aligned} \tag{3.45}$$

In the internal (viscous) layer, (2.30) implies that $\tilde{\tau}_1$ is constant. It follows from the matching procedure that this constant is equal to $\tau_1'\left(0\right)$ given by equation (3.44).

Next, the constitutive law and the external (shallow-water) effective viscosity (2.20) give

$$\tau_1' = 2z'^2\frac{\partial u_0'}{\partial z'}\frac{\partial u_1'}{\partial z'}. \tag{3.46}$$

The inversion of this relation gives

$$\frac{\partial u_1'}{\partial z'} = \frac{1}{2\kappa^2}\sqrt{\frac{\lambda_0}{h'}}\frac{\partial h'}{\partial x'}\frac{T_0\left(s\right)+\mu T_1\left(s\right)+\mu^2 T_2\left(s\right)}{\mu s\sqrt{1-s}}. \tag{3.47}$$

The expression of the velocity at order 1 in the external layer is found by integration:

$$u_1' = u_1'(h) + \frac{\sqrt{\lambda_0 h'}}{2\kappa^2}\frac{\partial h'}{\partial x'}\frac{1}{\mu}\int_1^s \frac{T_0(s) + \mu T_1(s) + \mu^2 T_2(s)}{s\sqrt{1-s}}\,\mathrm{d}s \tag{3.48}$$

The calculation yields

$$\int_1^s \frac{T_0(s)}{s\sqrt{1-s}}\,\mathrm{d}s = \left(1 - \frac{\cos\theta}{\lambda_0 F^2}\right)\left[2\sqrt{1-s} + j(s)\right], \tag{3.49}$$

$$\int_1^s \frac{T_1(s)}{s\sqrt{1-s}}\,\mathrm{d}s = (R - 1 + 3\ln 2 + \ln M)\left[2\sqrt{1-s} + j(s)\right] + \frac{4}{3}\ln s + \frac{2}{3}(s-1) - \sqrt{1-s}\,j(s), \tag{3.50}$$

$$\begin{aligned}\int_1^s \frac{T_2(s)}{s\sqrt{1-s}}\,\mathrm{d}s = {} & \left[\frac{R^2}{4} - \frac{R}{2} - \frac{2}{3} - \frac{3}{2}\ln 2 + \frac{3}{2}R\ln 2 + \frac{9}{4}\ln^2 2 \right.\\ & \left. + \left(\frac{R}{2} - \frac{1}{2} + \frac{3}{2}\ln 2 + \frac{\ln M}{4}\right)\ln M\right]\left[2\sqrt{1-s} + j(s)\right]\\ & + \left(3 + \frac{R}{6} - \frac{73}{6}\ln 2 + \frac{\ln M}{6}\right)\ln s + \left(\frac{R}{3} + \ln 2 + \frac{\ln M}{3}\right)(s-1)\\ & - \frac{5}{3}j(s)\ln s - \frac{19}{18}\pi^2 - 19\ln^2\frac{1+\sqrt{1-s}}{2} + \frac{8}{3}j(s)^2 - 2j(s)\ln\left(1+\sqrt{1-s}\right)\\ & + \frac{1}{3}\left[2\sqrt{1-s} + sj(s)\right] + \left(\frac{1}{2} + \frac{R}{4} + \frac{3}{4}\ln 2 + \frac{\ln M}{4}\right)\left[2\ln s - 2\sqrt{1-s}\,j(s)\right]\\ & - \sqrt{1-s}\,j(s)^2 + \frac{38}{3}\ln s\ln\left(1+\sqrt{1-s}\right) + \frac{38}{3}\mathrm{dilog}\frac{1+\sqrt{1-s}}{2}\\ & + \frac{76}{3}\ln^2 2 - \frac{76}{3}\ln 2\ln\left(1+\sqrt{1-s}\right),\end{aligned} \tag{3.51}$$

where the function dilog is defined by

$$\mathrm{dilog}(x) = \int_1^x \frac{\ln t}{1-t}\,\mathrm{d}t. \tag{3.52}$$

Because u_1' diverges when $s \to 0$, we proceed as for the zeroth order: the same calculation is done in the internal layer and both expressions are fitted in the overlap domain. With the constitutive law and the internal effective viscosity (2.34), we get

$$\tilde{\tau}_1 = \frac{\partial \tilde{u}_1}{\partial \tilde{z}}\left[1 + 2\tilde{z}^2\left(1 - \mathrm{e}^{-\tilde{z}\mu\sqrt{\lambda_0 h'}/(\kappa A^+)}\right)^2\frac{\partial \tilde{u}_0}{\partial \tilde{z}}\right]. \tag{3.53}$$

Since $\tilde{\tau}_1$ is constant and equal to $\tau_1'(0)$,

$$\frac{\partial \tilde{u}_1}{\partial \tilde{z}} = \frac{\lambda_0 h'}{\kappa^2}\frac{\partial h'}{\partial x'}\frac{T_0(0) + \mu T_1(0) + \mu^2 T_2(0)}{\sqrt{1 + \xi^2\left(1 - \mathrm{e}^{-\xi/A}\right)^2}}, \tag{3.54}$$

where $\xi = 2\sqrt{\lambda h'}\tilde{z}$ and $A = 2\kappa A^+$. Integrating this expression from 0 to ξ, taking into account the no-slip condition, we find

$$\tilde{u}_1 = \frac{\sqrt{\lambda_0 h'}}{2\kappa^2}\frac{\partial h'}{\partial x'}\frac{1}{\mu}\left[T_0(0) + \mu T_1(0) + \mu^2 T_2(0)\right]\left[\ln\left(\xi + \sqrt{1+\xi^2}\right) + \mathcal{R}_1(\xi)\right], \tag{3.55}$$

where the function $\mathcal{R}_1$ is defined by

$$\mathcal{R}_1(\xi) = \int_0^{\xi} \left[\frac{1}{\sqrt{1+Z^2\left(1-\mathrm{e}^{-Z/A}\right)^2}} - \frac{1}{\sqrt{1+Z^2}} \right] \mathrm{d}Z. \tag{3.56}$$

This function possesses a finite limit when $\xi \to \infty$ which will be denoted by R_1. The replacement of $\mathcal{R}_1(Mb/\sqrt{\eta})$ by R_1 in the matching procedure below is justified in a similar way as for R (see Appendix A).

To match the external velocity with the inner one, we solve, for $0 < b_1 < b < b_2$,

$$u_1'(s=\sqrt{\eta}b) = \tilde{u}_1\left(\xi = \frac{Mb}{\sqrt{\eta}}\right) + O(\sqrt{\eta}). \tag{3.57}$$

In the external layer, we have

$$u_1'(s=\sqrt{\eta}b) = u_1'(h') + \frac{\sqrt{\lambda_0 h'}}{2\kappa^2}\frac{\partial h'}{\partial x'}\frac{1}{\mu}\int_1^{\sqrt{\eta}b} \frac{T_0(s)+\mu T_1(s)+\mu^2 T_2(s)}{s\sqrt{1-s}}\,\mathrm{d}s. \tag{3.58}$$

We calculate firstly

$$\int_1^{\sqrt{\eta}b} \frac{T_0(s)}{s\sqrt{1-s}}\,\mathrm{d}s = -\frac{1}{\mu}\left(1-\frac{\cos\theta}{\lambda_0 F^2}\right) + (2-2\ln 2+\ln b)\left(1-\frac{\cos\theta}{\lambda_0 F^2}\right) + O(\sqrt{\eta}), \tag{3.59}$$

secondly,

$$\begin{aligned}\int_1^{\sqrt{\eta}b} \frac{T_1(s)}{s\sqrt{1-s}}\,\mathrm{d}s = &-\frac{1}{\mu}\left(R-\frac{2}{3}+3\ln 2+\ln M\right) + (R-1+3\ln 2+\ln M)(2-2\ln 2)\\ &+2\ln 2-\frac{2}{3}+\ln b\left(R-\frac{2}{3}+3\ln 2+\ln M\right)+O(\sqrt{\eta}),\end{aligned} \tag{3.60}$$

and thirdly

$$\begin{aligned}\int_1^{\sqrt{\eta}b} \frac{T_2(s)}{s\sqrt{1-s}}\,\mathrm{d}s = -\frac{1}{\mu}\Bigg[&\frac{7}{3}-\frac{R}{3}+\frac{R^2}{4}+\left(\frac{3}{2}R-\frac{19}{3}+\frac{9}{4}\ln 2\right)\ln 2\\ &+\left(\frac{R}{2}-\frac{1}{3}+\frac{3}{2}\ln 2+\frac{\ln M}{4}\right)\ln M\Bigg]+O(1)+O(\sqrt{\eta}).\end{aligned} \tag{3.61}$$

In the internal layer, we have

$$\tilde{u}_1\left(\frac{Mb}{\sqrt{\eta}}\right) = \frac{\sqrt{\lambda_0 h'}}{2\kappa^2}\frac{\partial h'}{\partial x'}\frac{1}{\mu}\left[T_0(0)+\mu T_1(0)+\mu^2 T_2(0)\right]\left[\ln(2Mb)+R_1+\frac{1}{\mu}\right]. \tag{3.62}$$

We calculate

$$\begin{aligned}&\left[T_0(0)+\mu T_1(0)+\mu^2 T_2(0)\right]\left[\ln(2Mb)+R_1+\frac{1}{\mu}\right] = \frac{1}{\mu}\left(1-\frac{\cos\theta}{\lambda_0 F^2}\right)\\ &\quad + R + R_1 - \frac{2}{3} + 4\ln 2 + 2\ln M - \frac{\cos\theta}{\lambda_0 F^2}(R_1+\ln 2+\ln M) + \ln b\left(1-\frac{\cos\theta}{\lambda_0 F^2}\right)\\ &+\mu\Bigg[\frac{7}{3}-\frac{R}{3}+\frac{R^2}{4}-\frac{2}{3}R_1+RR_1-7\ln 2+\frac{5}{2}R\ln 2+3R_1\ln 2+\frac{21}{4}\ln^2 2-\ln M+\frac{3}{2}R\ln M\\ &+R_1\ln M+\frac{11}{2}\ln 2\ln M+\frac{5}{4}\ln^2 M+\ln b\left(R-\frac{2}{3}+3\ln 2+\ln M\right)\Bigg]+O(\mu^2)+O(\sqrt{\eta}).\end{aligned} \tag{3.63}$$

Solving (3.57) using the previous calculations gives

$$u_1'(h) = \frac{\sqrt{\lambda_0 h'}}{\kappa^2}\frac{\partial h'}{\partial x'}\left[\frac{1}{\mu^2}\left(1-\frac{\cos\theta}{\lambda_0 F^2}\right)+\frac{V_1}{\mu}+V_2\right]+O(\mu), \tag{3.64}$$

where

$$\begin{aligned} V_1 &= \frac{1}{2}\left[2R+R_1-\frac{10}{3}+9\ln 2+3\ln M+\frac{\cos\theta}{\lambda_0 F^2}\left(2-R_1-3\ln 2-\ln M\right)\right], \\ V_2 &= \frac{1}{2}\left[\frac{22}{3}-\frac{8}{3}R+\frac{R^2}{2}-\frac{2}{3}R_1+RR_1+\left(6R+3R_1-\frac{70}{3}+\frac{27}{2}\ln 2\right)\ln 2\right. \\ &\quad \left.+\left(2R+R_1-\frac{10}{3}+9\ln 2+\frac{3}{2}\ln M\right)\ln M\right]. \end{aligned} \tag{3.65}$$

The final expression of u_1' is

$$\begin{aligned} u_1' = u_1'(h) + \frac{\sqrt{\lambda_0 h'}}{2\kappa^2}\frac{\partial h'}{\partial x'}\Bigg[&\left(1-\frac{\cos\theta}{\lambda_0 F^2}\right)\frac{2\sqrt{1-s}+j(s)}{\mu}+\frac{4}{3}\ln s+\frac{2}{3}(s-1) \\ &+\left(R-1+3\ln 2+\ln M\right)\left(2\sqrt{1-s}+j(s)\right)-\sqrt{1-s}\,j(s)\Bigg]+O(\mu). \end{aligned} \tag{3.66}$$

Even if this expression is not valid over the whole fluid depth (it diverges for $z \to 0$) and must be connected to the inner expression, u_1' possesses a finite average value over the whole depth. This first order average velocity, denoted by $U_1' = \langle u_1' \rangle$ is given by:

$$U_1' = u_1'(h) - \frac{\sqrt{\lambda_0 h'}}{3\kappa^2}\frac{\partial h'}{\partial x'}\left[\frac{1}{\mu}\left(1-\frac{\cos\theta}{\lambda_0 F^2}\right)+R+\frac{1}{2}+\ln 2+\ln M\right]+O(\mu). \tag{3.67}$$

We are now able to derive our reduce models of smooth turbulent free surface flow.