

Supplementary Material

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1. Detailed Derivation

The viscous potential flow approach employing a decomposition of the velocity field into a curl-free and vortical part used in our derivation and earlier (Lamb 1932; Miles 1968; Prosperetti 1976; Menikoff *et al.* 1978; Prosperetti 1981), relies on the following identity for a vector field $\mathbf{v}_p(\mathbf{x})$,

$$\nabla(\nabla \cdot \mathbf{v}_p) - \nabla \times (\nabla \times \mathbf{v}_p) = \nabla^2 \mathbf{v}_p. \quad (1.1)$$

If $\mathbf{v}_p = \nabla \phi_p$, then identity 1.1 reduces to

$$\nabla(\nabla^2 \phi_p) = \nabla^2(\nabla \phi_p). \quad (1.2)$$

This implies that a potential velocity field \mathbf{v}_p and a corresponding pressure field p_p obtained from the Bernoulli's equation, satisfies the Navier-Stokes equation exactly (Joseph 2003), since according to identity 1.2 the viscous term in the Navier-Stokes equation vanishes identically for a potential flow field. However this potential flow field \mathbf{v}_p and p_p , does not satisfy continuity of tangential stresses and of tangential velocities (slip) at an interface separating two fluids of different viscosities and densities. An additional velocity and pressure field, \mathbf{v}_v and p_v is needed and the composite fields viz. $\mathbf{v}_p + \mathbf{v}_v$ and $p_p + p_v$ are determined subject to the constraint that each field *individually* satisfies the linearised Navier-Stokes as well as continuity equations while the composite field satisfies the kinematic boundary condition, continuity of tangential and normal stresses and that of tangential velocities. Due to linearity, the composite solution will automatically satisfy the linearised Navier-Stokes equation as well. We determine \mathbf{v}_p and p_p from solving the Laplace and linearised Bernoulli's equation respectively.

1.1. Potential Flow

The potential solution satisfies the axisymmetric Laplace equation. From variable separation the radial part of the solution is given by Bessel function of the first kind (boundedness at $r = 0$, eliminates the Bessel function of the second kind)

$$\phi_p^{\mathcal{U}}(r, z, t) = F(z)J_0(kr)\dot{a}_k(t), \quad \phi_p^{\mathcal{L}}(r, z, t) = G(z)J_0(kr)\dot{a}_k(t), \quad (1.3)$$

with $\eta(r, t) = a_k(t)J_0(kr)$. We set the time dependence of ϕ_p equal to $\dot{a}_k(t)$ anticipating the linearised kinematic boundary condition at the interface Eq. 1.7. The axisymmetric Laplace equation is obtained from the continuity equation $\nabla \cdot \mathbf{v}_p = 0$ written in cylindrical axisymmetric coordinates with $u_p \equiv \frac{\partial \phi_p}{\partial r}$ and $v_p \equiv \frac{\partial \phi_p}{\partial z}$. This is (Kundu & Cohen 2002),

$$\frac{\partial^2 \phi_p}{\partial r^2} + \frac{1}{r} \frac{\partial \phi_p}{\partial r} + \frac{\partial^2 \phi_p}{\partial z^2} = 0. \quad (1.4)$$

Substituting Eq. 1.3 in this and using the equation for $J_0(\bar{r})$ ($\bar{r} \equiv kr$),

$$\frac{d^2 J_0(\bar{r})}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{dJ_0}{d\bar{r}} + J_0(\bar{r}) = 0, \quad (1.5)$$

we obtain

$$\frac{d^2 F}{dz^2} + k^2 F = 0, \quad \frac{d^2 G}{dz^2} + k^2 F = 0. \quad (1.6)$$

Using $\phi^{\mathcal{U}}(r, \infty, t) \rightarrow 0$ and $\phi^{\mathcal{L}}(r, -\infty, t) \rightarrow 0$, and the linearised kinematic boundary condition

$$\left. \frac{\partial \phi_p^{\mathcal{U}}}{\partial z} \right|_{z=0} = \left. \frac{\partial \phi_p^{\mathcal{L}}}{\partial z} \right|_{z=0} = \frac{\partial \eta}{\partial t} \quad (1.7)$$

we obtain

$$\phi_p^{\mathcal{U}}(r, z, t) = -k^{-1} \exp[-kz] J_0(kr) \dot{a}_k(t), \quad \phi_p^{\mathcal{L}}(r, z, t) = k^{-1} \exp[kz] J_0(kr) \dot{a}_k(t) \quad (1.8)$$

The potential part of pressure $p_p^{\mathcal{U}}$ and $p_p^{\mathcal{L}}$ is given by the linearised Bernoulli's equation,

$$\begin{aligned} p_p^{\mathcal{L}}(r, z, t) &= -\rho^{\mathcal{L}} \frac{\partial \phi_p^{\mathcal{L}}}{\partial t} - \rho^{\mathcal{L}} g z \\ p_p^{\mathcal{U}}(r, z, t) &= -\rho^{\mathcal{U}} \frac{\partial \phi_p^{\mathcal{U}}}{\partial t} - \rho^{\mathcal{U}} g z. \end{aligned} \quad (1.9)$$

If the flow is purely irrotational, then continuity of pressure at the linearised interface, leads to the following equation for $a_k(t)$,

$$\ddot{a}_k(t) + \left[\left(\frac{\rho^{\mathcal{L}} - \rho^{\mathcal{U}}}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \right) g k \right] a_k(t) = 0 \quad (1.10)$$

As we include viscous effects, instead of imposing continuity of pressure, we impose a condition on viscous normal stresses (accounting for jump due to surface tension) at the interface and this will lead to a modified viscous equation for $a_k(t)$. This equation is derived in the next section.

1.2. Viscous Flow

The viscous part of the flow satisfies the linearised Navier-Stokes equation in both the fluids

$$\frac{\partial \mathbf{v}_v^{\mathcal{U}}}{\partial t} = -\frac{1}{\rho^{\mathcal{U}}} \nabla p_v^{\mathcal{U}} + \nu^{\mathcal{U}} \nabla^2 \mathbf{v}_v^{\mathcal{U}}, \quad \frac{\partial \mathbf{v}_v^{\mathcal{L}}}{\partial t} = -\frac{1}{\rho^{\mathcal{L}}} \nabla p_v^{\mathcal{L}} + \nu^{\mathcal{L}} \nabla^2 \mathbf{v}_v^{\mathcal{L}}. \quad (1.11)$$

Note that gravity has already been included as a body force in the potential part of the calculation and hence is excluded in the vortical calculation. We solve the viscous part of the flow in stream-function vorticity formulation. The curl of Eqs. 1.5 gives us the vorticity equation

$$\frac{\partial \omega^{\mathcal{U}}}{\partial t} = \nu^{\mathcal{U}} \nabla^2 \omega^{\mathcal{U}}, \quad \frac{\partial \omega^{\mathcal{L}}}{\partial t} = \nu^{\mathcal{L}} \nabla^2 \omega^{\mathcal{L}}. \quad (1.12)$$

The Stokes stream function $r\psi_v$ (Miles 1968) is defined as

$$\begin{aligned} u_v^{\mathcal{U}} &\equiv \frac{\partial \psi_v^{\mathcal{U}}}{\partial z}, & v_v^{\mathcal{U}} &\equiv -\frac{1}{r} \frac{\partial (r\psi_v^{\mathcal{U}})}{\partial r} \\ u_v^{\mathcal{L}} &\equiv \frac{\partial \psi_v^{\mathcal{L}}}{\partial z}, & v_v^{\mathcal{L}} &\equiv -\frac{1}{r} \frac{\partial (r\psi_v^{\mathcal{L}})}{\partial r} \end{aligned} \quad (1.13)$$

The azimuthal vorticity $(\omega)_\theta \equiv \omega_\theta$, is given by

$$\omega_\theta^\mathcal{U} \equiv \frac{\partial u_v^\mathcal{U}}{\partial z} - \frac{\partial v_v^\mathcal{U}}{\partial r}, \quad \omega_\theta^\mathcal{L} \equiv \frac{\partial u_v^\mathcal{L}}{\partial z} - \frac{\partial v_v^\mathcal{L}}{\partial r}. \quad (1.14)$$

Combining Eq. 1.13 and 1.14, the relation between vorticity and streamfunction is written as

$$\frac{\partial^2 \psi_v^\mathcal{U}}{\partial r^2} - \frac{\psi_v^\mathcal{U}}{r^2} + \frac{1}{r} \frac{\partial \psi_v^\mathcal{U}}{\partial r} + \frac{\partial^2 \psi_v^\mathcal{U}}{\partial z^2} = \omega_\theta^\mathcal{U}, \quad \frac{\partial^2 \psi_v^\mathcal{L}}{\partial r^2} - \frac{\psi_v^\mathcal{L}}{r^2} + \frac{1}{r} \frac{\partial \psi_v^\mathcal{L}}{\partial r} + \frac{\partial^2 \psi_v^\mathcal{L}}{\partial z^2} = \omega_\theta^\mathcal{L}. \quad (1.15)$$

An equation for ω_θ can be obtained from Eq. 1.12 using the expression for the laplacian of a vector in cylindrical axisymmetric coordinates (Kundu & Cohen 2002)),

$$\frac{\partial \omega_\theta^\mathcal{U}}{\partial t} = \nu^\mathcal{U} \left(\frac{\partial^2 \omega_\theta^\mathcal{U}}{\partial r^2} - \frac{\omega_\theta^\mathcal{U}}{r^2} + \frac{1}{r} \frac{\partial \omega_\theta^\mathcal{U}}{\partial r} + \frac{\partial^2 \omega_\theta^\mathcal{U}}{\partial z^2} \right), \quad \frac{\partial \omega_\theta^\mathcal{L}}{\partial t} = \nu^\mathcal{L} \left(\frac{\partial^2 \omega_\theta^\mathcal{L}}{\partial r^2} - \frac{\omega_\theta^\mathcal{L}}{r^2} + \frac{1}{r} \frac{\partial \omega_\theta^\mathcal{L}}{\partial r} + \frac{\partial^2 \omega_\theta^\mathcal{L}}{\partial z^2} \right) \quad (1.16)$$

The solution of Eqs. 1.15 and 1.16 subject to boundary and initial conditions, determine the flow field in both fluids. We set the radial part of all quantities as Bessel function of the first kind $J_1(kr)$ viz.

$$\omega_\theta^\mathcal{U}(r, z, t) = \Omega^\mathcal{U}(z, t) J_1(kr), \quad \psi_v^\mathcal{U}(r, z, t) = \Psi^\mathcal{U}(z, t) J_1(kr), \quad (1.17)$$

$$\omega_\theta^\mathcal{L}(r, z, t) = \Omega^\mathcal{L}(z, t) J_1(kr), \quad \psi_v^\mathcal{L}(r, z, t) = \Psi^\mathcal{L}(z, t) J_1(kr). \quad (1.18)$$

Substitution of Eqs. 1.17 and 1.18 in Eq. 1.15, gives us an equation relating Ω to Ψ ,

$$\frac{\partial^2 \Psi^\mathcal{U}}{\partial z^2} - k^2 \Psi^\mathcal{U} = \Omega^\mathcal{U}(z, t), \quad \frac{\partial^2 \Psi^\mathcal{L}}{\partial z^2} - k^2 \Psi^\mathcal{L} = \Omega^\mathcal{L}(z, t). \quad (1.19)$$

Note that in deriving Eq. 1.19, we have used the equation governing $J_1(\bar{r})$ i.e.

$$\frac{d^2 J_1(\bar{r})}{d\bar{r}^2} + \frac{1}{\bar{r}} \frac{dJ_1}{d\bar{r}} + \left(1 - \frac{1}{\bar{r}^2}\right) J_1(\bar{r}) = 0. \quad (1.20)$$

An equation involving Ω alone is obtained by substituting expressions for vorticity from Eqs. 1.17 and 1.18 into Eqs. 1.16 and then using Eq. 1.20.

$$\frac{\partial \Omega^\mathcal{U}}{\partial t} = \nu \left(\frac{\partial^2 \Omega^\mathcal{U}}{\partial z^2} - k^2 \Omega^\mathcal{U} \right), \quad \frac{\partial \Omega^\mathcal{L}}{\partial t} = \nu \left(\frac{\partial^2 \Omega^\mathcal{L}}{\partial z^2} - k^2 \Omega^\mathcal{L} \right). \quad (1.21)$$

Eqs. 1.19 and 1.21 are the central equations. These turn out to be identical to those derived by Prosperetti (1981) for a planar geometry. We follow the approach of Prosperetti (1981) for solving these equations. These steps involve additional manipulations with Bessel functions not necessary in the planar case.

1.3. Viscous pressure

In order to obtain an equation for $a_k(t)$, we will need an expression for the viscous part of pressure p_v in both the fluids. This is obtained by integrating the vertical momentum equation. We demonstrate the algebra for the lower fluid. The (linearised) vertical momentum equation is (Kundu & Cohen 2002)

$$\frac{1}{\rho^\mathcal{L}} \frac{\partial p_v^\mathcal{L}}{\partial z} = \nu^\mathcal{L} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_v^\mathcal{L}}{\partial r} \right) + \frac{\partial^2 v_v^\mathcal{L}}{\partial z^2} \right] - \frac{\partial v_v^\mathcal{L}}{\partial t} \quad (1.22)$$

Note that gravity is already included in the potential flow and hence is not present in the viscous part of the calculation. With $v_v^\mathcal{L} = -\frac{1}{r} \frac{\partial(r\psi_v^\mathcal{L})}{\partial r}$ and Eq. 1.18, $-v_v^\mathcal{L} =$

$k\Psi^{\mathcal{L}}(dJ_1(\bar{r})/d\bar{r} + J_1/\bar{r})$. Substituting this in Eq. 1.22, using the equation governing J_0 and the identity $J_1'(x) + J_1(x)/x = J_0(x)$, prime indicating differentiation, we obtain

$$\frac{1}{\rho^{\mathcal{L}}} \frac{\partial p_v^{\mathcal{L}}}{\partial z} = kJ_0(kr) \left[\nu^{\mathcal{L}} \left(k^2 \Psi_v^{\mathcal{L}} - \frac{\partial^2 \Psi^{\mathcal{L}}}{\partial z^2} \right) + \frac{\partial \Psi^{\mathcal{L}}}{\partial t} \right] \quad (1.23)$$

Eq. 1.23 can be further simplified by using Eq. 1.19 on the right hand side to rewrite it as

$$\frac{1}{\rho^{\mathcal{L}}} \frac{\partial p_v^{\mathcal{L}}}{\partial z} = kJ_0(kr) \left[-\nu^{\mathcal{L}} \Omega^{\mathcal{L}} + \frac{\partial \Psi^{\mathcal{L}}}{\partial t} \right] \quad (1.24)$$

A similar equation can be written for the viscous pressure in the upper fluid.

2. Boundary/Initial conditions

In this section, we provide a derivation of the kinematic boundary condition, continuity of shear stress and tangential velocities at the interface.

2.1. Initial conditions

There is no motion in the fluid at $t = 0$ (we also assume $\hat{a}_k(0) = 0$ later). Hence,

$$\Omega^{\mathcal{U}}(z, 0) = \Omega^{\mathcal{L}}(z, 0) = \Psi^{\mathcal{U}}(z, 0) = \Psi^{\mathcal{L}}(z, 0) = 0. \quad (2.1)$$

2.2. Decay at $\pm\infty$

All quantities decay to zero at $\pm\infty$ and hence

$$\Psi^{\mathcal{U}}(\infty, t) = \Omega^{\mathcal{U}}(\infty, 0) = 0, \quad \Psi^{\mathcal{L}}(-\infty, t) = \Omega^{\mathcal{L}}(-\infty, 0) = 0 \quad (2.2)$$

2.3. Kinematic Boundary Condition

The (linearized) kinematic boundary condition is

$$v^{\mathcal{U}}(r, 0, t) = v^{\mathcal{L}}(r, 0, t) = \frac{\partial \eta}{\partial t}. \quad (2.3)$$

Writing the velocity as a sum of potential and viscous parts,

$$\left. \frac{\partial \phi^{\mathcal{U}}}{\partial z} \right|_{z=0} + v_v^{\mathcal{L}}(r, 0, t) = \left. \frac{\partial \phi^{\mathcal{L}}}{\partial z} \right|_{z=0} + v_v^{\mathcal{L}}(r, 0, t) = \frac{\partial \eta}{\partial t}. \quad (2.4)$$

Subtracting the kinematic boundary condition for potential flow viz. Eq. 1.7 from 2.4 we find that

$$v_v^{\mathcal{U}}(r, 0, t) = v_v^{\mathcal{L}}(r, 0, t) = 0 \quad (2.5)$$

Thus,

$$\Psi^{\mathcal{U}}(0, t) = \Psi^{\mathcal{L}}(0, t) = 0 \quad (2.6)$$

In deriving Eq. 2.6 we have used the identity $J_1'(x) + J_1(x)/x = J_0(x)$. Also note that Eq. 2.5 implies

$$\left. \frac{\partial v_v^{\mathcal{U}}}{\partial r} \right|_{z=0} = \left. \frac{\partial v_v^{\mathcal{L}}}{\partial r} \right|_{z=0}. \quad (2.7)$$

2.4. Continuity of shear stress

The shear stress τ_{rz} is continuous at the linearised interface implying,

$$\mu^{\mathcal{U}} \left(\frac{\partial v^{\mathcal{U}}}{\partial r} + \frac{\partial u^{\mathcal{U}}}{\partial z} \right) \Big|_{z=0} = \mu^{\mathcal{L}} \left(\frac{\partial v^{\mathcal{L}}}{\partial r} + \frac{\partial u^{\mathcal{L}}}{\partial z} \right) \Big|_{z=0}. \quad (2.8)$$

where $u^{\mathcal{U}} = u_p^{\mathcal{U}} + u_v^{\mathcal{U}}$, $v^{\mathcal{U}} = v_p^{\mathcal{U}} + v_v^{\mathcal{U}}$ and likewise for the lower fluid. Combining now Eq. 2.7 and Eq. 2.8 we find,

$$\mu^{\mathcal{U}} \left(\frac{\partial v_p^{\mathcal{U}}}{\partial r} + \frac{\partial u_p^{\mathcal{U}}}{\partial z} + \frac{\partial u_v^{\mathcal{U}}}{\partial z} \right) \Big|_{z=0} = \mu^{\mathcal{L}} \left(\frac{\partial v_p^{\mathcal{L}}}{\partial r} + \frac{\partial u_p^{\mathcal{L}}}{\partial z} + \frac{\partial u_v^{\mathcal{L}}}{\partial z} \right) \Big|_{z=0}. \quad (2.9)$$

Eq. 2.9 can be used to relate the vorticity on either sides of the interface at $z = 0$. As the interface acts as a source of vorticity, the vorticity field is discontinuous at the linearised interface $z = 0$. The potential part of the flow satisfies

$$\frac{\partial u_p^{\mathcal{U}}}{\partial z} = \frac{\partial v_p^{\mathcal{U}}}{\partial r}, \quad \frac{\partial u_p^{\mathcal{L}}}{\partial z} = \frac{\partial v_p^{\mathcal{L}}}{\partial r}, \quad (2.10)$$

everywhere in the flow. This can be used in Eq. 2.9 which simplifies to,

$$\mu^{\mathcal{U}} \left(2 \frac{\partial v_p^{\mathcal{U}}}{\partial r} + \frac{\partial u_v^{\mathcal{U}}}{\partial z} \right) \Big|_{z=0} = \mu^{\mathcal{L}} \left(2 \frac{\partial v_p^{\mathcal{L}}}{\partial r} + \frac{\partial u_v^{\mathcal{L}}}{\partial z} \right) \Big|_{z=0} \quad (2.11)$$

We now evaluate the following expression utilising Eq. 2.11.

$$(\mu^{\mathcal{U}} \omega_{\theta}^{\mathcal{U}} - \mu^{\mathcal{L}} \omega_{\theta}^{\mathcal{L}}) \Big|_{z=0} = \mu^{\mathcal{U}} \left(\frac{\partial u_v^{\mathcal{U}}}{\partial z} - \frac{\partial v_v^{\mathcal{U}}}{\partial r} \right) \Big|_{z=0} - \mu^{\mathcal{L}} \left(\frac{\partial u_v^{\mathcal{L}}}{\partial z} - \frac{\partial v_v^{\mathcal{L}}}{\partial r} \right) \Big|_{z=0} \quad (2.12)$$

Using Eq. 2.7, Eq. 2.12 can be simplified to obtain

$$(\mu^{\mathcal{U}} \Omega^{\mathcal{U}}(0, t) - \mu^{\mathcal{L}} \Omega^{\mathcal{L}}(0, t)) J_1(kr) = \left(\mu^{\mathcal{U}} \frac{\partial u_v^{\mathcal{U}}}{\partial z} - \mu^{\mathcal{L}} \frac{\partial u_v^{\mathcal{L}}}{\partial z} \right) \Big|_{z=0} \quad (2.13)$$

Using Eq. 2.11 to simplify the right hand side of Eq. 2.14, we obtain

$$\begin{aligned} (\mu^{\mathcal{U}} \Omega^{\mathcal{U}}(0, t) - \mu^{\mathcal{L}} \Omega^{\mathcal{L}}(0, t)) J_1(kr) &= 2 \left(\mu^{\mathcal{L}} \frac{\partial v_p^{\mathcal{L}}}{\partial r} - \mu^{\mathcal{U}} \frac{\partial v_p^{\mathcal{U}}}{\partial r} \right) \Big|_{z=0} \\ &= 2 \left(\mu^{\mathcal{L}} \frac{\partial^2 \phi_p^{\mathcal{L}}}{\partial r \partial z} - \mu^{\mathcal{U}} \frac{\partial^2 \phi_p^{\mathcal{U}}}{\partial r \partial z} \right) \Big|_{z=0} \end{aligned} \quad (2.14)$$

Eq. 2.14 can be further simplified to,

$$\mu^{\mathcal{U}} \Omega^{\mathcal{U}}(0, t) - \mu^{\mathcal{L}} \Omega^{\mathcal{L}}(0, t) = -2k(\mu^{\mathcal{L}} - \mu^{\mathcal{U}}) \dot{a}_k(t) \quad (2.15)$$

where we have used the relation $\frac{dJ_0(x)}{dx} = -J_1(x)$ to eliminate the radial dependence from both sides.

2.5. Continuity of tangential velocities

It is clear that the tangential velocities are not continuous at $z = 0$ for potential flow. Imposing the continuity of tangential velocities at the linearised interface implies,

$$\left(\frac{\partial \phi_p^{\mathcal{U}}}{\partial r} + \frac{\partial \psi_v^{\mathcal{U}}}{\partial z} \right) \Big|_{z=0} = \left(\frac{\partial \phi_p^{\mathcal{L}}}{\partial r} + \frac{\partial \psi_v^{\mathcal{L}}}{\partial z} \right) \Big|_{z=0} \quad (2.16)$$

Using expressions derived earlier, this can be written as

$$\dot{a}_k(t) + \frac{\partial \Psi^{\mathcal{U}}}{\partial z}(0, t) = -\dot{a}_k(t) + \frac{\partial \Psi^{\mathcal{L}}}{\partial z}(0, t) \quad (2.17)$$

where we have used $\frac{dJ_0(x)}{dx} = -J_1(x)$. Eq. 2.17 can be rearranged to obtain

$$\frac{\partial \Psi^{\mathcal{L}}}{\partial z}(0, t) - \frac{\partial \Psi^{\mathcal{U}}}{\partial z}(0, t) = 2\dot{a}_k(t). \quad (2.18)$$

3. Laplace Transforms

3.1. Equations in the Laplace domain

Eqs. 1.19 and 1.21 are easily solved in the Laplace domain. We define the Laplace transform of $\Omega(z, t)$ and $\Psi(z, t)$ as

$$L(\Omega(z, t)) \equiv \tilde{\Omega}(z, s) = \int_0^\infty \Omega(z, t) \exp[-st] dt, \quad \tilde{\Psi}(z, s) \equiv \int_0^\infty \Psi(z, t) \exp[-st] dt.$$

Note that all quantities with a tilde on top are in the Laplace domain. The Laplace transform of Eqs. 1.19 leads to,

$$\frac{\partial^2 \tilde{\Psi}^{\mathcal{U}}}{\partial z^2} - k^2 \tilde{\Psi}^{\mathcal{U}} = \tilde{\Omega}^{\mathcal{U}}(z, s), \quad \frac{\partial^2 \tilde{\Psi}^{\mathcal{L}}}{\partial z^2} - k^2 \tilde{\Psi}^{\mathcal{L}} = \tilde{\Omega}^{\mathcal{L}}(z, s). \quad (3.1)$$

Laplace transforming Eqs. 1.21 and using initial conditions in Eq. 2.1, we obtain

$$\begin{aligned} \frac{\partial^2 \tilde{\Omega}^{\mathcal{U}}(z, s)}{\partial z^2} - \left(k^2 + \frac{s}{\nu^{\mathcal{U}}}\right) \tilde{\Omega}^{\mathcal{U}}(z, s) &= 0 \\ \frac{\partial^2 \tilde{\Omega}^{\mathcal{L}}(z, s)}{\partial z^2} - \left(k^2 + \frac{s}{\nu^{\mathcal{L}}}\right) \tilde{\Omega}^{\mathcal{L}}(z, s) &= 0 \end{aligned} \quad (3.2)$$

Laplace transform of the viscous contribution to pressure in Eq. 1.23 leads to

$$\frac{1}{\rho^{\mathcal{L}}} \frac{\partial \tilde{p}_v^{\mathcal{L}}}{\partial z} = kJ_0(kr) \left[-\nu^{\mathcal{L}} \tilde{\Omega}^{\mathcal{L}} + s \tilde{\Psi}^{\mathcal{L}} \right] \quad (3.3)$$

where $\Psi^{\mathcal{L}}(z, 0) = 0$ is taken as initial condition. Similarly we can find an expression for pressure for the upper fluid,

$$\frac{1}{\rho^{\mathcal{U}}} \frac{\partial \tilde{p}_v^{\mathcal{U}}}{\partial z} = kJ_0(kr) \left[-\nu^{\mathcal{U}} \tilde{\Omega}^{\mathcal{U}} + s \tilde{\Psi}^{\mathcal{U}} \right] \quad (3.4)$$

Laplace transforming Eqs. 2.2, 2.6, 2.15 and 2.18, we obtain

$$\tilde{\Psi}(\infty, s) = \tilde{\Omega}^{\mathcal{U}}(\infty, 0) = 0, \quad \tilde{\Psi}^{\mathcal{L}}(-\infty, t) = \tilde{\Omega}^{\mathcal{L}}(-\infty, 0) = 0 \quad (3.5)$$

$$\mu^{\mathcal{U}} \tilde{\Omega}^{\mathcal{U}}(0, s) - \mu^{\mathcal{L}} \tilde{\Omega}^{\mathcal{L}}(0, s) = -2k(\mu^{\mathcal{L}} - \mu^{\mathcal{U}}) L(\dot{a}_k(t)), \quad (3.6)$$

$$\frac{\partial \tilde{\Psi}^{\mathcal{L}}}{\partial z}(0, s) - \frac{\partial \tilde{\Psi}^{\mathcal{U}}}{\partial z}(0, s) = 2L(\dot{a}_k(t)), \quad (3.7)$$

$$\tilde{\Psi}^{\mathcal{U}}(0, s) = \tilde{\Psi}^{\mathcal{L}}(0, s) = 0. \quad (3.8)$$

3.2. Solution in Laplace Domain

Our task is to first solve Eqs. 3.1, 3.2 using Eqs. 3.5 and 3.6. The solution of Eq. 3.2 is

$$\tilde{\Omega}^{\mathcal{U}}(z, s) = \tilde{A}^{\mathcal{U}}(s) \exp[-z\lambda_{\mathcal{U}}], \quad \tilde{\Omega}^{\mathcal{L}}(z, s) = \tilde{A}^{\mathcal{L}}(s) \exp[z\lambda_{\mathcal{L}}]. \quad (3.9)$$

Here $A^{\mathcal{L}}(s)$ and $A^{\mathcal{U}}(s)$ are constants of integration and following Prosperetti (1981), we define $\lambda_{\mathcal{U}} \equiv \sqrt{k^2 + \frac{s}{\mu^{\mathcal{U}}}}$ and so on. Eqs. 3.1 are solved using 3.5 and 3.6 to obtain,

$$\tilde{\Psi}^{\mathcal{U}}(z, s) = \frac{\tilde{A}^{\mathcal{U}}(s)}{\lambda_{\mathcal{U}}^2 - k^2} (\exp[-z\lambda_{\mathcal{U}}] - \exp[-kz]) \quad (3.10)$$

$$\tilde{\Psi}^{\mathcal{L}}(z, s) = \frac{\tilde{A}^{\mathcal{L}}(s)}{\lambda_{\mathcal{L}}^2 - k^2} (\exp[z\lambda_{\mathcal{L}}] - \exp[kz]) \quad (3.11)$$

Eqs 3.7 and 3.8 can now be written as

$$\mu^{\mathcal{U}} \tilde{A}^{\mathcal{U}}(s) - \mu^{\mathcal{L}} \tilde{A}^{\mathcal{L}}(s) = -2k(\mu^{\mathcal{L}} - \mu^{\mathcal{U}})L[\dot{a}_k(t)], \quad (3.12)$$

$$\frac{\tilde{A}^{\mathcal{L}}(s)}{k + \lambda_{\mathcal{L}}} + \frac{\tilde{A}^{\mathcal{U}}(s)}{k + \lambda_{\mathcal{U}}} = 2L[\dot{a}_k(t)] \quad (3.13)$$

Eqs. 3.12 and 3.13 can be solved for $\tilde{A}^{\mathcal{L}}(s)$ and $\tilde{A}^{\mathcal{U}}(s)$ to obtain,

$$\tilde{A}^{\mathcal{U}}(s) = \frac{2L(\dot{a}_k(t))(k + \lambda_{\mathcal{U}})(\mu^{\mathcal{U}}k + \mu^{\mathcal{L}}\lambda_{\mathcal{L}})}{\mu^{\mathcal{L}}(k + \lambda_{\mathcal{L}}) + \mu^{\mathcal{U}}(k + \lambda_{\mathcal{U}})}, \quad (3.14)$$

$$\text{and } \tilde{A}^{\mathcal{L}}(s) = \frac{2L(\dot{a}_k(t))(k + \lambda_{\mathcal{L}})(\mu^{\mathcal{L}}k + \mu^{\mathcal{U}}\lambda_{\mathcal{U}})}{\mu^{\mathcal{L}}(k + \lambda_{\mathcal{L}}) + \mu^{\mathcal{U}}(k + \lambda_{\mathcal{U}})}. \quad (3.15)$$

The expression for the viscous part of pressure can also be simplified further. Eq. 3.3 can be written as

$$\frac{1}{\rho^{\mathcal{L}}} \frac{\partial \tilde{p}_v^{\mathcal{L}}}{\partial z} = kJ_0(kr) \left[-\nu^{\mathcal{L}} \tilde{A}^{\mathcal{L}}(s) \exp[z\lambda_{\mathcal{L}}] + s\tilde{\Psi}^{\mathcal{L}} \right] \quad (3.16)$$

Eq. 3.11 can be rewritten as

$$s\tilde{\Psi}^{\mathcal{L}}(z, s) = \tilde{A}^{\mathcal{L}}(s)\nu^{\mathcal{L}} (\exp[z\lambda_{\mathcal{L}}] - \exp[kz]) \quad (3.17)$$

Using Eq. 3.17 in 3.16 we obtain

$$\frac{1}{\rho^{\mathcal{L}}} \frac{\partial \tilde{p}_v^{\mathcal{L}}}{\partial z} = -kJ_0(kr)\tilde{A}^{\mathcal{L}}(s)\nu^{\mathcal{L}} \exp[kz] \quad (3.18)$$

which can be integrated from $z' = -\infty$ to $z' = z$ (z' being dummy variable) to obtain

$$\tilde{p}_v^{\mathcal{L}}(r, z, s) = -\mu^{\mathcal{L}} \tilde{A}^{\mathcal{L}}(s)J_0(kr) \exp[kz]. \quad (3.19)$$

Using a similar procedure and integrating from $z' = z$ to $z' = \infty$ for the upper fluid,

$$\tilde{p}_v^{\mathcal{U}}(r, z, s) = \mu^{\mathcal{U}} \tilde{A}^{\mathcal{U}}(s)J_0(kr) \exp[-kz]. \quad (3.20)$$

The potential part of the flow in Laplace domain is written below.

$$\tilde{\phi}_p^{\mathcal{U}}(r, z, s) = -k^{-1} \exp[-kz]J_0(kr)L(\dot{a}_k(t)), \quad \tilde{\phi}_p^{\mathcal{L}}(r, z, s) = k^{-1} \exp[kz]J_0(kr)L(\dot{a}_k(t)) \quad (3.21)$$

$$\tilde{p}_p^{\mathcal{L}}(r, z, s) = -\rho^{\mathcal{L}} \left(s\tilde{\phi}_p^{\mathcal{L}} - \phi_p^{\mathcal{L}}(r, z, 0) \right) - \rho^{\mathcal{L}}gz \quad (3.22)$$

$$p_p^{\mathcal{U}}(r, z, s) = -\rho^{\mathcal{U}} \left(s\tilde{\phi}_p^{\mathcal{U}} - \phi_p^{\mathcal{U}}(r, z, 0) \right) - \rho^{\mathcal{U}}gz. \quad (3.23)$$

3.3. Final assembly

We can now obtain an equation for $\tilde{a}_k(s)$ by taking into account the jump in normal stresses across the interface due to surface tension. This is (cf. Bush (2013)),

$$\sigma_{zz}^{\mathcal{U}}(r, 0, t) - \sigma_{zz}^{\mathcal{L}}(r, 0, t) = T(\nabla \cdot \mathbf{n}). \quad (3.24)$$

where \mathbf{n} is the local unit normal to an axisymmetric interface $z = \eta(r, t) = a_k(t)J_0(kr)$. The local radius of curvature is given by (Bush 2013),

$$\nabla \cdot \mathbf{n} = \frac{-r\eta_r - r^2\eta_{rr}}{r^2(1 + \eta_r^2)^{3/2}} \quad (3.25)$$

Eq. 3.25 when linearised for small-amplitude oscillations becomes,

$$-\nabla \cdot \mathbf{n} \approx \eta_{rr} + \frac{1}{r}\eta_r = a_k(t) \left(\frac{d^2 J_0}{dr^2} + \frac{1}{r} \frac{dJ_0}{dr} \right) = -k^2 a_k(t) J_0(kr). \quad (3.26)$$

Eq. 3.24 and 3.26 can be combined to obtain,

$$p^\mathcal{L}(r, 0, t) - p^\mathcal{U}(r, 0, t) + 2\mu^\mathcal{U} \frac{\partial v^\mathcal{U}}{\partial z} \Big|_{(r,0,t)} - 2\mu^\mathcal{L} \frac{\partial v^\mathcal{L}}{\partial z} \Big|_{(r,0,t)} = Tk^2 a_k(t) J_0(kr) \quad (3.27)$$

Note that we have used the Newtonian constitutive relation $\sigma_{zz} = -p + 2\mu \frac{\partial v}{\partial z}$. Using the decomposition $p = p_p + p_v$ and $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_v$ for upper and lower fluids, Eq. 3.27 can be rewritten after Laplace transformation as,

$$\begin{aligned} & \tilde{p}_p^\mathcal{L}(r, 0, s) + \tilde{p}_v^\mathcal{L}(r, 0, s) - \tilde{p}_p^\mathcal{U}(r, 0, s) - \tilde{p}_v^\mathcal{U}(r, 0, s) + 2\mu^\mathcal{U} \frac{\partial \tilde{v}_p^\mathcal{U}}{\partial z} \Big|_{(r,0,s)} + 2\mu^\mathcal{U} \frac{\partial \tilde{v}_v^\mathcal{U}}{\partial z} \Big|_{(r,0,s)} \\ & - 2\mu^\mathcal{L} \frac{\partial \tilde{v}_p^\mathcal{L}}{\partial z} \Big|_{(r,0,s)} - 2\mu^\mathcal{L} \frac{\partial \tilde{v}_v^\mathcal{L}}{\partial z} \Big|_{(r,0,s)} = Tk^2 L(a_k(t)) J_0(kr) \end{aligned} \quad (3.28)$$

From Eqs. 3.22 and 3.23 we have

$$\begin{aligned} \tilde{p}_p^\mathcal{L}(r, 0, s) &= -\rho^\mathcal{L} \left(s\tilde{\phi}_p^\mathcal{L} - \phi_l^\mathcal{L}(r, 0, 0) \right) - \rho^\mathcal{L} g J_0(kr) L(a_k(t)) \\ &= \rho^\mathcal{L} \left[-sk^{-1} J_0(kr) L(\dot{a}_k(t)) + k^{-1} J_0(kr) \dot{a}_k(0) \right] - \rho^\mathcal{L} g J_0(kr) L(a_k(t)) \end{aligned} \quad (3.29)$$

Similarly,

$$\begin{aligned} \tilde{p}_p^\mathcal{U}(r, 0, s) &= -\rho^\mathcal{U} \left(s\tilde{\phi}_p^\mathcal{U} - \phi_l^\mathcal{U}(r, 0, 0) \right) - \rho^\mathcal{U} g J_0(kr) L(a_k(t)) \\ &= \rho^\mathcal{U} \left[sk^{-1} J_0(kr) L(\dot{a}_k(t)) - k^{-1} J_0(kr) \dot{a}_k(0) \right] - \rho^\mathcal{U} g J_0(kr) L(a_k(t)) \end{aligned} \quad (3.30)$$

From Eq. 3.19 and 3.20, we obtain

$$\tilde{p}_v^\mathcal{L}(r, 0, t) - \tilde{p}_v^\mathcal{U}(r, 0, t) = - \left(\mu^\mathcal{L} \tilde{A}^\mathcal{L} + \mu^\mathcal{U} \tilde{A}^\mathcal{U} \right) J_0(kr) \quad (3.31)$$

Also,

$$2\mu^\mathcal{U} \frac{\partial \tilde{v}_p^\mathcal{U}}{\partial z} \Big|_{z=0} - 2\mu^\mathcal{L} \frac{\partial \tilde{v}_p^\mathcal{L}}{\partial z} \Big|_{z=0} = -2k(\mu^\mathcal{U} + \mu^\mathcal{L}) J_0(kr) L(\dot{a}_k(t)) \quad (3.32)$$

It can be shown that

$$\tilde{v}_v^\mathcal{L} = -k\tilde{\Psi}^\mathcal{L}(z, s) J_0(kr) \quad (3.33)$$

from which we obtain

$$\begin{aligned} 2\mu^\mathcal{U} \frac{\partial \tilde{v}_v^\mathcal{U}}{\partial z} \Big|_{z=0} - 2\mu^\mathcal{L} \frac{\partial \tilde{v}_v^\mathcal{L}}{\partial z} \Big|_{z=0} &= -2k\mu^\mathcal{U} \frac{\partial \tilde{\Psi}^\mathcal{U}}{\partial z} \Big|_{z=0} J_0(kr) + 2k\mu^\mathcal{L} \frac{\partial \tilde{\Psi}^\mathcal{L}}{\partial z} \Big|_{z=0} J_0(kr) \\ &= 2kJ_0(kr) \left(\frac{\mu^\mathcal{U}}{\lambda_\mathcal{U} + k} \tilde{A}^\mathcal{U}(s) + \frac{\mu^\mathcal{L}}{\lambda_\mathcal{L} + k} \tilde{A}^\mathcal{L}(s) \right) \end{aligned} \quad (3.34)$$

In the final assembly, we substitute expressions from Eqs. 3.29, 3.30, 3.31, 3.32 and 3.34 into Eq. 3.28 and divide throughout by $J_0(kr)$ to obtain,

$$\begin{aligned} & -(\rho^{\mathcal{L}} + \rho^{\mathcal{U}})k^{-1}s[s\tilde{a}_k(s) - a_k(0)] + (\rho^{\mathcal{L}} + \rho^{\mathcal{U}})k^{-1}\dot{a}_k(0) - (\rho^{\mathcal{L}} - \rho^{\mathcal{U}})g\tilde{a}_k(s) \\ & - \left(\mu^{\mathcal{L}}\tilde{A}^{\mathcal{L}} + \mu^{\mathcal{U}}\tilde{A}^{\mathcal{U}}\right) - 2k(\mu^{\mathcal{U}} + \mu^{\mathcal{L}})[s\tilde{a}_k(s) - a_k(0)] \\ & + 2k\left(\frac{\mu^{\mathcal{U}}\tilde{A}^{\mathcal{U}}}{\lambda_{\mathcal{U}} + k} + \frac{\mu^{\mathcal{L}}\tilde{A}^{\mathcal{L}}}{\lambda_{\mathcal{L}} + k}\right) = Tk^2\tilde{a}_k(s) \end{aligned} \quad (3.35)$$

which can be rewritten as,

$$\begin{aligned} & (\rho^{\mathcal{L}} + \rho^{\mathcal{U}})s^2\tilde{a}_k - (\rho^{\mathcal{L}} + \rho^{\mathcal{U}})sa_k(0) + 2k^2(\mu^{\mathcal{U}} + \mu^{\mathcal{L}})s\tilde{a}_k - (\rho^{\mathcal{L}} + \rho^{\mathcal{U}})\dot{a}_k(0) - 2k^2(\mu^{\mathcal{U}} + \mu^{\mathcal{L}})a_k(0) \\ & + [(\rho^{\mathcal{L}} - \rho^{\mathcal{U}})gk + Tk^3]\tilde{a}_k + k\left(\mu^{\mathcal{L}}\tilde{A}^{\mathcal{L}} + \mu^{\mathcal{U}}\tilde{A}^{\mathcal{U}}\right) - 2k^2\left(\frac{\mu^{\mathcal{U}}\tilde{A}^{\mathcal{U}}}{\lambda_{\mathcal{U}} + k} + \frac{\mu^{\mathcal{L}}\tilde{A}^{\mathcal{L}}}{\lambda_{\mathcal{L}} + k}\right) = 0 \end{aligned} \quad (3.36)$$

We define $\xi(s)$ and $\zeta(s)$ from Eqs. 3.14 and 3.15 as

$$\begin{aligned} \tilde{A}^{\mathcal{U}}(s) &= \zeta(s)(s\tilde{a}_k - a_k(0)) \\ \tilde{A}^{\mathcal{L}}(s) &= \xi(s)(s\tilde{a}_k - a_k(0)) \end{aligned} \quad (3.37)$$

Thus

$$\zeta(s) \equiv \frac{2(k + \lambda_{\mathcal{U}})(\mu^{\mathcal{U}}k + \mu^{\mathcal{L}}\lambda_{\mathcal{L}})}{\mu^{\mathcal{L}}(k + \lambda_{\mathcal{L}}) + \mu^{\mathcal{U}}(k + \lambda_{\mathcal{U}})}, \quad \xi(s) \equiv \frac{2(k + \lambda_{\mathcal{L}})(\mu^{\mathcal{L}}k + \mu^{\mathcal{U}}\lambda_{\mathcal{U}})}{\mu^{\mathcal{L}}(k + \lambda_{\mathcal{L}}) + \mu^{\mathcal{U}}(k + \lambda_{\mathcal{U}})} \quad (3.38)$$

For this study, we will consider $\dot{a}_k(0) = 0$. Hence using Eqs. 3.37, Eq. 3.36 can be rewritten as

$$\begin{aligned} & [(\rho^{\mathcal{U}} + \rho^{\mathcal{L}})s^2 + 2k^2(\mu^{\mathcal{U}} + \mu^{\mathcal{L}})s + (\rho^{\mathcal{L}} - \rho^{\mathcal{U}})gk + Tk^3 + k(\mu^{\mathcal{L}}\xi + \mu^{\mathcal{U}}\zeta)s \\ & - 2k^2\left(\frac{\mu^{\mathcal{U}}\zeta}{\lambda_{\mathcal{U}} + k} + \frac{\mu^{\mathcal{L}}\xi}{\lambda_{\mathcal{L}} + k}\right)s]\tilde{a}_k(s) = [(\rho^{\mathcal{L}} + \rho^{\mathcal{U}})s + 2k^2(\mu^{\mathcal{U}} + \mu^{\mathcal{L}}) \\ & + k(\mu^{\mathcal{L}}\xi + \mu^{\mathcal{U}}\zeta) - 2k^2\left(\frac{\mu^{\mathcal{U}}\zeta}{\lambda_{\mathcal{U}} + k} + \frac{\mu^{\mathcal{L}}\xi}{\lambda_{\mathcal{L}} + k}\right)]a_k(0) \end{aligned} \quad (3.39)$$

which can be expressed as

$$\tilde{a}_k(s) = \frac{\left[s + \frac{1}{\rho^{\mathcal{U}} + \rho^{\mathcal{L}}} \left\{ 2k^2(\mu^{\mathcal{U}} + \mu^{\mathcal{L}}) + k(\mu^{\mathcal{U}}\zeta + \mu^{\mathcal{L}}\xi) - 2k^2\left(\frac{\mu^{\mathcal{U}}\zeta}{\lambda_{\mathcal{U}} + k} + \frac{\mu^{\mathcal{L}}\xi}{\lambda_{\mathcal{L}} + k}\right) \right\}\right]}{s^2 + \frac{1}{\rho^{\mathcal{U}} + \rho^{\mathcal{L}}} \left\{ 2k^2(\mu^{\mathcal{U}} + \mu^{\mathcal{L}}) + k(\mu^{\mathcal{U}}\zeta + \mu^{\mathcal{L}}\xi) - 2k^2\left(\frac{\mu^{\mathcal{U}}\zeta}{\lambda_{\mathcal{U}} + k} + \frac{\mu^{\mathcal{L}}\xi}{\lambda_{\mathcal{L}} + k}\right) \right\} s + \omega_0^2} a_k(0) \quad (3.40)$$

Thus we can write

$$\tilde{a}_k(s) = \frac{s + \Lambda(s)}{s^2 + \Lambda(s)s + \omega_0^2} a_k(0) \quad (3.41)$$

in a form analogous to that obtained in Prosperetti (1981), where

$$\omega_0^2 = gk \left(\frac{\rho^{\mathcal{L}} - \rho^{\mathcal{U}}}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \right) + \frac{Tk^3}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \quad (3.42)$$

and,

$$A(s) = \frac{1}{\rho^{\mathcal{U}} + \rho^{\mathcal{L}}} \left\{ 2k^2(\mu^{\mathcal{U}} + \mu^{\mathcal{L}}) + k(\mu^{\mathcal{U}}\zeta + \mu^{\mathcal{L}}\xi) - 2k^2 \left(\frac{\mu^{\mathcal{U}}\zeta}{\lambda_{\mathcal{U}} + k} + \frac{\mu^{\mathcal{L}}\xi}{\lambda_{\mathcal{L}} + k} \right) \right\} \quad (3.43)$$

3.4. Analytical expressions for $\nu^{\mathcal{L}} = \nu^{\mathcal{U}}$

Analytical expressions are possible (Chandrasekhar 1961; Menikoff *et al.* 1978; Prosperetti 1981) in the limit of equal kinematic viscosities of both fluids. In this limit $\lambda_{\mathcal{L}} = \lambda_{\mathcal{U}} = \lambda$, and we obtain

$$\zeta(s) = \frac{2(\rho^{\mathcal{U}}k + \rho^{\mathcal{L}}\lambda)}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}}, \quad \xi(s) = \frac{2(\rho^{\mathcal{L}}k + \rho^{\mathcal{U}}\lambda)}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}}. \quad (3.44)$$

3.4.1. Amplitude

Eq. 3.40, can now be written as

$$\tilde{a}_k(s) = \frac{\left\{ s + \frac{2\nu k^2}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \left[(\rho^{\mathcal{L}} + \rho^{\mathcal{U}}) + \frac{1}{2k} (\rho^{\mathcal{L}}\xi + \rho^{\mathcal{U}}\zeta) - \left(\frac{\rho^{\mathcal{L}}\xi + \rho^{\mathcal{U}}\zeta}{\lambda + k} \right) \right] \right\} a_k(0)}{s^2 + \frac{2\nu k^2}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \left[(\rho^{\mathcal{L}} + \rho^{\mathcal{U}}) + \frac{1}{2k} (\rho^{\mathcal{L}}\xi + \rho^{\mathcal{U}}\zeta) - \left(\frac{\rho^{\mathcal{L}}\xi + \rho^{\mathcal{U}}\zeta}{\lambda + k} \right) \right] s + \omega_0^2} \quad (3.45)$$

It is algebraically easier to invert this expression post non-dimensionalisation as follows

$$\hat{a}_k \equiv \frac{\tilde{a}_k(s)\omega_0}{a_0}, \quad q \equiv \frac{s}{\omega_0}, \quad \theta \equiv \frac{\nu k^2}{\omega_0}, \quad \Delta \equiv \frac{\rho^{\mathcal{L}}}{\rho^{\mathcal{U}}}. \quad (3.46)$$

Using these Eq. 3.45 can be simplified and written as

$$\hat{a}_k(q) = \frac{1}{q} \left[1 - \frac{(1 + \Delta)^2}{\tilde{G}(q)} \right] \quad (3.47)$$

$$\begin{aligned} \tilde{G}(q) = (1 + \Delta)^2 & \left[q^2 + 2\theta q + 2\theta q \frac{1 + \Delta^2}{(1 + \Delta)^2} + 4q\sqrt{\theta(q + \theta)} \frac{\Delta}{(1 + \Delta)^2} - 4\theta^{3/2}\sqrt{q + \theta} \frac{1 + \Delta^2}{(1 + \Delta)^2} \right. \\ & \left. - 8\theta(q + \theta) \frac{\Delta}{(1 + \Delta)^2} + 4\theta^2 \frac{1 + \Delta^2}{(1 + \Delta)^2} + 8\theta^{3/2}\sqrt{q + \theta} \frac{\Delta}{(1 + \Delta)^2} + 1 \right] \end{aligned}$$

In order to compare our final expression with Prosperetti (1981), we introduce the non-dimensional parameter $\beta \equiv \frac{\rho^{\mathcal{U}}\rho^{\mathcal{L}}}{(\rho^{\mathcal{L}} + \rho^{\mathcal{U}})^2} = \frac{\Delta}{(1 + \Delta)^2}$. In terms of β , the expression for $\tilde{G}(q)$ becomes

$$\begin{aligned} \tilde{G}(q) = (1 + \Delta)^2 & \left[(q + \theta)^2 + 2\theta(1 - 6\beta)(q + \theta) - 4\theta^{3/2}(1 - 3\beta)\sqrt{q + \theta} \right. \\ & \left. + 4\theta^{1/2}\beta(q + \theta)^{3/2} + \theta^2(1 - 4\beta) + 1 \right] \end{aligned} \quad (3.48)$$

Further define

$$\tilde{I}(q) \equiv \left[q^2 + 2\theta(1 - 6\beta)q - 4\theta^{3/2}(1 - 3\beta)\sqrt{q} + 4\theta^{1/2}\beta q^{3/2} + \theta^2(1 - 4\beta) + 1 \right]^{-1} \quad (3.49)$$

using which we can write Eq. 3.47 as

$$\hat{a}_k(q) = \frac{1}{q} \left(1 - \tilde{I}(q + \theta) \right) \quad (3.50)$$

In order to invert the above, we non-dimensionalise the following Laplace transform,

$$\tilde{a}_k(s) = \int_0^\infty \exp[-st] a_k(t) dt \quad (3.51)$$

by

$$\hat{a}_k \equiv \frac{\tilde{a}_k(s)\omega_0}{a_k(0)}, \quad \bar{a}_k \equiv \frac{a_k(t)}{a_k(0)}, \quad \tau = t\omega_0 \quad (3.52)$$

to obtain

$$\hat{a}_k(q) = \int_0^\infty \exp[-q\tau] \bar{a}_k(\tau) d\tau \quad (3.53)$$

The inversion of Eq. 3.50 is

$$\bar{a}_k(\tau) = 1 - \int_0^\tau \exp[-\theta m] \Gamma(m) dm \quad (3.54)$$

where $\Gamma(\tau)$ and $\tilde{\Gamma}(q)$ are non-dimensionalised Laplace transform pairs and m is a dummy variable. Eq. 3.54 can be further evaluated once $\Gamma(m)$ is known. For this we need to invert Eq. 3.49. Using $h \equiv -\sqrt{q}$, we transform this equation into a quartic,

$$\tilde{\Gamma} = \frac{1}{h^4 - 4\theta^{1/2}\beta h^3 + 2\theta(1 - 6\beta)h^2 + 4\theta^{3/2}(1 - 3\beta)h + \theta^2(1 - 4\beta) + 1} \quad (3.55)$$

The denominator is the same quartic which was obtained by Prosperetti (1981) earlier. Define $P(h)$ as

$$P(h) \equiv h^4 - 4\theta^{1/2}\beta h^3 + 2\theta(1 - 6\beta)h^2 + 4\theta^{3/2}(1 - 3\beta)h + \theta^2(1 - 4\beta) + 1 \quad (3.56)$$

Let the roots of the quartic be h_i such that $P(h) = \sum_{i=1}^4 (h - h_i)$ and using the partial fraction decomposition $P(h)^{-1} = \sum_{i=0}^4 A_i(h - h_i)^{-1}$, Eq. 3.55 can be written as

$$\tilde{\Gamma}(q) = \sum_{i=1}^4 \frac{-A_i}{\sqrt{q} + h_i} \quad (3.57)$$

Eq. 3.57 can be inverted to obtain a non-dimensional time-domain expression (Abramowitz & Segun 1970) (page 1023)

$$\Gamma(\tau) = \sum_{i=1}^4 -A_i \left[\frac{1}{\sqrt{\pi\tau}} - h_i \exp[h_i^2 \tau] \operatorname{Erfc}[h_i \sqrt{\tau}] \right] \quad (3.58)$$

Substituting this in Eq. 3.54, and using the fact that $\sum_{i=1}^4 A_i = 0$ we obtain

$$\bar{a}_k(\tau) = 1 - \sum_{i=1}^4 h_i A_i \int_0^\tau \exp[(h_i^2 - \theta)m] \operatorname{Erfc}[h_i \sqrt{m}] dm \quad (3.59)$$

where Erfc is the complementary error function. We need the following result,

$$\int_0^\tau \exp[(h_i^2 - \theta)m] \operatorname{Erfc}(h_i \sqrt{m}) dm = \frac{1 - \frac{h_i \operatorname{Erf}(\sqrt{\theta\tau})}{\sqrt{\theta}} - \exp[(h_i^2 - \theta)\tau] \operatorname{Erfc}(h_i \sqrt{\tau})}{\theta - h_i^2} \quad (3.60)$$

Substituting Eq. 3.60 into 3.59, we obtain

$$\bar{a}_k(\tau) = 1 - \sum_{i=1}^4 \frac{h_i A_i}{\theta - h_i^2} + \theta^{-1/2} \text{Erf} \left(\sqrt{\theta \tau} \right) \sum_{i=1}^4 \frac{h_i^2 A_i}{\theta - h_i^2} + \sum_{i=1}^4 \frac{A_i h_i \exp[(h_i^2 - \theta)\tau] \text{Erfc}(z_i \sqrt{\tau})}{\theta - h_i^2} \quad (3.61)$$

The following can be proven (for $\beta = 0$, they reduce to the one fluid expressions given by Prosperetti (1976))

$$\sum_{i=1}^4 \frac{A_i h_i}{\theta - h_i^2} = \frac{4\theta^2(1 - 4\beta) + 1}{8\theta^2(1 - 4\beta) + 1}, \quad \sum_{i=1}^4 \frac{A_i h_i^2}{\theta - h_i^2} = \frac{-4\theta^{5/2}(1 - 4\beta)}{8\theta^2(1 - 4\beta) + 1}. \quad (3.62)$$

Using these the first two terms in Eq. 3.61 can be combined as

$$\bar{a}_k(\tau) = \frac{4\theta^2(1 - 4\beta)}{8\theta^2(1 - 4\beta) + 1} \text{Erfc} \left(\sqrt{\theta \tau} \right) + \sum_{i=1}^4 \frac{A_i h_i \exp[(h_i^2 - \theta)\tau] \text{Erfc}(h_i \sqrt{\tau})}{\theta - h_i^2} \quad (3.63)$$

We will now re-dimensionalise our equation and write the final answer

$$\frac{a_k(t)}{a_k(0)} = \frac{4(\nu k^2)^2(1 - 4\beta)}{8(\nu k^2)^2(1 - 4\beta) + \omega_0^2} \text{Erfc} \left(\sqrt{\nu k^2 t} \right) + \sum_{i=1}^4 \frac{\hat{A}_i \hat{h}_i \omega_0^2 \exp[(\hat{h}_i^2 - \nu k^2)t] \text{Erfc}(\hat{h}_i \sqrt{t})}{\nu k^2 - \hat{h}_i^2}$$

(3.64a)

where $\hat{h}_i \equiv h_i \sqrt{\omega_0}$ is the root of the equation

$$\hat{h}^4 - 4(\nu k^2)^{1/2} \beta \hat{h}^3 + 2(\nu k^2)(1 - 6\beta) \hat{h}^2 + 4(\nu k^2)^{3/2}(1 - 3\beta) \hat{h} + (\nu k^2)^2(1 - 4\beta) + \omega_0^2 = 0$$

Note that $A_i \equiv \hat{A}_i \omega_0^{3/2}$ as A_i appears in the dimensional equation

$$\begin{aligned} & \frac{\omega_0^2}{\hat{h}^4 - 4(\nu k^2)^{1/2} \beta \hat{h}^3 + 2(\nu k^2)(1 - 6\beta) \hat{h}^2 + 4(\nu k^2)^{3/2}(1 - 3\beta) \hat{h} + (\nu k^2)^2(1 - 4\beta) + \omega_0^2} \\ &= \sum_{i=0}^4 \frac{A_i \sqrt{\omega_0}}{\hat{h} - \hat{h}_i} = \sum_{i=0}^4 \frac{\hat{A}_i \omega_0^2}{\hat{h} - \hat{h}_i} \end{aligned} \quad (3.65)$$

Finally note that if we know the roots \hat{h}_i , \hat{A}_i for $i = 1 \dots 4$ can be calculated using the partial fractions formula

$$\hat{A}_1 = \left((\hat{h}_1 - \hat{h}_2) (\hat{h}_1 - \hat{h}_3) (\hat{h}_1 - \hat{h}_4) \right)^{-1} \quad (3.66)$$

Other formulae for \hat{A}_2 , \hat{A}_3 and \hat{A}_4 arise similarly.

3.4.2. *Vorticity*

Thus the vorticity field becomes

$$\tilde{\Omega}^u(z, s) = \frac{2L(\dot{a}_k(t))(\rho^u k + \rho^L \sqrt{k^2 + s/\nu})}{\rho^L + \rho^u} \exp \left[-z \sqrt{k^2 + s/\nu} \right] \quad (3.67)$$

With the help of following inverse Laplace Transform (see Abramowitz & Segun (1970), page no. 1026)

$$L^{-1} \left[s^{\frac{n-1}{2}} \exp[-\alpha\sqrt{s}] \right] = \frac{\exp[-\frac{\alpha^2}{4t}]}{2^n \sqrt{\pi t^{n+1}}} H_n \left(\frac{\alpha}{2\sqrt{t}} \right), \quad (3.68)$$

and the first shifting theorem, Eq. 3.67 becomes

$$\begin{aligned} \tilde{\Omega}^{\mathcal{U}}(z, s) = & \left(\frac{\nu^{-1/2}}{\rho^{\mathcal{U}} + \rho^{\mathcal{L}}} \right) L(\dot{a}_k(t)) \left[k \rho^{\mathcal{U}} L \left(\frac{z \exp \left[- \left(\nu k^2 t + \frac{z^2}{4\nu t} \right) \right]}{\sqrt{\pi t^3}} \right) \right. \\ & \left. + \frac{\rho^{\mathcal{L}}}{2} L \left(\frac{\exp \left[- \left(\nu k^2 t + \frac{z^2}{4\nu t} \right) \right]}{\sqrt{\pi t^3}} H_2 \left(\frac{z}{2\sqrt{\nu t}} \right) \right) \right] \end{aligned} \quad (3.69)$$

where $H_i(x)$ ($i = 1, 2, \dots$) are the Hermite polynomials with $H_1(x) = 2x$ and $H_2(x) \equiv 4x^2 - 2$ (Abramowitz & Segun 1970). The linearity of the Laplace transform together with the convolution theorem allows us to invert Eq. 3.69 and obtain,

$$\boxed{\Omega^{\mathcal{U}}(z, t) = \left(\frac{\nu^{-1/2}}{\rho^{\mathcal{U}} + \rho^{\mathcal{L}}} \right) \left[\int_0^t dm \frac{\dot{a}_k(m) \exp \left[- \left(\nu k^2 (t-m) + \frac{z^2}{4\nu(t-m)} \right) \right]}{\sqrt{\pi(t-m)^3}} \left(z k \rho^{\mathcal{U}} + \frac{\rho^{\mathcal{L}}}{2} H_2 \left(\frac{z}{2\sqrt{\nu(t-m)}} \right) \right) \right]} \quad (3.70a)$$

We can thus write an expression for the vorticity in the upper fluid

$$\omega_{\theta}^{\mathcal{U}}(r, z, t) = \Omega^{\mathcal{U}}(z, t) J_1(kr), \quad (3.71)$$

where Eq. 3.69 gives the expression for $\Omega^{\mathcal{U}}(r, z, t)$. A similar expression can be obtained for the lower fluid as

$$\boxed{\Omega^{\mathcal{L}}(z, t) = \left(\frac{\nu^{-1/2}}{\rho^{\mathcal{U}} + \rho^{\mathcal{L}}} \right) \left[\int_0^t dm \frac{\dot{a}_k(m) \exp \left[- \left(\nu k^2 (t-m) + \frac{z^2}{4\nu(t-m)} \right) \right]}{\sqrt{\pi(t-m)^3}} \left(-z k \rho^{\mathcal{L}} + \frac{\rho^{\mathcal{U}}}{2} H_2 \left(\frac{-z}{2\sqrt{\nu(t-m)}} \right) \right) \right]} \quad (3.72a)$$

3.4.3. *Pressure*

We present here an expression by inverting the viscous part of pressure. The inversion of the potential part is easy and is not presented here. From Eq. 3.19, we have

$$\tilde{p}_v^{\mathcal{L}}(r, z, s) = -\mu^{\mathcal{L}} \tilde{A}^{\mathcal{L}}(s) J_0(kr) \exp[kz]. \quad (3.73)$$

In order to invert Eq. 3.73, we need to simply invert $A^{\mathcal{L}}(s)$, which from Eqs. 3.37 & 3.44 has the expression,

$$\begin{aligned} \tilde{A}^{\mathcal{L}}(s) &= \xi(s) (s\tilde{a}_k(s) - a_k(0)) \\ &= \frac{2(\rho^{\mathcal{L}}k + \rho^{\mathcal{U}}\lambda)}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} L[\dot{a}_k(t)] \\ &= \frac{2\rho^{\mathcal{L}}k}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} L[\dot{a}_k(t)] + \frac{2\rho^{\mathcal{U}}}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \sqrt{k^2 + s/\nu} (s\tilde{a}_k(s) - a_k(0)) \\ &\equiv \tilde{A}_1^{\mathcal{L}}(s) + \tilde{A}_2^{\mathcal{L}}(s) \end{aligned} \quad (3.74)$$

Note that in defining $\tilde{A}_2^{\mathcal{L}}(s)$, we have used the definition of λ . $\tilde{A}_1^{\mathcal{L}}(s)$ in Eq. 3.74 is directly invertible. We provide the algebra for inverting $\tilde{A}_2^{\mathcal{L}}(s)$. The algebra is easier to do in non-dimensional space. Using Eq. 3.46 we obtain

$$\tilde{A}_2^{\mathcal{L}}(q) = \frac{2\rho^{\mathcal{U}}a_k(0)}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \sqrt{\frac{\omega_0}{\nu}} \sqrt{q + \theta} (q\hat{a} - 1) \quad (3.75)$$

Using Eq. 3.50, Eq. 3.75 can be written as

$$\tilde{A}_2^{\mathcal{L}}(q) = -\frac{2\rho^{\mathcal{U}}a_k(0)}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \sqrt{\frac{\omega_0}{\nu}} \sqrt{q + \theta} \tilde{\Gamma}(q + \theta) = -\frac{2\rho^{\mathcal{U}}a_k(0)}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \sqrt{\frac{\omega_0}{\nu}} \tilde{M}(q + \theta) \quad (3.76)$$

where $\tilde{M}(q) \equiv \sqrt{q} \tilde{\Gamma}(q)$. With the help of first shifting theorem, we can invert 3.76 to obtain

$$A_2^{\mathcal{L}}(\tau) = -\frac{2\rho^{\mathcal{U}}a_k(0)}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \sqrt{\frac{\omega_0}{\nu}} \exp[-\theta\tau] M(\tau) \quad (3.77)$$

where $M(\tau)$, $\tilde{M}(q)$ and $A_2^{\mathcal{L}}(\tau)$, $\tilde{A}_2^{\mathcal{L}}(q)$ are Laplace transform pairs. Using $h = -\sqrt{q}$, similar to Eq. 3.55 we can write,

$$\tilde{M} = \frac{-h}{h^4 - 4\theta^{1/2}\beta h^3 + 2\theta(1 - 6\beta)h^2 + 4\theta^{3/2}(1 - 3\beta)h + \theta^2(1 - 4\beta) + 1} = \frac{-h}{P(h)} \quad (3.78)$$

Using the partial fraction decomposition $-hP(h)^{-1} = \sum_{i=1}^4 B_i(h - h_i)^{-1}$, we obtain

$$\tilde{M}(q) = \sum_{i=1}^4 \frac{-B_i}{\sqrt{q} + h_i} \quad (3.79)$$

Using Eq. 3.58 in Eq. 3.79, we obtain

$$M(\tau) = \sum_{i=1}^4 -B_i \left[\frac{1}{\sqrt{\pi\tau}} - h_i \exp[h_i^2\tau] \operatorname{Erfc}[h_i\sqrt{\tau}] \right] \quad (3.80)$$

Substituting this in expression 3.77 and using $\sum_{i=1}^4 B_i = 0$, we obtain

$$A_2^{\mathcal{L}}(\tau) = -\frac{2\rho^{\mathcal{U}}a_k(0)}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \sqrt{\frac{\omega_0}{\nu}} \exp[-\theta\tau] \sum_{i=1}^4 B_i h_i \exp[h_i^2\tau] \operatorname{Erfc}[h_i\sqrt{\tau}] \quad (3.81)$$

Note that $A_2^{\mathcal{L}}(\tau)$ is dimensionless and hence the dimensional version of it viz. $A_{2d}^{\mathcal{L}}(t) \equiv A_2^{\mathcal{L}}(\tau)\omega_0$ can be written as

$$\frac{A_{2d}^{\mathcal{L}}(t)}{\omega_0} = -\frac{2\rho^{\mathcal{U}}a_k(0)}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \sqrt{\frac{\omega_0}{\nu}} \exp[-\nu k^2 t] \sum_{i=1}^4 \hat{B}_i \hat{h}_i \omega_0^{1/2} \exp\left[\hat{h}_i^2 t\right] \text{Erfc}\left[\hat{h}_i \sqrt{t}\right] \quad (3.82)$$

where similar to earlier algebra, $B_i \equiv \hat{B}_i \omega_0$ as \hat{B}_i appears in the dimensional equation

$$\begin{aligned} & \frac{-\hat{h}\omega_0^{3/2}}{\hat{h}^4 - 4(\nu k^2)^{1/2}\beta\hat{h}^3 + 2(\nu k^2)(1-6\beta)\hat{h}^2 + 4(\nu k^2)^{3/2}(1-3\beta)\hat{h} + (\nu k^2)^2(1-4\beta) + \omega_0^2} \\ &= \sum_{i=0}^4 \frac{B_i \sqrt{\omega_0}}{\hat{h} - \hat{h}_i} = \sum_{i=0}^4 \frac{\hat{B}_i \omega_0^{3/2}}{\hat{h} - \hat{h}_i} \end{aligned} \quad (3.83)$$

and $\hat{h} = h\sqrt{\omega_0}$. The expression for the viscous part of pressure in the lower fluid in time domain can be written as

$$p_v^{\mathcal{L}}(r, z, t) = -\mu^{\mathcal{L}} J_0(kr) \exp[kz] \left(A_1^{\mathcal{L}}(t) + A_{2d}^{\mathcal{L}}(t) \right) \quad (3.84)$$

which can be rewritten as

$$\boxed{\begin{aligned} \frac{p_v^{\mathcal{L}}(r, z, t)}{\mu^{\mathcal{L}}\omega_0} &= -J_0(kr) \exp[kz] \left[\frac{2\rho^{\mathcal{L}}k\omega_0^{-1}}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \dot{a}_k(t) - \frac{2\rho^{\mathcal{U}}a_k(0)\omega_0^{1/2}\nu^{-1/2}}{\rho^{\mathcal{L}} + \rho^{\mathcal{U}}} \right. \\ &\quad \left. \exp[-\nu k^2 t] \sum_{i=1}^4 \hat{B}_i \hat{h}_i \omega_0^{1/2} \exp\left[\hat{h}_i^2 t\right] \text{Erfc}\left[\hat{h}_i \sqrt{t}\right] \right]. \end{aligned}} \quad (3.85a)$$

where \hat{h}_i are the roots of the equation,

$$\hat{h}^4 - 4(\nu k^2)^{1/2}\beta\hat{h}^3 + 2(\nu k^2)(1-6\beta)\hat{h}^2 + 4(\nu k^2)^{3/2}(1-3\beta)\hat{h} + (\nu k^2)^2(1-4\beta) + \omega_0^2 = 0$$

B_i 's can be evaluated using partial fractions. A similar expression for viscous part of pressure can be obtained for the upper fluid.

4. Additional Data

Axisymmetric simulations were also conducted for pure capillary waves. Data collapse using the scales suggested in Denner (2016) is shown below.

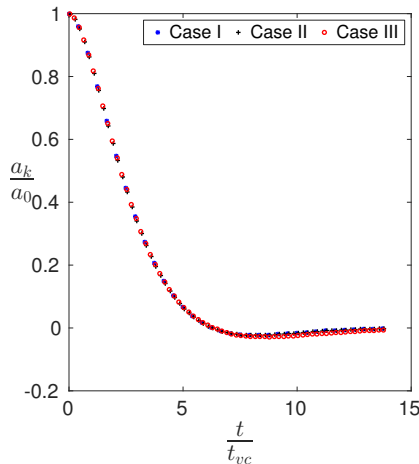


Figure 1: Collapse of the amplitude versus time curves for pure capillary waves using the time scale t_{vc} proposed in Denner (2016). The non-dimensional wave number in our axisymmetric DNS was chosen to be $\hat{k} = 0.5$ where \hat{k} is defined in Denner (2016). The parameters for cases I, II and III are provided in the data sheet available at Farsoiyya *et al.* (2017).

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