Supplementary material

Pattern selection in ternary mushy layers

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A. Nonlinear terms in the disturbance equations

The nonlinear terms $\hat{\Phi}$, $\hat{\Theta}_j$, $\hat{U}$, $\hat{V}$ and $\hat{W}$ which appear on the right-hand sides of the disturbance equations (2.19) are given by

$$\begin{aligned}\hat{\Phi} = &-\epsilon\big[m_1\boldsymbol{\nabla}\cdot\big(\hat{\phi}\boldsymbol{\nabla}\hat{C}_1\big) + m_2\boldsymbol{\nabla}\cdot\big(\hat{\phi}\boldsymbol{\nabla}\hat{C}_2\big)\big] + \epsilon\big[m_1\left(1-\Phi-Le_1\right)\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\hat{C}_1 + m_2\left(1-\Phi-Le_2\right)\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\hat{C}_2\big]\\ &+\epsilon\hat{\phi}\left(m_1 Le_1\frac{\partial\hat{C}_1}{\partial t} + m_2 Le_2\frac{\partial\hat{C}_2}{\partial t}\right),\end{aligned} \tag{A.1a}$$

$$\begin{aligned}\hat{\Theta}_1 = &\ \epsilon\frac{1}{Le_1}m_2\boldsymbol{\nabla}\cdot\big(\hat{\phi}\boldsymbol{\nabla}\hat{C}_1 - \hat{\phi}\boldsymbol{\nabla}\hat{C}_2\big) + \epsilon\frac{1}{Le_1}\big\{\left[m_1\left(1-\Phi-Le_1\right)+Le_1\right]\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\hat{C}_1 + m_2\left(1-\Phi-Le_2\right)\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\hat{C}_2\big\}\\ &+\epsilon\frac{1}{Le_1}m_2\hat{\phi}\left(Le_2\frac{\partial\hat{C}_2}{\partial t} - Le_1\frac{\partial\hat{C}_1}{\partial t}\right),\end{aligned} \tag{A.1b}$$

$$\begin{aligned}\hat{\Theta}_2 = &\ \epsilon\frac{1}{Le_2}m_1\boldsymbol{\nabla}\cdot\big(\hat{\phi}\boldsymbol{\nabla}\hat{C}_2 - \hat{\phi}\boldsymbol{\nabla}\hat{C}_1\big) + \epsilon\frac{1}{Le_2}\big\{\left[m_2\left(1-\Phi-Le_2\right)+Le_2\right]\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\hat{C}_2 + m_1\left(1-\Phi-Le_1\right)\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\hat{C}_1\big\}\\ &+\epsilon\frac{1}{Le_2}m_1\hat{\phi}\left(Le_1\frac{\partial\hat{C}_1}{\partial t} - Le_2\frac{\partial\hat{C}_2}{\partial t}\right),\end{aligned} \tag{A.1c}$$

$$\hat{U} = \epsilon\frac{\partial}{\partial x}\big[\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{K}\big(\hat{\phi}\big)\big] - \epsilon\nabla^2\big[\bar{K}\big(\hat{\phi}\big)\hat{u}\big], \tag{A.1d}$$

$$\hat{V} = \epsilon\frac{\partial}{\partial y}\big[\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{K}\big(\hat{\phi}\big)\big] - \epsilon\nabla^2\big[\bar{K}\big(\hat{\phi}\big)\hat{v}\big], \tag{A.1e}$$

$$\hat{W} = \epsilon\frac{\partial}{\partial z}\big[\hat{\boldsymbol{u}}\boldsymbol{\cdot}\boldsymbol{\nabla}\bar{K}(\hat{\phi})\big] - \epsilon\nabla^2\big[\bar{K}(\hat{\phi})\hat{w}\big]. \tag{A.1f}$$

B. Expansion of disturbance equations

We first describe the time scalings employed in a derivation of amplitude equations near criticality. From the general time-dependent linear stability results (§ 3), we know that the real part of the growth rate σ is proportional to $Ra_{Cj} - Ra_{Cj}^0$; that is, in view of the expansions (2.23e), the rate at which the critical disturbances evolve is slow. For a pitchfork bifurcation to rolls or squares (§ 4.1), we find $Ra_{C1}^1 = Ra_{C2}^1 = 0$ so that the difference $Ra_{Cj} - Ra_{Cj}^0 = O(\epsilon^2)$, and therefore employ a slow time scale $\tau = \epsilon^2 t$. For a transcritical bifurcation to hexagons (§ 4.2), we anticipate that $Ra_{Cj} - Ra_{Cj}^0 = O(\epsilon)$ in general, implying an appropriate time scale $\tau = \epsilon t$. A consistent derivation of the evolution amplitude equations in this case requires the use of the method of multiple scales involving both $O(\epsilon)$ and $O(\epsilon^2)$ time scales, as described in e.g. Fujimura (1991) and Kuske & Milewski (1999). However, for a special case where the hexagonal bifurcation in nearly vertical, it is appropriate to consider modulations over the time scale $\tau = \epsilon^2 t$ only, as in the pitchfork case. This special case is particularly interesting since the bifurcation to rolls is then supercritical, allowing transitions to *stable* states through secondary bifurcations. This is the procedure followed in § 4.2.

With this scaling employed, the set of linear equations which arise from substituting the expansions (2.23) into the nonlinear system (2.19), (2.16) and (2.17) is

$$\frac{\partial\phi^k}{\partial z} + \frac{1}{\Gamma}w^k = \Phi^k, \tag{B.1a}$$

$$\frac{1-\Phi}{Le_j}\nabla^2 C_j^k - \frac{(1-\Phi)\,\Omega_j}{Le_j}w^k = \Theta_j^k, \qquad \text{for} \quad j=1,2, \tag{B.1b}$$

$$\nabla^2 u^k + \frac{\partial^2}{\partial x\partial z}\big(Ra_{C1}^0 C_1^k + Ra_{C2}^0 C_2^k\big) = U^k, \tag{B.1c}$$

$$\nabla^2 v^k + \frac{\partial^2}{\partial y\partial z}\big(Ra_{C1}^0 C_1^k + Ra_{C2}^0 C_2^k\big) = V^k, \tag{B.1d}$$

$$\nabla^2 w^k - \nabla_2^2\big(Ra_{C1}^0 C_1^k + Ra_{C2}^0 C_2^k\big) = W^k, \tag{B.1e}$$

subject to

$$C_1^k = 0, \quad C_2^k = 0, \quad \phi^k = 0, \quad w^k = 0 \qquad \text{at} \quad z = 1, \tag{B.2a–d}$$

$$C_1^k = 0, \quad C_2^k = 0, \quad w^k = 0 \qquad \text{at} \quad z = 0. \tag{B.3a–c}$$

For $k = 0$,

$$\Phi^0 = \Theta_j^0 = U^0 = V^0 = W^0 = 0. \tag{B.4a–e}$$

For $k = 1$,

$$\Phi^1 = m_1 (1 - \Phi - Le_1) \boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_1^0 + m_2 (1 - \Phi - Le_2) \boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_2^0 - \big[m_1 \boldsymbol{\nabla} \boldsymbol{\cdot} \big(\phi^0 \boldsymbol{\nabla} C_1^0\big) + m_2 \boldsymbol{\nabla} \boldsymbol{\cdot} \big(\phi^0 \boldsymbol{\nabla} C_2^0\big)\big], \tag{B.5a}$$

$$\Theta_1^1 = \frac{1}{Le_1} \big\{[m_1 (1 - \Phi - Le_1) + Le_1] \boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_1^0 + m_2 (1 - \Phi - Le_2) \boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_2^0\big\} + \frac{1}{Le_1} m_2 \boldsymbol{\nabla} \boldsymbol{\cdot} \big(\phi^0 \boldsymbol{\nabla} C_1^0 - \phi^0 \boldsymbol{\nabla} C_2^0\big), \tag{B.5b}$$

$$\Theta_2^1 = \frac{1}{Le_2} \big\{[m_2 (1 - \Phi - Le_2) + Le_2] \boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_2^0 + m_1 (1 - \Phi - Le_1) \boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_1^0\big\} + \frac{1}{Le_2} m_1 \boldsymbol{\nabla} \boldsymbol{\cdot} \big(\phi^0 \boldsymbol{\nabla} C_2^0 - \phi^0 \boldsymbol{\nabla} C_1^0\big), \tag{B.5c}$$

$$U^1 = -\frac{\partial^2}{\partial x \partial z} \left(Ra_{C1}^1 C_1^0 + Ra_{C2}^1 C_2^0\right) + \frac{\partial}{\partial x} \left[\boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} \left(K_1 \phi^0\right)\right] - \nabla^2 \left(K_1 \phi^0 u^0\right), \tag{B.5d}$$

$$V^1 = -\frac{\partial^2}{\partial y \partial z} \left(Ra_{C1}^1 C_1^0 + Ra_{C2}^1 C_2^0\right) + \frac{\partial}{\partial y} \left[\boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} \left(K_1 \phi^0\right)\right] - \nabla^2 \left(K_1 \phi^0 v^0\right), \tag{B.5e}$$

$$W^1 = \nabla_2^2 \left(Ra_{C1}^1 C_1^0 + Ra_{C2}^1 C_2^0\right) + \frac{\partial}{\partial z} \left[\boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} \left(K_1 \phi^0\right)\right] - \nabla^2 \left(K_1 \phi^0 w^0\right). \tag{B.5f}$$

For $k = 2$,

$$\begin{aligned}
\Theta_1^2 = {} & \frac{1}{Le_1} \big\{[m_1 (1 - \Phi - Le_1) + Le_1] \big(\boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_1^1 + \boldsymbol{u}^1 \boldsymbol{\cdot} \boldsymbol{\nabla} C_1^0\big) + m_2 (1 - \Phi - Le_2) \big(\boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_2^1 + \boldsymbol{u}^1 \boldsymbol{\cdot} \boldsymbol{\nabla} C_2^0\big)\big\} \\
& + \frac{1}{Le_1} m_2 \boldsymbol{\nabla} \boldsymbol{\cdot} \big(\phi^0 \boldsymbol{\nabla} C_1^1 + \phi^1 \boldsymbol{\nabla} C_1^0 - \phi^0 \boldsymbol{\nabla} C_2^1 - \phi^1 \boldsymbol{\nabla} C_2^0\big) \\
& - \frac{1 - \Phi}{Le_1} \left[[m_1 (Le_1 - 1) - Le_1] \frac{\partial C_1^0}{\partial \tau} + m_2 (Le_2 - 1) \frac{\partial C_2^0}{\partial \tau}\right],
\end{aligned} \tag{B.6a}$$

$$\begin{aligned}
\Theta_2^2 = {} & \frac{1}{Le_2} \big\{[m_2 (1 - \Phi - Le_2) + Le_2] \big(\boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_2^1 + \boldsymbol{u}^1 \boldsymbol{\cdot} \boldsymbol{\nabla} C_2^0\big) + m_1 (1 - \Phi - Le_1) \big(\boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} C_1^1 + \boldsymbol{u}^1 \boldsymbol{\cdot} \boldsymbol{\nabla} C_1^0\big)\big\} \\
& + \frac{1}{Le_2} m_1 \boldsymbol{\nabla} \boldsymbol{\cdot} \big(\phi^0 \boldsymbol{\nabla} C_2^1 + \phi^1 \boldsymbol{\nabla} C_2^0 - \phi^0 \boldsymbol{\nabla} C_1^1 - \phi^1 \boldsymbol{\nabla} C_1^0\big), \\
& - \frac{1 - \Phi}{Le_2} \left[[m_2 (Le_2 - 1) - Le_2] \frac{\partial C_2^0}{\partial \tau} + m_1 (Le_1 - 1) \frac{\partial C_1^0}{\partial \tau}\right],
\end{aligned} \tag{B.6b}$$

$$\begin{aligned}
W^2 = {} & \nabla_2^2 \left(Ra_{C1}^1 C_1^1 + Ra_{C1}^2 C_1^0 + Ra_{C2}^1 C_2^1 + Ra_{C2}^2 C_2^0\right) \\
& + \frac{\partial}{\partial z} \left\{\boldsymbol{u}^0 \boldsymbol{\cdot} \boldsymbol{\nabla} \left[K_1 \phi^1 + K_2 (\phi^0)^2\right] + \boldsymbol{u}^1 \boldsymbol{\cdot} \boldsymbol{\nabla} \left(K_1 \phi^0\right)\right\} - \nabla^2 \big[K_1 \phi^0 w^1 + K_1 \phi^1 w^0 + K_2 (\phi^0)^2 w^0\big].
\end{aligned} \tag{B.6c}$$

Expressions for Φ^2, U^2 and V^2 will not be needed explicitly.

C. Adjoint problem

The linear stability problem for the neutrally-stable real modes may be represented succinctly as

$$\mathsf{L}\boldsymbol{\phi}^0 = \mathbf{0}, \tag{C.1a}$$

$$\boldsymbol{\phi}^0 = \mathbf{0} \quad \text{at} \quad z = 0, 1, \tag{C.1b}$$

where the linear differential operator, from (B.1b) and (B.1e), is written as

$$\mathsf{L} = \begin{bmatrix} \dfrac{1-\varPhi}{Le_1}\nabla^2 & 0 & -\dfrac{(1-\varPhi)\,\varOmega_1}{Le_1} \\ 0 & \dfrac{1-\varPhi}{Le_2}\nabla^2 & -\dfrac{(1-\varPhi)\,\varOmega_2}{Le_2} \\ -Ra_{C1}^0\nabla_2^2 & -Ra_{C2}^0\nabla_2^2 & \nabla^2 \end{bmatrix} \tag{C.2}$$

and $\boldsymbol{\phi}^0 = \left[C_1^0, C_2^0, w^0\right]^{\mathsf{T}}$. This system of equations is not self-adjoint, and it is necessary to calculate the adjoint eigensolution explicitly. The inner product is defined by

$$\langle \boldsymbol{\psi}, \boldsymbol{\phi} \rangle \equiv \int_0^{2\pi/k_1}\int_0^{2\pi/k_2}\int_0^1 \boldsymbol{\psi}^{\mathsf{T}*}\cdot\boldsymbol{\phi}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where k_1 and k_2 are chosen so as the range of horizontal integration extends over a period of the integrand. For squares $k_1 = k_2 = \pi$; for hexagons $k_1 = \frac{\sqrt{3}}{2}\pi$ and $k_2 = \frac{1}{2}\pi$.

The adjoint problem is found to take the form

$$\mathsf{L}^\dagger\boldsymbol{\phi}^\dagger = \mathbf{0}, \tag{C.3a}$$

$$\boldsymbol{\phi}^\dagger = \mathbf{0} \quad \text{at} \quad z = 0, 1, \tag{C.3b}$$

where $\mathsf{L}^\dagger = \mathsf{L}^{\mathsf{T}}$ and $\boldsymbol{\phi}^\dagger = \left[C_1^\dagger, C_2^\dagger, w^\dagger\right]^{\mathsf{T}}$. The adjoint boundary conditions (C.3b) ensure that the adjoint solution satisfies $\langle\boldsymbol{\phi}^\dagger, \mathsf{L}\boldsymbol{\phi}^0\rangle = \langle\mathsf{L}^\dagger\boldsymbol{\phi}^\dagger, \boldsymbol{\phi}^0\rangle$, as deduced on integration by parts. For a single roll with a single horizontal wavevector $\boldsymbol{k}$, the adjoint eigenfunction $\boldsymbol{\phi}^\dagger$ is given by

$$\begin{bmatrix} C_j^\dagger \\ w^\dagger \end{bmatrix} = \begin{bmatrix} \dfrac{Le_j\,Ra_{Cj}^0}{2\,(1-\varPhi)} \\ 1 \end{bmatrix} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\sin\pi z \tag{C.4}$$

in conjunction with (3.5). Now the inhomogeneous linear problems arising at higher orders have a solution provided

$$\langle \boldsymbol{\phi}^\dagger, \boldsymbol{F}^k \rangle = 0 \quad \text{for} \quad n = 1, 2, \ldots,$$

where the components of $\boldsymbol{F}^k \equiv \left[\Theta_1^k, \Theta_2^k, W^k\right]^\intercal$ are determined from the nonlinear terms in (2.19b,c,f) as detailed in Appendix B.

D. A summary of the $O(\epsilon)$ solutions

D.1. Roll/square interaction

With $Ra_{C1}^1 = Ra_{C2}^1 = 0$, an $O(\epsilon)$ correction to the eigensolution is *

$$\begin{bmatrix} C_j^1 \\ w^1 \end{bmatrix} = \begin{bmatrix} C_j^{10}(z) \\ w^{10}(z) \end{bmatrix} \eta^{10}(\tau) + \begin{bmatrix} C_j^{11}(z) \\ w^{11}(z) \end{bmatrix} \eta^{11}(x, y, \tau) + \begin{bmatrix} C_j^{12}(z) \\ w^{12}(z) \end{bmatrix} \eta^{12}(x, y, \tau), \tag{D.1}$$

where

$$\eta^{10}(\tau) = A_1(\tau)A_1^*(\tau) + A_2(\tau)A_2^*(\tau), \tag{D.2a}$$

$$\eta^{11}(x, y, \tau) = A_1(\tau)A_2(\tau)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{r}} + A_1(\tau)A_2^*(\tau)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r}} + \text{c.c.}, \tag{D.2b}$$

$$\eta^{12}(x, y, \tau) = \sum_{i=1}^{2} A_i^2(\tau)\mathrm{e}^{\mathrm{i}2\boldsymbol{k}_i\cdot\boldsymbol{r}} + \text{c.c.} \tag{D.2c}$$

and

$$C_1^{10}(z) = -\frac{4\pi m_2 \mathscr{O}}{(1-\Phi)\,\Gamma}\sin\pi z + \left(\pi + \frac{\pi m_2 \mathscr{L}}{1-\Phi} - \frac{\pi m_2 \mathscr{O}}{(1-\Phi)\,\Gamma}\right)\sin 2\pi z, \tag{D.3a}$$

$$\begin{aligned} C_1^{11}(z) = &\left(-\frac{24\pi m_2 \mathscr{O}}{(1-\Phi)\,\Gamma} + \frac{24\pi \Omega_1 K_1}{\Gamma} - \frac{16\mathscr{O} Ra_{C2}^0}{3\pi\,(1-\Phi)\,\Gamma}\right)\sin\pi z \\ &+ \left[\frac{3\pi}{7} + \frac{3\pi m_2 \mathscr{L}}{7\,(1-\Phi)} - \frac{9\pi m_2 \mathscr{O}}{7\,(1-\Phi)\,\Gamma} + \frac{4\pi\Omega_1 K_1}{7\Gamma} - \left(\frac{\mathscr{O}}{42\pi} + \frac{\mathscr{O}}{14\pi\,(1-\Phi)\,\Gamma} - \frac{\mathscr{L}}{42\pi\,(1-\Phi)}\right) Ra_{C2}^0\right]\sin 2\pi z, \end{aligned} \tag{D.3b}$$

$$C_1^{12}(z) = \left(-\frac{10\pi m_2 \mathscr{O}}{3\,(1-\Phi)\,\Gamma} + \frac{8\pi\Omega_1 K_1}{3\Gamma} - \frac{8\mathscr{O} Ra_{C2}^0}{15\pi\,(1-\Phi)\,\Gamma}\right)\sin\pi z - \left(\frac{2\pi m_2 \mathscr{O}}{3\,(1-\Phi)\,\Gamma} - \frac{\pi\Omega_1 K_1}{3\Gamma} + \frac{\mathscr{O} Ra_{C2}^0}{24\pi\,(1-\Phi)\,\Gamma}\right)\sin 2\pi z, \tag{D.3c}$$

* We could add to (D.1) a complementary solution with the same form as $\boldsymbol{\phi}^0$ but the amplitudes $A_i(\tau)$ replaced by another undetermined amplitudes, say $B_i(\tau)$. This solution, however, has no influence on a derivation of amplitude equations at $O(\epsilon^2)$, and hence is omitted.

$$w^{10}(z) = 0, \tag{D.3d}$$

$$w^{11}(z) = -\left(\frac{16\pi\mathscr{O}\mathscr{R}}{(1-\Phi)\,\Gamma} + \frac{72\pi^3 K_1}{\Gamma}\right)\sin\pi z + \left(\frac{\pi\mathscr{T}}{7} + \frac{\pi\mathscr{L}}{7\,(1-\Phi)} - \frac{3\pi\mathscr{O}}{7\,(1-\Phi)\,\Gamma} - \frac{24\pi^3 K_1}{7\Gamma}\right)\sin 2\pi z, \tag{D.3e}$$

$$w^{12}(z) = -\left(\frac{8\pi\mathscr{O}\mathscr{R}}{3\,(1-\Phi)\,\Gamma} + \frac{40\pi^3 K_1}{3\Gamma}\right)\sin\pi z - \left(\frac{\pi\mathscr{O}\mathscr{R}}{3\,(1-\Phi)\,\Gamma} + \frac{8\pi^3 K_1}{3\Gamma}\right)\sin 2\pi z. \tag{D.3f}$$

Expressions for C_2^{10}, C_2^{11} and C_2^{12} follow from (D.3a–c) on using the symmetry between the indices 1 and 2. The remaining quantities are given by

$$\phi^1 = \phi^{10}(z)\eta^{10}(\tau) + \phi^{11}(z)\eta^{11}(x,y,\tau) + \phi^{12}(z)\eta^{12}(x,y,\tau), \tag{D.4a}$$

$$u^1 = u^{11}(z)\frac{\partial\eta^{11}(x,y,\tau)}{\partial x} + u^{12}(z)\frac{\partial\eta^{12}(x,y,\tau)}{\partial x}, \tag{D.4b}$$

$$v^1 = v^{11}(z)\frac{\partial\eta^{11}(x,y,\tau)}{\partial y} + v^{12}(z)\frac{\partial\eta^{12}(x,y,\tau)}{\partial y} \tag{D.4c}$$

where

$$\phi^{10}(z) = \frac{4\pi^2}{\Gamma}(1+\cos\pi z) + \left(2\pi^2(1-\Phi) - 2\pi^2\mathscr{M} + \frac{2\pi^2}{\Gamma}\right)(\cos 2\pi z - 1), \tag{D.5a}$$

$$\phi^{11}(z) = \left[\frac{8\pi^2}{\Gamma} - \frac{1}{\Gamma}\left(\frac{16\mathscr{O}\mathscr{R}}{1-\Phi} + 72\pi^2 K_1\right)\right](1+\cos\pi z) + \left[\pi^2(1-\Phi) - \pi^2\mathscr{M} + \frac{1}{\Gamma}\left(3\pi^2 + \frac{\mathscr{T}}{14} + \frac{\mathscr{L}\mathscr{R}}{14\,(1-\Phi)}\right) - \frac{1}{\Gamma^2}\left(\frac{3\mathscr{O}\mathscr{R}}{14\,(1-\Phi)} + \frac{12\pi^2 K_1}{7}\right)\right](\cos 2\pi z - 1), \tag{D.5b}$$

$$\phi^{12}(z) = \left[\frac{6\pi^2}{\Gamma} - \frac{1}{\Gamma^2}\left(\frac{8\mathscr{O}\mathscr{R}}{3\,(1-\Phi)} + \frac{40\pi^2 K_1}{3}\right)\right](1+\cos\pi z) + \left[\frac{2\pi^2}{\Gamma} - \frac{1}{\Gamma^2}\left(\frac{\mathscr{O}\mathscr{R}}{6\,(1-\Phi)} + \frac{4\pi^2 K_1}{3}\right)\right](\cos 2\pi z - 1), \tag{D.5c}$$

$$u^{11}(z) = -\left(\frac{8\mathscr{O}\mathscr{R}}{(1-\Phi)\,\Gamma} + \frac{36\pi^2 K_1}{\Gamma}\right)\cos\pi z + \left(\frac{\mathscr{T}}{7} + \frac{\mathscr{L}\mathscr{R}}{7\,(1-\Phi)} - \frac{3\mathscr{O}\mathscr{R}}{7\,(1-\Phi)\,\Gamma} - \frac{24\pi^2 K_1}{7\Gamma}\right)\cos 2\pi z, \tag{D.5d}$$

$$u^{12}(z) = -\left(\frac{2\mathscr{O}\mathscr{R}}{3\,(1-\Phi)\,\Gamma} + \frac{10\pi^2 K_1}{3\Gamma}\right)\cos\pi z - \left(\frac{\mathscr{O}\mathscr{R}}{6\,(1-\Phi)\,\Gamma} + \frac{4\pi^2 K_1}{3\Gamma}\right)\cos 2\pi z, \tag{D.5e}$$

and $v^{11}(z) = u^{11}(z)$ and $v^{12}(z) = u^{12}(z)$. To minimize the verbiage we have introduced a shorthand notation

$$\mathscr{O} = \Omega_1 - \Omega_2, \tag{D.6a}$$

$$\mathscr{L} = Le_1\Omega_1 - Le_2\Omega_2, \tag{D.6b}$$

$$\mathscr{M} = m_1 Le_1\Omega_1 + m_2 Le_2\Omega_2, \tag{D.6c}$$

$$\mathscr{R} = m_2 Ra_{C1}^0 - m_1 Ra_{C2}^0, \tag{D.6d}$$

$$\mathscr{T} = Ra^0_{C1} + Ra^0_{C2}. \tag{D.6e}$$

In writing these results, we have made repeated use of the relation $m_1 + m_2 = 1$ and the result (3.5).

D.2. Roll/hexagon interaction

An $O(\epsilon^1)$ correction to the eigensolution is

$$\begin{bmatrix} C_j^1 \\ w^1 \end{bmatrix} = \begin{bmatrix} C_j^{10}(z) \\ w^{10}(z) \end{bmatrix} \eta^{10}(\tau) + \begin{bmatrix} C_j^{12}(z) \\ w^{12}(z) \end{bmatrix} \eta^{12}(x,y,\tau) + \begin{bmatrix} C_j^{13}(z) \\ w^{13}(z) \end{bmatrix} \eta^{13}(x,y,\tau) + \begin{bmatrix} C_j^{14}(z) \\ w^{14}(z) \end{bmatrix} \eta^{14}(x,y,\tau), \tag{D.7}$$

where

$$\eta^{10}(\tau) = A_1(\tau)A_1^*(\tau) + A_2(\tau)A_2^*(\tau) + A_3(\tau)A_3^*(\tau), \tag{D.8a}$$

$$\eta^{12}(x,y,\tau) = \sum_{i=1}^{3} A_i^2(\tau)\mathrm{e}^{\mathrm{i}2\boldsymbol{k}_i\cdot\boldsymbol{r}} + \text{c.c.}, \tag{D.8b}$$

$$\eta^{13}(x,y,\tau) = A_1(\tau)A_2(\tau)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_2)\cdot\boldsymbol{r}} + A_1^*(\tau)A_3(\tau)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_3-\boldsymbol{k}_1)\cdot\boldsymbol{r}} + A_2(\tau)A_3(\tau)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2+\boldsymbol{k}_3)\cdot\boldsymbol{r}} + \text{c.c.}, \tag{D.8c}$$

$$\eta^{14}(x,y,\tau) = A_1^*(\tau)A_2(\tau)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_1)\cdot\boldsymbol{r}} + A_1(\tau)A_3(\tau)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_1+\boldsymbol{k}_3)\cdot\boldsymbol{r}} + A_2(\tau)A_3^*(\tau)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_2-\boldsymbol{k}_3)\cdot\boldsymbol{r}} + \text{c.c.}. \tag{D.8d}$$

Here C_j^{10}, C_j^{12}, w^{10} and w^{12} have the same form as in the roll/square case (Appendix D.1) but with K_1 replaced by K_{1c}, and

$$\begin{aligned} C_1^{13}(z) = &-\left(\frac{10\pi m_2\mathscr{O}}{(1-\varPhi)\,\varGamma} - \frac{9\pi\varOmega_1 K_{1c}}{\varGamma} + \frac{15\mathscr{O} Ra^0_{C2}}{8\pi\,(1-\varPhi)\,\varGamma}\right)\sin\pi z \\ &+\left[\frac{7\pi}{37} + \frac{7\pi m_2\mathscr{L}}{37\,(1-\varPhi)} - \frac{49\pi m_2\mathscr{O}}{37\,(1-\varPhi)\,\varGamma} + \frac{24\pi\varOmega_1 K_1}{37\varGamma} - \left(\frac{3\mathscr{O}}{259\pi} - \frac{3\mathscr{L}}{259\pi\,(1-\varPhi)} + \frac{3\mathscr{O}}{37\pi\,(1-\varPhi)\,\varGamma}\right) Ra^0_{C2}\right]\sin 2\pi z, \end{aligned} \tag{D.9a}$$

$$\begin{aligned} C_1^{14}(z) = &-\frac{3\pi m_2\mathscr{O}}{(1-\varPhi)\,\varGamma}\sin\pi z \\ &+\left[\frac{5\pi}{7} + \frac{5\pi m_2\mathscr{L}}{7\,(1-\varPhi)} - \frac{25\pi m_2\mathscr{O}}{21\,(1-\varPhi)\,\varGamma} + \frac{8\pi\varOmega_1 K_{1c}}{21\varGamma} - \left(\frac{\mathscr{O}}{35\pi} + \frac{\mathscr{L}}{35\pi\,(1-\varPhi)} - \frac{\mathscr{O}}{21\pi\,(1-\varPhi)\,\varGamma}\right) Ra^0_{C2}\right]\sin 2\pi z, \end{aligned} \tag{D.9b}$$

$$w^{13}(z) = -\left(\frac{15\pi\mathscr{O}\mathscr{R}}{2\,(1-\varPhi)\,\varGamma} + \frac{36\pi^3 K_{1c}}{\varGamma}\right)\sin\pi z - \left(\frac{168\pi^3 K_{1c}}{37\varGamma} - \frac{3\pi\mathscr{T}}{37} - \frac{3\pi\mathscr{L}\mathscr{R}}{37\,(1-\varPhi)} + \frac{21\pi\mathscr{O}\mathscr{R}}{37\,(1-\varPhi)\,\varGamma}\right)\sin 2\pi z, \tag{D.9c}$$

$$w^{14}(z) = \left(\frac{\pi\mathscr{T}}{7} + \frac{\pi\mathscr{L}\mathscr{R}}{7\,(1-\varPhi)} - \frac{5\pi\mathscr{O}\mathscr{R}}{21\,(1-\varPhi)\,\varGamma} - \frac{40\pi^3 K_{1c}}{21\varGamma}\right)\sin 2\pi z. \tag{D.9d}$$

Expressions for C_2^{13} and C_2^{14} follow from (D.9a, b) on using the symmetry between the indices 1 and 2. The remaining quantities are given by

$$\phi^1 = \phi^{10}(z)\eta^{10}(\tau) + \phi^{12}(z)\eta^{12}(x,y,\tau) + \phi^{13}(z)\eta^{13}(x,y,\tau) + \phi^{14}(z)\eta^{14}(x,y,\tau), \tag{D.10a}$$

$$u^1 = u^{12}(z)\frac{\partial\eta^{12}(x,y,\tau)}{\partial x} + u^{13}(z)\frac{\partial\eta^{13}(x,y,\tau)}{\partial x} + u^{14}(z)\frac{\partial\eta^{14}(x,y,\tau)}{\partial x}, \tag{D.10b}$$

$$v^1 = v^{12}(z)\frac{\partial\eta^{12}(x,y,\tau)}{\partial y} + v^{13}(z)\frac{\partial\eta^{13}(x,y,\tau)}{\partial y} + v^{14}(z)\frac{\partial\eta^{14}(x,y,\tau)}{\partial y} \tag{D.10c}$$

where ϕ^{10}, ϕ^{12}, u^{12} and v^{12} have the same form as in the roll/square case (Appendix D.1) but with K_1 replaced by K_{1c}, and

$$\begin{aligned}\phi^{13}(z) = &\left[\frac{10\pi^2}{\mathit{\Gamma}} - \frac{1}{\mathit{\Gamma}^2}\left(\frac{15\mathscr{O}\mathscr{R}}{2(1-\mathit{\Phi})} + 36\pi^2 K_{1c}\right)\right](1+\cos\pi z)\\ &+ \left[\frac{\pi^2(1-\mathit{\Phi})}{2} - \frac{\pi^2\mathscr{M}}{2} + \frac{1}{\mathit{\Gamma}}\left(\frac{7\pi^2}{2} + \frac{3\mathscr{T}}{74} + \frac{3\mathscr{L}\mathscr{R}}{74(1-\mathit{\Phi})}\right) - \frac{1}{\mathit{\Gamma}^2}\left(\frac{21\mathscr{O}\mathscr{R}}{74(1-\mathit{\Phi})} + \frac{84\pi^2 K_{1c}}{37}\right)\right](\cos 2\pi z - 1),\end{aligned} \tag{D.11a}$$

$$\begin{aligned}\phi^{14}(z) = &\frac{6\pi^2}{\mathit{\Gamma}}(1+\cos\pi z)\\ &+ \left[\frac{3\pi^2(1-\mathit{\Phi})}{2} - \frac{3\pi^2\mathscr{M}}{2} + \frac{1}{\mathit{\Gamma}}\left(\frac{5\pi^2}{2} + \frac{\mathscr{T}}{14} + \frac{\mathscr{L}\mathscr{R}}{14(1-\mathit{\Phi})}\right) - \frac{1}{\mathit{\Gamma}^2}\left(\frac{5\mathscr{O}\mathscr{R}}{42(1-\mathit{\Phi})} + \frac{20\pi^2 K_{1c}}{21}\right)\right](\cos 2\pi z - 1),\end{aligned} \tag{D.11b}$$

$$u^{13}(z) = -\left(\frac{5\mathscr{O}\mathscr{R}}{2(1-\mathit{\Phi})\mathit{\Gamma}} + \frac{12\pi^2 K_{1c}}{\mathit{\Gamma}}\right)\cos\pi z + \left(\frac{2\mathscr{T}}{37} + \frac{2\mathscr{L}\mathscr{R}}{37(1-\mathit{\Phi})} - \frac{14\mathscr{O}\mathscr{R}}{37(1-\mathit{\Phi})\mathit{\Gamma}} - \frac{112\pi^2 K_{1c}}{37\mathit{\Gamma}}\right)\cos 2\pi z, \tag{D.11c}$$

$$u^{14}(z) = \left(\frac{2\mathscr{T}}{7} + \frac{2\mathscr{L}\mathscr{R}}{7(1-\mathit{\Phi})} - \frac{10\mathscr{O}\mathscr{R}}{21(1-\mathit{\Phi})\mathit{\Gamma}} - \frac{80\pi^2 K_{1c}}{21\mathit{\Gamma}}\right)\cos 2\pi z, \tag{D.11d}$$

and $v^{13}(z) = u^{13}(z)$ and $v^{14}(z) = u^{14}(z)$. In writing these results, we have made repeated use of the relation $m_1 + m_2 = 1$ and the result (3.5).

E. Coefficients in the amplitude equations

E.1. Roll/square interaction

The coefficients of the cubic terms in (4.6) are

$$c = 6\pi^4\mathscr{O}\mathscr{R} + 2\pi^4\mathscr{T} - \frac{98\pi^4}{3}\frac{\mathscr{O}\mathscr{R}}{\mathit{\Gamma}^2(1-\mathit{\Phi})^2} + \frac{\pi^2}{6}\frac{\mathscr{O}^2\mathscr{R}^2}{\mathit{\Gamma}^2(1-\mathit{\Phi})^2} - \frac{101\pi^2}{30}\frac{\mathscr{O}\mathscr{R}\mathscr{T}}{\mathit{\Gamma}^2(1-\mathit{\Phi})^2} + 2\pi^4\frac{\mathscr{L}\mathscr{R}}{\mathit{\Gamma}(1-\mathit{\Phi})^2}$$

$$
\begin{aligned}
&- 4\pi^4 \frac{\mathscr{O}\mathscr{M}\mathscr{R}}{1-\Phi} - 6\pi^4 \frac{\mathscr{O}\mathscr{R}}{\Gamma(1-\Phi)} + 2\pi^4 \frac{(m_1 Le_2 + m_2 Le_1)\,\mathscr{L}\mathscr{R}}{(1-\Phi)^2} - 2\pi^4 \frac{(m_1 Le_2 + m_2 Le_1)\,\mathscr{O}\mathscr{R}}{\Gamma(1-\Phi)^2} \\
&+ \left(-\frac{8\pi^6}{\Gamma} + \frac{106\pi^4}{3}\frac{\mathscr{O}\mathscr{R}}{\Gamma^2(1-\Phi)} + 16\pi^6(1-\Phi-\mathscr{M})\right) K_1 + \frac{256\pi^6}{3}\frac{K_1^2}{\Gamma^2} - 88\pi^6 \frac{K_2}{\Gamma^2},
\end{aligned} \tag{E.1a}
$$

$$
\begin{aligned}
d = {} & \frac{55\pi^4}{7}\mathscr{O}\mathscr{R} + \frac{20\pi^4}{7}\mathscr{T} + \frac{\pi^2}{21}\mathscr{O}\mathscr{R}\mathscr{T} - \frac{\pi^2}{7}\frac{\mathscr{O}\mathscr{R}\mathscr{U}}{\Gamma(1-\Phi)^2} - \frac{\pi^2}{21}\frac{\mathscr{O}\mathscr{R}\mathscr{U}}{1-\Phi} + \frac{\pi^2}{21}\frac{\mathscr{L}\mathscr{R}\mathscr{U}}{(1-\Phi)^2} - \frac{1468\pi^4}{7}\frac{\mathscr{O}\mathscr{R}}{\Gamma^2(1-\Phi)^2} \\
&+ \frac{445\pi^2}{14}\frac{\mathscr{O}^2\mathscr{R}^2}{\Gamma^2(1-\Phi)^2} - \frac{905\pi^2}{21}\frac{\mathscr{O}\mathscr{R}\mathscr{T}}{\Gamma^2(1-\Phi)^2} + \frac{32\pi^4}{7}\frac{\mathscr{L}\mathscr{R}}{\Gamma(1-\Phi)^2} + \frac{\pi^2}{14}\frac{\mathscr{L}\mathscr{O}\mathscr{R}^2}{\Gamma(1-\Phi)^2} + \frac{\pi^2}{7}\frac{\mathscr{L}\mathscr{R}\mathscr{T}}{\Gamma(1-\Phi)^2} - 5\pi^4\frac{\mathscr{O}\mathscr{R}\mathscr{M}}{1-\Phi} \\
&- \frac{\pi^2}{21}\frac{\mathscr{L}\mathscr{R}\mathscr{T}}{1-\Phi} - 17\pi^4\frac{\mathscr{O}\mathscr{R}}{\Gamma(1-\Phi)} + \frac{\pi^2}{14}\frac{\mathscr{O}\mathscr{R}\mathscr{T}}{\Gamma(1-\Phi)} + \frac{20\pi^4}{7}\frac{(m_1 Le_2 + m_2 Le_1)\,\mathscr{L}\mathscr{R}}{(1-\Phi)^2} - \frac{32\pi^4}{7}\frac{(m_1 Le_2 + m_2 Le_1)\,\mathscr{O}\mathscr{R}}{\Gamma(1-\Phi)^2} \\
&+ \left(-\frac{44\pi^6}{\Gamma} + \frac{6\pi^4}{7}\frac{\mathscr{T}}{\Gamma} + \frac{4610\pi^4}{7}\frac{\mathscr{O}\mathscr{R}}{\Gamma^2(1-\Phi)} + \frac{6\pi^4}{7}\frac{\mathscr{L}\mathscr{R}}{\Gamma(1-\Phi)} + 28\pi^6(1-\Phi-\mathscr{M})\right) K_1 \\
&+ \frac{10128\pi^6}{7}\frac{K_1^2}{\Gamma^2} - 176\pi^6\frac{K_2}{\Gamma^2}.
\end{aligned} \tag{E.1b}
$$

Here $\mathscr{O}$, $\mathscr{L}$, $\mathscr{M}$, $\mathscr{R}$ and $\mathscr{T}$ are defined in (D.6), while $\mathscr{U} = Le_1 Ra^0_{C2} + Le_2 Ra^0_{C1}$.

E.2. Roll/hexagon interaction

The coefficients of the cubic terms in (4.16) are

$$
\begin{aligned}
c = {} & 6\pi^4\mathscr{O}\mathscr{R} + 2\pi^4\mathscr{T} - \frac{98\pi^4}{3}\frac{\mathscr{O}\mathscr{R}}{\Gamma^2(1-\Phi)^2} - \frac{9\pi^2}{2}\frac{\mathscr{O}^2\mathscr{R}^2}{\Gamma^2(1-\Phi)^2} - \frac{101\pi^2}{30}\frac{\mathscr{O}\mathscr{R}\mathscr{T}}{\Gamma^2(1-\Phi)^2} + 2\pi^4\frac{\mathscr{L}\mathscr{R}}{\Gamma(1-\Phi)^2} \\
&- 4\pi^4\frac{\mathscr{O}\mathscr{M}\mathscr{R}}{1-\Phi} - 6\pi^4\frac{\mathscr{O}\mathscr{R}}{\Gamma(1-\Phi)} + 2\pi^4\frac{(m_1 Le_2 + m_2 Le_1)\,\mathscr{L}\mathscr{R}}{(1-\Phi)^2} - 2\pi^4\frac{(m_1 Le_2 + m_2 Le_1)\,\mathscr{O}\mathscr{R}}{\Gamma(1-\Phi)^2},
\end{aligned} \tag{E.2a}
$$

$$
\begin{aligned}
d = {} & \frac{9237\pi^4}{518}\mathscr{O}\mathscr{R} + \frac{1640\pi^4}{259}\mathscr{T} + \frac{18\pi^2}{185}\mathscr{O}\mathscr{R}\mathscr{T} - \frac{58\pi^2}{259}\frac{\mathscr{O}\mathscr{R}\mathscr{U}}{\Gamma(1-\Phi)^2} - \frac{18\pi^2}{185}\frac{\mathscr{O}\mathscr{R}\mathscr{U}}{1-\Phi} + \frac{18\pi^2}{185}\frac{\mathscr{L}\mathscr{R}\mathscr{U}}{(1-\Phi)^2} \\
&- \frac{119054\pi^4}{777}\frac{\mathscr{O}\mathscr{R}}{\Gamma^2(1-\Phi)^2} - \frac{6593\pi^2}{259}\frac{\mathscr{O}^2\mathscr{R}^2}{\Gamma^2(1-\Phi)^2} - \frac{60779\pi^2}{3108}\frac{\mathscr{O}\mathscr{R}\mathscr{T}}{\Gamma^2(1-\Phi)^2} + \frac{2304\pi^4}{259}\frac{\mathscr{L}\mathscr{R}}{\Gamma(1-\Phi)^2} \\
&- \frac{75\pi^2}{259}\frac{\mathscr{L}\mathscr{O}\mathscr{R}^2}{\Gamma(1-\Phi)^2} + \frac{58\pi^2}{259}\frac{\mathscr{L}\mathscr{R}\mathscr{T}}{\Gamma(1-\Phi)^2} - \frac{23\pi^4}{2}\frac{\mathscr{O}\mathscr{R}\mathscr{M}}{1-\Phi} - \frac{18\pi^2}{185}\frac{\mathscr{L}\mathscr{R}\mathscr{T}}{1-\Phi} - \frac{63\pi^4}{2}\frac{\mathscr{O}\mathscr{R}}{\Gamma(1-\Phi)} - \frac{75\pi^2}{259}\frac{\mathscr{O}\mathscr{R}\mathscr{T}}{\Gamma(1-\Phi)} \\
&+ \frac{1640\pi^4}{259}\frac{(m_1 Le_2 + m_2 Le_1)\,\mathscr{L}\mathscr{R}}{(1-\Phi)^2} - \frac{2304\pi^4}{259}\frac{(m_1 Le_2 + m_2 Le_1)\,\mathscr{O}\mathscr{R}}{\Gamma(1-\Phi)^2}.
\end{aligned} \tag{E.2b}
$$

Here $\mathscr{O}$, $\mathscr{L}$, $\mathscr{M}$, $\mathscr{R}$ and $\mathscr{T}$ are defined in (D.6), while $\mathscr{U} = Le_1 Ra^0_{C2} + Le_2 Ra^0_{C1}$.

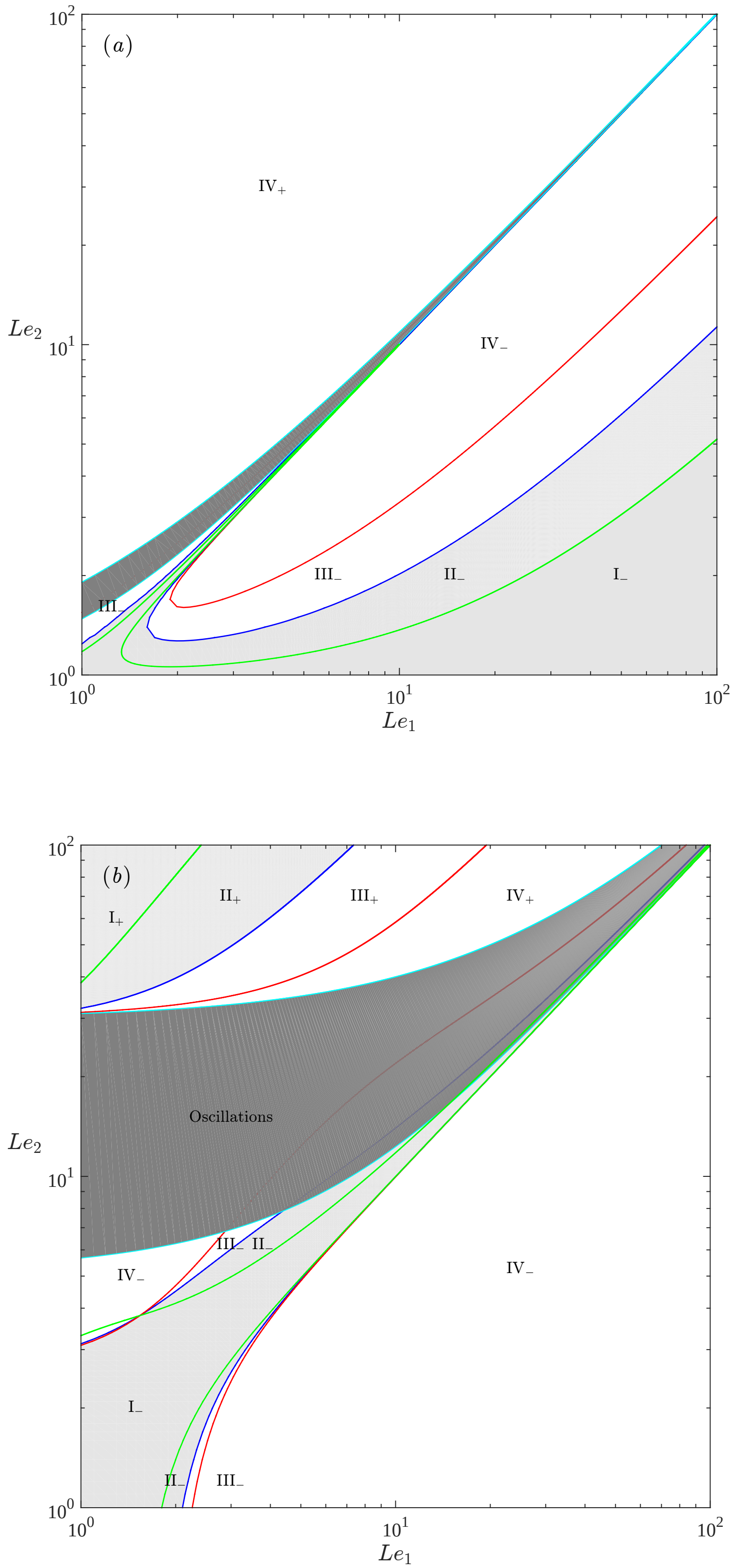

FIGURE S1: Parameter regimes for roll/square interaction. (a) $m_1 = 0.01$, $m_2 = 0.99$, (b) $m_1 = 0.97$, $m_2 = 0.03$. The remaining parameter values are the same as those for figure 5.