

# Supplementary material: Subsonic flow past localised heating elements in boundary layers

A. F. Aljohani<sup>1,2</sup> and J. S. B. Gajjar<sup>2†</sup>

<sup>1</sup>Department of Mathematics, Faculty of Science, University of Tabuk, Saudi Arabia

<sup>2</sup>School of Mathematics, University of Manchester, Manchester, M13 9PL, UK

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## Supplementary material: Solution of the linearised subsonic flow problem

The expressions for the perturbed pressure and wall shear are given by

$$\frac{\partial \tilde{P}}{\partial x_b} = \frac{\text{Ai}'(0)}{2\pi \text{Ai}(0)} \int_{-\infty}^{\infty} \frac{(i\omega)^{\frac{5}{3}} \hat{f}(\omega) + (i\omega)^2 \hat{H}(\omega) \Lambda}{i \text{sgn}(\omega) \Theta + (i\omega)^{\frac{4}{3}}} e^{i\omega x_b} d\omega \quad (0.1)$$

and

$$\frac{\partial \hat{U}}{\partial y_b}(x_b, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(i\omega)^{\frac{4}{3}} \hat{f}(\omega) + (i\omega)^{\frac{5}{3}} \hat{H}(\omega) \Lambda}{i \text{sgn}(\omega) \Theta + (i\omega)^{\frac{4}{3}}} e^{i\omega x_b} d\omega \quad (0.2)$$

where  $\Theta = -3\text{Ai}'(0)$  and  $\Lambda = 3\text{Ai}(0)$

In order to express (0.1) and (0.2) in forms that are easier to deal with numerically, we first write the integrals as

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial x_b} = \frac{\text{Ai}'(0)}{2\pi \text{Ai}(0)} & \left[ \int_0^{\infty} \frac{r^{\frac{5}{3}} e^{-\frac{i5\pi}{6}} \hat{f}(-r) + r^2 e^{-i\pi} \hat{H}(r) \Lambda}{-i\Theta + r^{\frac{4}{3}} e^{-\frac{i2\pi}{3}}} e^{-irx_b} dr \right. \\ & \left. + \int_0^{\infty} \frac{r^{\frac{5}{3}} e^{\frac{i5\pi}{6}} \hat{f}(r) + r^2 e^{i\pi} \hat{H}(r) \Lambda}{i\Theta + r^{\frac{4}{3}} e^{\frac{i2\pi}{3}}} e^{irx_b} dr \right] \end{aligned} \quad (0.3)$$

and

$$\begin{aligned} \frac{\partial \hat{U}}{\partial y_b}(x_b, 0) = \frac{1}{2\pi} & \left[ \int_0^{\infty} \frac{r^{\frac{4}{3}} e^{-\frac{i2\pi}{3}} \hat{f}(-r) + r^{\frac{5}{3}} e^{-\frac{i5\pi}{6}} \hat{H}(r) \Lambda}{-i\Theta + r^{\frac{4}{3}} e^{-\frac{i2\pi}{3}}} e^{-irx_b} dr \right. \\ & \left. + \int_0^{\infty} \frac{r^{\frac{4}{3}} e^{\frac{i2\pi}{3}} \hat{f}(r) + r^{\frac{5}{3}} e^{\frac{i5\pi}{6}} \hat{H}(r) \Lambda}{i\Theta + r^{\frac{4}{3}} e^{\frac{i2\pi}{3}}} e^{irx_b} dr \right], \end{aligned} \quad (0.4)$$

where we have written  $r = |\omega|$ . We assume the heating region is of size 1 and located in a domain where  $x_b \in [-0.5, 0.5]$  and that the wall temperature

† Email address for correspondence: jitesh.gajjar@manchester.ac.uk

distribution is given by

$$\tilde{T}_w(x_b) = \begin{cases} 0.2 & |x_b| < 0.5 \\ 0 & |x_b| > 0.5 \end{cases}. \quad (0.5)$$

Applying the Fourier transform to (0.5), yields that

$$\hat{f}(\omega) = \frac{0.4}{\omega} \sin\left(\frac{\omega}{2}\right). \quad (0.6)$$

We also assume the hump shape is given by

$$\tilde{H}(x_b) = \tilde{h} \exp(-5x_b^2),$$

and the Fourier transform of  $\tilde{H}(x_b)$  is

$$\mathcal{F}\{\tilde{H}(x_b)\} = \hat{H}(\omega) = \tilde{h} \frac{\sqrt{5\pi}}{5} \exp\left(-\frac{\omega^2}{20}\right). \quad (0.7)$$

Clearly, the integrand in (0.3) can be expressed as twice the real part of

$$\frac{r^{\frac{5}{3}} e^{\frac{i5\pi}{6}} \hat{f}(r) + r^2 e^{i\pi} \hat{H}(r) \Lambda}{i\Theta + r^{\frac{4}{3}} e^{\frac{i2\pi}{3}}} e^{irx_b}.$$

To find the real part and using (0.6) and (0.7) it can be shown that,

$$\begin{aligned} & \frac{r^{\frac{5}{3}} e^{\frac{i5\pi}{6}} \hat{f}(r) + r^2 e^{i\pi} \hat{H}(r) \Lambda}{i\Theta + r^{\frac{4}{3}} e^{\frac{i2\pi}{3}}} e^{irx_b} \\ &= \left[ \frac{0.4r^{\frac{2}{3}} \sin(\frac{r}{2})}{\sqrt{[\frac{1}{2}\Theta + r^{\frac{4}{3}} \frac{\sqrt{3}}{2}]^2 + [-r^{\frac{4}{3}} \frac{1}{2} - \frac{\sqrt{3}}{2}\Theta]^2}} e^{i\phi} + \frac{r^2 (\tilde{h} \frac{\sqrt{5\pi}}{5} e^{-\frac{r^2}{20}}) \Lambda}{\sqrt{[r^{\frac{4}{3}} \frac{1}{2}]^2 + [-\Theta - r^{\frac{4}{3}} \frac{\sqrt{3}}{2}]^2}} e^{i\alpha} \right] e^{irx_b} \end{aligned}$$

where  $\phi = \arctan\left(\frac{-r^{\frac{4}{3}} \frac{1}{2} - \frac{\sqrt{3}}{2}\Theta}{\frac{1}{2}\Theta + r^{\frac{4}{3}} \frac{\sqrt{3}}{2}}\right)$  and  $\alpha = \arctan\left(\frac{-\Theta - r^{\frac{4}{3}} \frac{\sqrt{3}}{2}}{\frac{1}{2}r^{\frac{4}{3}}}\right)$ .

Taking the real part, we obtain

$$\begin{aligned} & \Re\left\{ \frac{r^{\frac{5}{3}} e^{\frac{i5\pi}{6}} \hat{f}(r) + r^2 e^{i\pi} (\tilde{h} \frac{\sqrt{5\pi}}{5} e^{-\frac{r^2}{20}}) \Lambda}{i\Theta + r^{\frac{4}{3}} e^{\frac{i2\pi}{3}}} e^{irx_b} \right\} \\ &= \frac{0.4r^{\frac{2}{3}} \sin(\frac{r}{2}) \cos(rx_b - \phi)}{\sqrt{[\frac{1}{2}\Theta + r^{\frac{4}{3}} \frac{\sqrt{3}}{2}]^2 + [-r^{\frac{4}{3}} \frac{1}{2} - \frac{\sqrt{3}}{2}\Theta]^2}} + \frac{r^2 (\tilde{h} \frac{\sqrt{5\pi}}{5} \exp(-\frac{r^2}{20})) \Lambda \cos(rx_b - \alpha)}{\sqrt{[-r^{\frac{4}{3}} \frac{1}{2}]^2 + [\Theta + r^{\frac{4}{3}} \frac{\sqrt{3}}{2}]^2}}. \end{aligned}$$

Therefore, the gradient pressure distribution is given by

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial x_b} &= \frac{\text{Ai}'(0)}{\pi \text{Ai}(0)} \int_0^\infty \left[ \frac{0.4r^{\frac{2}{3}} \sin(\frac{r}{2}) \cos(rx_b - \phi)}{[\Theta^2 + \sqrt{3}\Theta r^{\frac{4}{3}} + r^{\frac{8}{3}}]^{\frac{1}{2}}} \right. \\ & \quad \left. + \frac{r^2 \tilde{h} \frac{\sqrt{5\pi}}{5} \exp(-\frac{r^2}{20}) \Lambda \cos(rx_b - \alpha)}{[\Theta^2 + \sqrt{3}\Theta r^{\frac{4}{3}} + r^{\frac{8}{3}}]^{\frac{1}{2}}} \right] dr \end{aligned} \quad (0.8)$$

where  $\phi = \arctan(-\frac{r^{\frac{4}{3}}\frac{1}{2} + \frac{\sqrt{3}}{2}\Theta}{\frac{1}{2}\Theta + r^{\frac{4}{3}}\frac{\sqrt{3}}{2}})$  and  $\alpha = \arctan(-\frac{\Theta + r^{\frac{4}{3}}\frac{\sqrt{3}}{2}}{\frac{1}{2}r^{\frac{4}{3}}})$ .

The perturbed wall shear given by (0.4), can be dealt with in a similar way and we find that

$$\begin{aligned} & \Re\left\{\frac{r^{\frac{4}{3}}e^{\frac{i2\pi}{3}}\hat{f}(r) + r^{\frac{5}{3}}e^{\frac{i5\pi}{6}}\hat{H}(r)\Lambda}{i\Theta + r^{\frac{4}{3}}e^{\frac{i2\pi}{3}}}e^{irx_b}\right\} \\ &= \left[\frac{0.4r^{\frac{1}{3}}\sin(\frac{r}{2})\cos(rx_b - \phi_1)}{[\Theta^2 + \sqrt{3}\Theta r^{\frac{4}{3}} + r^{\frac{8}{3}}]^{\frac{1}{2}}} + \frac{r^{\frac{5}{3}}(\tilde{h}\frac{\sqrt{5\pi}}{5}e^{-\frac{r^2}{20}})\Lambda\cos(rx_b - \alpha_1)}{[\Theta^2 + \sqrt{3}\Theta r^{\frac{4}{3}} + r^{\frac{8}{3}}]^{\frac{1}{2}}}\right]. \end{aligned}$$

Therefore, the wall shear distribution is given by

$$\begin{aligned} \frac{\partial \hat{U}}{\partial y_b}(x_b, 0) &= \frac{1}{\pi} \int_0^\infty \left[ \frac{0.4r^{\frac{1}{3}}\sin(\frac{r}{2})\cos(rx_b - \phi_1)}{[\Theta^2 + \sqrt{3}\Theta r^{\frac{4}{3}} + r^{\frac{8}{3}}]^{\frac{1}{2}}} + \right. \\ & \quad \left. \frac{r^{\frac{5}{3}}(\tilde{h}\frac{\sqrt{5\pi}}{5}e^{-\frac{r^2}{20}})\Lambda\cos(rx_b - \alpha_1)}{[\Theta^2 + \sqrt{3}\Theta r^{\frac{4}{3}} + r^{\frac{8}{3}}]^{\frac{1}{2}}} \right] dr \end{aligned} \quad (0.9)$$

where  $\phi_1 = \arctan(\frac{-\frac{\Theta}{2}}{\frac{\sqrt{3}}{2}\Theta + r^{\frac{4}{3}}})$  and  $\alpha_1 = \arctan(\frac{-\frac{\sqrt{3}}{2}\Theta - \frac{1}{2}r^{\frac{4}{3}}}{\frac{\sqrt{3}}{2}r^{\frac{4}{3}} + \frac{1}{2}\Theta})$ .