This document contains supplementary material associated with the *J. Fluid Mech.* paper: "Perturbation analysis of subphase gas and meniscus curvature effects for longitudinal flows over superhydrophobic surfaces" by Darren Crowdy.

# 1 Geometry of a protruding meniscus

We assume that the meniscus protrudes into the upper working fluid; the details for the depressed meniscus just require a few minor changes of sign but lead to the same result. The depressed meniscus case requires that a "continuation" of Philip's solution into the region beneath the surfaces be effected, but this can be done in a natural way by various reflection arguments. To avoid such technical issues, we focus on the protruding interface case.

It is convenient to work within a complex variable formulation. The complexification of the unit outward normal on the meniscus is

$$\mathbf{n} \mapsto -\left[\frac{z+\mathrm{i}C}{R}\right], \tag{1}$$

where the symbol "$\mapsto$" denotes the process of taking a vector in $\mathbb{R}^2$ and writing its complex-valued analogue. From simple geometry and trigonometric arguments based on Figure 1 we deduce that

$$\frac{b}{C} = \tan\theta, \qquad \frac{b}{R} = \sin\theta, \qquad \frac{C}{R} = \cos\theta. \tag{2}$$

To leading order in $\theta \ll 1$,

$$z = x + \mathrm{i}\theta\eta(x), \qquad \eta(x) = \frac{1}{2b}(b^2 - x^2). \tag{3}$$

To derive this, notice that the equation of the meniscus is

$$(z+\mathrm{i}C)(\overline{z}-\mathrm{i}C) = R^2. \tag{4}$$

On setting $z = x + \mathrm{i}y$ we find

$$x^2 + y^2 + 2yC = R^2 - C^2. \tag{5}$$

On substitution from (2) this becomes

$$x^2 + y^2 + \frac{2yb}{\tan\theta} = \frac{b^2}{\sin^2\theta} - \frac{b^2}{\tan^2\theta} = \frac{b^2}{\sin^2\theta}(1-\cos^2\theta). \tag{6}$$

Expanding in powers of $\theta$ produces

$$x^2 + y^2 + \frac{2yb}{\theta} = b^2 + \mathcal{O}(\theta^2), \tag{7}$$

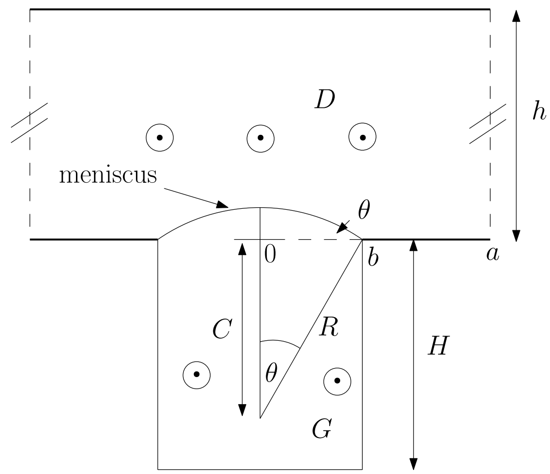


Figure 1: Single period window, in an $(x, y)$ plane, for longitudinal flow in a channel of height $h$ with a periodic array of rectangular grooves of height $H$. The upper wall is a no-slip surface. The meniscus protrusion angle $\theta$ is assumed to be small. Domain $D$ is occupied by fluid of viscosity $\eta_1$, the groove $G$ by fluid of viscosity $\eta_2$ with $\epsilon = \eta_2/\eta_1 \ll 1$.

or

$$y = \frac{\theta}{2b}(b^2 - x^2 - y^2 + \mathcal{O}(\theta^2)) = \frac{\theta}{2b}(b^2 - x^2) + \mathcal{O}(\theta^3). \tag{8}$$

Hence, to leading order in $\theta$, the meniscus is the parabola $y = \theta\eta(x)$. It is worth noting that

$$\eta'(x) = -\frac{x}{b} \tag{9}$$

and that, on the meniscus,

$$ds = [dz\overline{dz}]^{1/2} = [(1 + \mathrm{i}\theta\eta'(x))(1 - \mathrm{i}\theta\eta'(x))]^{1/2}dx = dx + \mathcal{O}(\theta^2). \tag{10}$$

The linearized complex normal is

$$\mathbf{n} \mapsto -\left[\frac{z + \mathrm{i}C}{R}\right] = -\left[\mathrm{i} + \frac{z\theta}{b}\right] + \mathcal{O}(\theta^2) = -\left[\mathrm{i} + \frac{x\theta}{b}\right] + \mathcal{O}(\theta^2). \tag{11}$$

# 2 Slip length formula

We write Philip's channel flow solution [3] in the form

$$w_P = -\frac{Sy^2}{2} + \hat{w}, \tag{12}$$

where the harmonic function $\hat{w}$ is given by

$$\hat{w} = \mathrm{Im}[h_0(z)] \tag{13}$$

and where $h_0(z)$ is a known complex potential (see [1] for example). This solution is relevant to a flat meniscus on which

$$\frac{\partial \hat{w}}{\partial y} = 0 \tag{14}$$

which is equivalent to

$$\mathrm{Re}[h_0'(x)] = 0, \qquad x \in \mathbb{R}. \tag{15}$$

If we define

$$\mathcal{P} = \frac{Sy}{2}(h - y). \tag{16}$$

it follows that

$$w_P + \mathcal{P} = -Sy^2 + \frac{Shy}{2} + \mathrm{Im}[h_0(z)], \tag{17}$$

or, in terms of $z$ and $\overline{z}$,

$$w_P + \mathcal{P} = \frac{S}{4}\left[z^2 - 2z\overline{z} + \overline{z}^2\right] + \frac{Sh}{4\mathrm{i}}(z - \overline{z}) + \mathrm{Im}[h_0(z)]. \tag{18}$$

First note that, on the meniscus where $z = x + \mathrm{i}y = x + \mathrm{i}\theta\eta(x)$,

$$\begin{aligned}
w_P(z,\overline{z}) &= \hat{w}(z,\overline{z}) + \mathcal{O}(\theta^2) \\
&= \frac{1}{2\mathrm{i}}\left[h_0(x + \mathrm{i}\theta\eta(x)) - \overline{h_0}(x - \mathrm{i}\theta\eta(x))\right] + \mathcal{O}(\theta^2) \\
&= \frac{1}{2\mathrm{i}}\left[h_0(x) - \overline{h_0}(x) + \mathrm{i}\theta\eta(x)h_0'(x) + \mathrm{i}\theta\eta(x)\overline{h_0}'(x)\right] + \mathcal{O}(\theta^2) \\
&= \hat{w}(x,0) + \theta\eta(x)\frac{\partial \hat{w}}{\partial y}(x,0) + \mathcal{O}(\theta^2) \\
&= \hat{w}(x,0) + \mathcal{O}(\theta^2),
\end{aligned} \tag{19}$$

where, in the third equality, we have used the Cauchy-Riemann equations and the fact that $\partial\hat{w}/\partial y = 0$ on $y = 0$.

Next, in complex notation, we find

$$\frac{\partial(w_P + \mathcal{P})}{\partial n} = \mathrm{Re}\left[2\frac{\partial(w_P + \mathcal{P})}{\partial z}\left\{-\left[\mathrm{i} + \frac{x\theta}{b}\right]\right\}\right] + \mathcal{O}(\theta^2). \tag{20}$$

It follows from (18) that

$$\begin{aligned}
\frac{\partial(w_P + \mathcal{P})}{\partial z} &= \frac{S(z - \overline{z})}{2} + \frac{Sh}{4\mathrm{i}} + \frac{h_0'(z)}{2\mathrm{i}} \\
&= \mathrm{i}S\theta\eta(x) + \frac{Sh}{4\mathrm{i}} + \frac{1}{2\mathrm{i}}(h_0'(x) + \mathrm{i}\theta\eta(x)h_0''(x)) + \mathcal{O}(\theta^2) \\
&= \frac{Sh}{4\mathrm{i}} + \frac{h_0'(x)}{2\mathrm{i}} + \mathrm{i}\theta\eta(x)\left(S + \frac{h_0''(x)}{2\mathrm{i}}\right) + \mathcal{O}(\theta^2).
\end{aligned} \tag{21}$$

Hence

$$
\begin{aligned}
2\frac{\partial(w_P+\mathcal{P})}{\partial z}\left\{-\left[\mathrm{i}+\frac{x\theta}{b}\right]\right\} = -2\Bigg\{&\left[\frac{Sh}{4}+\frac{h_0'(x)}{2}\right] \\
&-\theta\eta(x)\left(S+\frac{h_0''(x)}{2\mathrm{i}}\right)+\frac{x\theta}{b}\left(\frac{Sh}{4\mathrm{i}}+\frac{h_0'(x)}{2\mathrm{i}}\right)\Bigg\}+\mathcal{O}(\theta^2).
\end{aligned}
\tag{22}
$$

On taking the real part,

$$
\begin{aligned}
\frac{\partial(w_P+\mathcal{P})}{\partial n} &= -\frac{Sh}{2}+\mathrm{Re}\left[2\theta\left[\eta(x)\left(S+\frac{h_0''(x)}{2\mathrm{i}}\right)-\frac{x}{b}\frac{h_0'(x)}{2\mathrm{i}}\right]\right]\Bigg\}+\mathcal{O}(\theta^2) \\
&= -\frac{Sh}{2}+2S\theta\eta(x)+2\theta\mathrm{Re}\left[\frac{d}{dx}\left[\frac{\eta(x)h_0'(x)}{2\mathrm{i}}\right]\right]+\mathcal{O}(\theta^2),
\end{aligned}
\tag{23}
$$

where we have used both (9) and (15). On substitution of these quantities into the expression for the slip length correction given in the main body of the paper we find

$$
\frac{ah^2S\lambda}{2}=\frac{1}{S}\int_{-b}^{b}\hat{w}(x,0)\left[\frac{Sh}{2}-2S\theta\eta(x)-2\theta\mathrm{Re}\left[\frac{d}{dx}\left[\frac{\eta(x)h_0'(x)}{2\mathrm{i}}\right]\right]\right]dx+\mathcal{O}(\theta^2). \tag{24}
$$

On writing

$$
\lambda=\lambda_P+\theta\lambda^{(\theta)}+\mathcal{O}(\theta^2) \tag{25}
$$

and substituting into the left hand side of (24) we find, at leading order,

$$
\frac{ah^2S\lambda_P}{2}=\int_{-b}^{b}\frac{h}{2}\hat{w}(x,0)dx, \qquad \text{or} \qquad \lambda_P=\frac{1}{ahS}\int_{-b}^{b}\hat{w}(x,0)dx, \tag{26}
$$

and, to first order in $\theta$,

$$
\frac{ah^2S\lambda^{(\theta)}}{2}=-2\int_{-b}^{b}\hat{w}(x,0)\left\{\eta(x)+\frac{1}{S}\mathrm{Re}\left[\frac{d}{dx}\left[\frac{\eta(x)h_0'(x)}{2\mathrm{i}}\right]\right]\right\}dx \tag{27}
$$

or

$$
\begin{aligned}
\lambda^{(\theta)} &= -\frac{4}{ah^2S}\left[\int_{-b}^{b}\hat{w}(x,0)\eta(x)dx+\frac{1}{S}\int_{-b}^{b}\hat{w}(x,0)\frac{d}{dx}\left[\eta(x)\mathrm{Re}\left[\frac{h_0'(x)}{2\mathrm{i}}\right]\right]dx\right] \\
&= -\frac{4}{ah^2S}\left[\int_{-b}^{b}\hat{w}(x,0)\eta(x)dx-\frac{1}{S}\int_{-b}^{b}\frac{\partial\hat{w}(x,0)}{\partial x}\eta(x)\mathrm{Re}\left[\frac{h_0'(x)}{2\mathrm{i}}\right]dx\right]
\end{aligned}
\tag{28}
$$

where, in the last equality, we have used integration by parts and the facts that both $\eta$ and $\hat{w}$ vanish at $x=\pm b$. But

$$
\hat{w}(x,0)=\frac{h_0(x)-\overline{h_0}(x)}{2\mathrm{i}}, \qquad \frac{\partial\hat{w}(x,0)}{\partial x}=\frac{h_0'(x)-\overline{h_0}'(x)}{2\mathrm{i}} \tag{29}
$$

and

$$
\mathrm{Re}\left[\frac{h_0'(x)}{2\mathrm{i}}\right]=\frac{1}{2}\left[\frac{h_0'(x)}{2\mathrm{i}}-\frac{\overline{h_0}'(x)}{2\mathrm{i}}\right]=\frac{1}{2}\frac{\partial\hat{w}(x,0)}{\partial x}. \tag{30}
$$

Finally we arrive at

$$\lambda^{(\theta)} = -\frac{4}{ah^2 S}\int_{-b}^{b}\eta(x)\left[\hat{w}(x,0) - \frac{1}{2S}\left(\frac{\partial \hat{w}(x,0)}{\partial x}\right)^2\right]dx = \lambda_1^{(\theta)} + \lambda_2^{(\theta)}, \tag{31}$$

where we separate the final result into the two contributions

$$\begin{aligned}\lambda_1^{(\theta)} &\equiv -\frac{2}{ah^2 S}\int_{-b}^{b}\eta(x)\left[\hat{w}(x,0) - \frac{1}{S}\left(\frac{\partial \hat{w}(x,0)}{\partial x}\right)^2\right]dx,\\ \lambda_2^{(\theta)} &\equiv -\frac{2}{ah^2 S}\int_{-b}^{b}\eta(x)\hat{w}(x,0)dx.\end{aligned} \tag{32}$$

The quantity $\partial\hat{w}(x,0)/\partial x$ has inverse square root singularities at the integration end-points $x = \pm b$ but the integrand in the expression for $\lambda_1^{(\theta)}$ is nevertheless regular since $\eta(x)$ has simple zeros at those points.

From (12) we notice that, on $y = 0$, $\hat{w}(x,0) = w_P(x,0)$ and $\partial\hat{w}(x,0)/\partial x = \partial w_P(x,0)/\partial x$ which leads, after substitution of the expression (3) for $\eta(x)$, to the formulas reported in the main paper.

Sbragaglia and Prosperetti [2] state their (non-dimensionalized) slip length in the form

$$\lambda^{(L)} = \lambda^{(0,L)} + \tilde{\epsilon}\left[\lambda_1^{(1,L)} + \lambda_2^{(1,L)}\right] + o(\tilde{\epsilon}), \tag{33}$$

where the leading order slip length is also decomposed into two parts. In their paper all lengths are non-dimensionalized with respect to $a^*/\pi$. The non-dimensional expansion parameter $\tilde{\epsilon}$ is related to angle $\theta$ via

$$\tilde{\epsilon}^* = \frac{1}{2R^*} \approx -\frac{\theta}{2b^*} = \frac{\pi}{a^*}\tilde{\epsilon}, \tag{34}$$

where we have added asterisks to emphasize dimensional quantities. We have also added a minus sign in front of angle $\theta$ because a depressed meniscus (rather than a protruding one) was studied in [2]. It follows that the non-dimensional expansion parameters used here and in [2] are related by

$$\tilde{\epsilon} = -\frac{a^*\theta}{2\pi b^*}. \tag{35}$$

The contribution $\lambda_2^{(1)}$ found here precisely corresponds to the quantity $\lambda_2^{(1,L)}$ found by [2] after appropriate non-dimensionalization. Indeed, it can be verified that

$$\tilde{\epsilon}\lambda_2^{(1,L)} = -\theta\left(\frac{\pi}{a^*}\right)\lambda_2^{(1)*}, \tag{36}$$

where, on the right hand side, we have non-dimensionalized our result $\lambda_2^{(1)*}$ in the same way adopted in [2]. To check (36), on substituting from (35), it becomes

$$\frac{a^*\theta}{2\pi b^*}\lambda_2^{(1,L)} = -\theta\left(\frac{\pi}{a^*}\right)\lambda_2^{(1)*} \tag{37}$$

implying that

$$\lambda_2^{(1,L)} = \frac{\pi^2}{a^{*2}}(2b^*\lambda_2^{(1)*}) = \frac{\pi^2}{a^{*2}}\left[\frac{2}{a^*h^{*2}S}\right]\int_{-b^*}^{b^*}(b^{*2}-x^{*2})w_P^*(x,0)dx^*. \tag{38}$$

With the non-dimensionalizations used in [2],

$$b^* = \frac{a^*}{\pi}b, \qquad h^* = \frac{a^*}{\pi}h, \qquad x^* = \frac{a^*}{\pi}x, \qquad w_P^*(x,0) = S\left(\frac{a^*}{\pi}\right)^2\tilde{w}_P(x,0), \tag{39}$$

and (38) reduces to

$$\lambda_2^{(1,L)} = \frac{2}{\pi h^2}\int_{-b}^{b}(b^2-x^2)\tilde{w}_P(x,0)dx \tag{40}$$

which is precisely formula (42) of [2] (once we rename $h$ as $L$ and $b$ as $c$). It follows similarly that

$$\lambda_1^{(1,L)} = \frac{2}{\pi h^2}\int_{-b}^{b}(b^2-x^2)\left[\tilde{w}_P(x,0) - \left(\frac{\partial\tilde{w}_P(x,0)}{\partial x}\right)^2\right]dx, \tag{41}$$

where, in previous work [2], this was computed from the numerical solution of a truncated set of dual series equations.

# 3 Semi-infinite shear flow

In this section we show how to rederive, using reciprocity arguments, the integral expression for the first order slip length correction for semi-infinite shear flow (i.e. when the channel height $h \to \infty$) derived in [2]; the latter paper used quite different methods.

The region $D$ is again the region shown in Figure 1 but now with $h \to \infty$. Since there is no imposed pressure gradient Green's second identity implies

$$0 = \int\int_D \left[w\nabla^2 w_P - w_P\nabla^2 w\right]dA = \oint_{\partial D}\left[w\frac{\partial w_P}{\partial n} - w_P\frac{\partial w}{\partial n}\right]ds. \tag{42}$$

The solution $w$ is that for linear shear flow with unit shear rate $\dot{\gamma} = 1$ over the weakly curved meniscus satisfying $w = 0$ on the no-slip surfaces with

$$\frac{\partial w}{\partial n} = 0 \tag{43}$$

on the meniscus. This solution $w_P$ is that for semi-infinite shear flow over a periodic array of no-shear slots found by Philip [3]. As $y \to \infty$,

$$w \to y + \lambda, \qquad w_P \to y + \lambda_P. \tag{44}$$

The quantity $\lambda^{(\theta)}$ in the small-$\theta$ expansion

$$\lambda = \lambda_P + \theta\lambda^{(\theta)} + \mathcal{O}(\theta^2) \tag{45}$$

is to be found. On the interval $x \in [-b, b]$ we have [3]

$$w_P(x,0) = \frac{2a}{\pi}\cosh^{-1}\left[\frac{\cos(\pi x/2a)}{\cos(\pi b/2a)}\right], \tag{46}$$

implying

$$\frac{\partial w_P}{\partial x}(x,0) = -\frac{\sin(\pi x/2a)}{[\cos^2(\pi x/2a) - \cos^2(\pi b/2a)]^{1/2}}. \tag{47}$$

The associated slip length is

$$\lambda_P = \frac{2a}{\pi}\log\sec\left(\frac{\pi b}{2a}\right). \tag{48}$$

These formulas will be useful later.

Evaluation of the boundary integral in (42) gives

$$0 = \int_a^{-a}[(y+\lambda) - (y+\lambda_P](-dx) + \int_{\text{meniscus}}\left[w\frac{\partial w_P}{\partial n} - w_P\frac{\partial w}{\partial n}\right]ds, \tag{49}$$

where the first integral is the contribution as $y \to \infty$. It follows that

$$\lambda - \lambda_P = -\frac{1}{2a}\int_{\text{meniscus}} w\frac{\partial w_P}{\partial n}ds = -\frac{1}{2a}\int_{\text{meniscus}} w_P\frac{\partial w_P}{\partial n}ds + \mathcal{O}(\theta^2), \tag{50}$$

where we have used (43).

To isolate $\lambda^{(\theta)}$ it only remains to linearize the integral on the right hand side of (50). Introducing the complex potential

$$w_P(x,y) = \mathrm{Im}[h_0(z)] \tag{51}$$

it follows from steps similar to those carried out earlier that

$$\frac{\partial w_P}{\partial n} = \mathrm{Re}\left[-h_0'(z) + \frac{\mathrm{i}x\theta h_0'(z)}{b}\right] + \mathcal{O}(\theta^2). \tag{52}$$

Since $\mathrm{Re}[h_0'(x)] = 0$ then on the meniscus where $z = x + \mathrm{i}\theta\eta(x)$ we find

$$\frac{\partial w_P}{\partial n} = \mathrm{Re}\left[-\mathrm{i}\theta\eta h_0''(x) + \frac{\mathrm{i}x\theta h_0'(x)}{b}\right] + \mathcal{O}(\theta^2) = \mathrm{Re}\left[-\mathrm{i}\theta\frac{d}{dx}(\eta h_0'(x))\right] + \mathcal{O}(\theta^2). \tag{53}$$

On substitution into (50)

$$\lambda^{(\theta)} = \frac{\theta}{2a}\int_{-b}^{b} w_P(x,0)\mathrm{Re}\left[\frac{\theta}{\mathrm{i}}\frac{d}{dx}(\eta h_0'(x))\right]. \tag{54}$$

But, from (30),

$$\lambda^{(\theta)} = \frac{\theta}{2a} \int_{-b}^{b} \eta \left( \frac{\partial w_P}{\partial x} \right)^2 dx = \frac{\theta}{2a} \int_{-b}^{b} \eta(x) \frac{\sin^2(\pi x/2a)}{\cos^2(\pi x/2a) - \cos^2(\pi b/2a)} dx, \tag{55}$$

where we have used (47).

On substituting for $\eta(x)$ from (3), on use of some trigonometric double angle identities, a change of integration variable $s = x/b$ and exploiting the even nature of the integrand, we find

$$\lambda^{(\theta)} = \frac{b\delta}{2} \int_0^1 (1 - s^2) \left[ \frac{1 - \cos(\pi s\delta)}{\cos(\pi s\delta) - \cos(\pi\delta)} \right] ds = \frac{bF(\delta)}{2}, \qquad \delta = \frac{b}{a}, \tag{56}$$

with $F(\delta)$ given explicitly by

$$F(\delta) = \delta \int_0^1 (1 - x^2) \frac{[1 - \cos(x\pi\delta)]dx}{\cos(x\pi\delta) - \cos(\pi\delta)}. \tag{57}$$

Finally, to rescale in order make a connection with [2] where the first order correction to the slip length for this case is denoted by $\lambda^{(1,\infty)}$, as in (36) we must have

$$\epsilon\lambda^{(1,\infty)} = -\theta \left( \frac{\pi}{a} \right) \lambda^{(\theta)} = -\theta \left( \frac{\pi}{a} \right) \frac{bF(\delta)}{2}, \tag{58}$$

where we have used (56) in the second equality. It follows that

$$\frac{\epsilon\lambda^{(1,\infty)}}{2\pi} = -\theta \left( \frac{b}{a} \right) \frac{F(\delta)}{4} = -\frac{b^2}{4Ra} F(\delta) \tag{59}$$

which is exactly the result found in equation (45) of [2].

# 4 High viscosity in the subphase groove

To illustrate the versatility of the theoretical approach of combining perturbation theory with reciprocity ideas, we also include an analogous study of the case of high viscosity contrast, i.e.,

$$\epsilon \gg 1 \tag{60}$$

so we now assume that the subphase fluid is much more viscous than the working fluid. We will carry out a formal perturbation procedure in powers of $1/\epsilon \ll 1$. This situation would be relevant to liquid-infused surfaces where the grooves are filled with oil, say, with water serving as the working fluid. The work of Wexler *et al* [6] involves a theoretical study of this scenario although the focus there was on the induced flow in the subphase groove rather than the effective slip of the working fluid. The following analysis therefore complements that prior work. Here we study only open ended grooves, but the analysis can be generalized to other scenarios. In this section we also restrict attention to flat menisci.

## 4.1 Semi-infinite shear flow

Using $w$ to denote the flow in the upper working fluid and $W$ to denote the subphase fluid velocity, we develop the expansions

$$w = w_0 + \frac{1}{\epsilon} w_1 + \dots \qquad W = W_0 + \frac{1}{\epsilon} W_1 + \dots \tag{61}$$

On substitution into the boundary conditions on the meniscus we find

$$\frac{1}{\epsilon}\left[\frac{\partial w_0}{\partial y} + \frac{1}{\epsilon}\frac{\partial w_1}{\partial y} + \dots\right] = \frac{\partial W_0}{\partial y} + \frac{1}{\epsilon}\frac{\partial W_1}{\partial y} + \dots \tag{62}$$

At leading order this gives

$$\frac{\partial W_0}{\partial y} = 0. \tag{63}$$

The only consistent solution that also satisfies the no-slip condition on all the other channel walls is

$$W_0 = 0. \tag{64}$$

Hence the leading order subphase velocity vanishes; it is too viscous to be moved by the shear stress from the upper fluid implying

$$w_0 = 0 \tag{65}$$

which ensures continuity of velocity on the meniscus at leading order. The leading order flow in the upper fluid, and satisfying the far-field shear condition, is therefore the pure Couette flow:

$$w_0 = \dot{\gamma} y. \tag{66}$$

At $\mathcal{O}(1/\epsilon)$ the shear stress boundary condition (62) on the meniscus now implies

$$\frac{\partial W_1}{\partial y} = \frac{\partial w_0}{\partial y} = \dot{\gamma} \tag{67}$$

or, on use of the Cauchy-Riemann equations,

$$\chi_1 = \dot{\gamma} x, \tag{68}$$

where $\chi_1$ is the harmonic conjugate of $W_1$ (recall that both $w$ and $W$ are harmonic). At this order, continuity of velocity implies

$$w_1 = W_1, \tag{69}$$

where the right hand side is known from the solution for $W_1$ (to be found in §4.6).

## 4.2 Pressure driven channel flow

A similar analysis pertains to pressure driven channel flow with the difference that now the leading order flow in the upper fluid is the Poiseuille flow

$$w_0 = \frac{Sy(h-y)}{2}. \tag{70}$$

At $\mathcal{O}(1/\epsilon)$ the shear stress boundary condition (62) on the meniscus now implies

$$\frac{\partial W_1}{\partial y} = \frac{\partial w_0}{\partial y} = \frac{Sh}{2} \tag{71}$$

or, on use of the Cauchy-Riemann equations,

$$\chi_1 = \frac{Sh}{2}x. \tag{72}$$

At this order the continuity of velocity implies

$$w_1 = W_1, \tag{73}$$

where the right hand side is known from the solution for $W_1$ (which can be found by adapting the methods of §4.6).

## 4.3 Reciprocal formulas for the slip length

Now we give the implications of this perturbation analysis when used in conjunction with the reciprocity relations deriving from Green's identity.

## 4.4 Semi-infinite channel flow

Green's second identity with $w_1$ and the comparison field given by the leading order Couette flow $w_0$ gives

$$0 = \int\int_D \left[w_1 \nabla^2 w_0 - w_0 \nabla^2 w_1\right] dA = \oint_{\partial D} \left(w_1 \frac{\partial w_0}{\partial n} - w_0 \frac{\partial w_1}{\partial n}\right) ds \tag{74}$$

which, after use of all the boundary conditions on $w_1$, leads to

$$\lambda_1 = \frac{1}{2a\dot{\gamma}} \int_{-b}^{b} w_1 dx = \frac{1}{2a\dot{\gamma}} \int_{-b}^{b} W_1 dx, \tag{75}$$

where we have also used the condition (69).

## 4.5 Pressure driven channel flow

Green's second identity with $w_1$ and the comparison field given by the leading order Poiseuille flow $w_0$ gives

$$\int\int_D \left[w_1\nabla^2 w_0 - w_0\nabla^2 w_1\right] dA = \oint_{\partial D}\left(w_1\frac{\partial w_0}{\partial n} - w_0\frac{\partial w_1}{\partial n}\right) ds \tag{76}$$

which, after use of all the boundary conditions on $w_1$, leads to

$$SQ_1 = \int_{-b}^{b} w_1\frac{\partial w_0}{\partial y} ds = \frac{Sh}{2}\int_{-b}^{b} w_1 ds = \frac{Sh}{2}\int_{-b}^{b} W_1 ds, \tag{77}$$

where we have used (71) and (73) and where $Q_1$ is the volume flux associated with $w_1$, i.e.,

$$Q_1 \equiv \int\int_D w_1(x,y)dxdy = \frac{h}{2}\left[\int_{-b}^{b} W_1 dx\right]. \tag{78}$$

To deduce the correction to the slip length we set the total volume flux equal to the flux $Q_{\rm eff}$ through a channel, corresponding to the same pressure gradient, but now with a flat meniscus satisfying the Navier-slip condition on the lower wall with slip parameter $\lambda$, i.e.,

$$Q_{\rm eff} = \frac{ah^3S}{6}\left(1+\frac{3\lambda}{h}\right) = \frac{ah^3S}{6}\left(1+\frac{3(\lambda_0+\epsilon\lambda_1+\ldots)}{h}\right) = Q_0+\epsilon Q_1+\ldots \tag{79}$$

Hence, at first order and on use of (78),

$$\frac{ah^2S\lambda_1}{2} = \frac{h}{2}\left[\int_{-b}^{b} W_1 dx\right], \qquad \text{or} \quad \lambda_1 = \frac{1}{ahS}\left[\int_{-b}^{b} W_1 dx\right]. \tag{80}$$

## 4.6 The solution for $W_1$

To find the slip length correction we must find the complex potential

$$h_1(z) = \chi_1 + \mathrm{i}W_1. \tag{81}$$

On introduction of the composed complex potential

$$\mathcal{H}_1(\xi) \equiv h_1(z(\xi)) = \chi_1 + \mathrm{i}W_1 \tag{82}$$

we find from the no-slip condition on the three subphase walls of the rectangle, corresponding to the real diameter $\xi \in [-1,1]$, that

$$\overline{\mathcal{H}_1}(\xi) = \mathcal{H}_1(\xi). \tag{83}$$

For $\xi \in C_\xi^+$ we have

$$\mathrm{Re}[\mathcal{H}_1(\xi)] = \mathrm{Re}[z(\xi)] = Cx(\xi,\overline{\xi}), \tag{84}$$

where the constant C is

$$C = \begin{cases} \dot{\gamma}, & \text{for shear flow,} \\ Sh/2, & \text{for pressure driven flow.} \end{cases} \tag{85}$$

Hence, for $\xi \in C_\xi^-$ we have

$$\mathrm{Re}[\mathcal{H}_1(\xi)] = \mathrm{Re}[\overline{\mathcal{H}_1(\xi)}] = \mathrm{Re}[\mathcal{H}_1(\overline{\xi})] = Cx(\overline{\xi}, \xi). \tag{86}$$

This implies that

$$\mathrm{Re}[\mathcal{H}_1(\xi)] = \begin{cases} Cx(\xi, \overline{\xi}), & \xi \in C_\xi^+, \\ Cx(\overline{\xi}, \xi), & \xi \in C_\xi^-. \end{cases} \tag{87}$$

On substitution of the series expansion

$$\mathcal{H}_1(\xi) = a_0 + \sum_{n \geq 1} a_n \xi^n \tag{88}$$

into (87) we find that

$$\mathrm{Re}[a_0] = p_0, \qquad a_n = 2p_n, \qquad n \geq 1, \tag{89}$$

where $\{p_n\}$ is the set of Laurent coefficients given by

$$p_n = I_n + J_n \tag{90}$$

with

$$I_n = \frac{1}{2\pi\mathrm{i}} \int_{C_\xi^+} \frac{Cx(\xi, \overline{\xi}) d\xi}{\xi^{n+1}}, \qquad J_n = \frac{1}{2\pi\mathrm{i}} \int_{C_\xi^-} \frac{Cx(\overline{\xi}, \xi) d\xi}{\xi^{n+1}}. \tag{91}$$

By exploiting the result that for $|\xi| = 1$ then $x(\overline{\xi}, \xi) = x(\xi, \overline{\xi})$ it can be shown that

$$\overline{J_n} = I_n \tag{92}$$

implying that

$$p_n = 2\mathrm{Re}[I_n] \tag{93}$$

and, hence, that

$$\mathcal{H}_1(\xi) = 2\mathrm{Re}[I_0] + 4 \sum_{n \geq 1} \mathrm{Re}[I_n] \xi^n, \tag{94}$$

where we have not added any purely imaginary constant in order that the no-slip condition is satisfied at $x = \pm b$. Now, for any integer $n \geq 1$, integration by parts yields

$$I_n = -\frac{C}{2\pi\mathrm{i}} \int_{-b}^{b} \frac{x d\xi}{\xi^{n+1}} = \frac{C}{2\pi\mathrm{i}n} \int_{-b}^{b} x d\left(\frac{1}{\xi^n}\right) = \frac{C}{2\pi\mathrm{i}n} \left[ b((-1)^n + 1) - \int_{-b}^{b} \frac{dx}{\xi^n} \right] \tag{95}$$

and

$$\overline{I_n} = -\frac{C}{2\pi\mathrm{i}n} \left[ b((-1)^n + 1) - \int_{-b}^{b} \xi^n dx \right], \tag{96}$$

where we have used the fact that $\overline{\xi} = 1/\xi$ on the interval $x \in [-b, b]$.

## 4.7 Slip length correction

The leading order slip length is zero so it is the first order correction that is of interest. For semi-infinite shear flow from (75) we have

$$\lambda_1 = \frac{1}{2a\dot{\gamma}} \int_{-b}^{b} \mathrm{Im}\left[\mathcal{H}_1(\xi)\right] dx = \frac{1}{2a\dot{\gamma}} \int_{-b}^{b} \mathrm{Im}\left[2\mathrm{Re}[I_0] + 4\sum_{n\geq 1} \mathrm{Re}[I_n]\xi^n\right] dx. \tag{97}$$

Hence

$$\lambda_1 = \frac{2}{a\dot{\gamma}} \sum_{n\geq 1} \mathrm{Re}[I_n] \mathrm{Im} \int_{-b}^{b} \xi^n dx = \frac{2}{a\dot{\gamma}C} \sum_{n\geq 1} \mathrm{Re}[I_n] \mathrm{Im}[2\pi \mathrm{i} n \overline{I_n}], \tag{98}$$

where we have used (96) in the final step. Since $\mathrm{Im}[2\pi \mathrm{i} n\overline{I_n}] = \mathrm{Im}[2\pi \mathrm{i} n I_n] = \mathrm{Re}[2\pi n I_n]$ we find

$$\lambda_1 = \frac{4\pi}{aC^2} \sum_{n\geq 1} n \mathrm{Re}[I_n]^2, \qquad C = \dot{\gamma}, \tag{99}$$

where $I_n$ is given by formula (91) with $C = \dot{\gamma}$.

Figure 2 shows a graph of this (normalized) slip length correction as a function of aspect ratio of the groove for $b/a = 0.5$ together with the results given from the from large-$\epsilon$ expansions of the slip lengths given by the models of [4] and [5].

For pressure-driven channel flow, from (80) the analogous formula is

$$\lambda_1 = \frac{4\pi}{aC^2} \sum_{n\geq 1} n \mathrm{Re}[I_n]^2, \tag{100}$$

with $I_n$ again given by formula (91) but now with $C = Sh/2$.

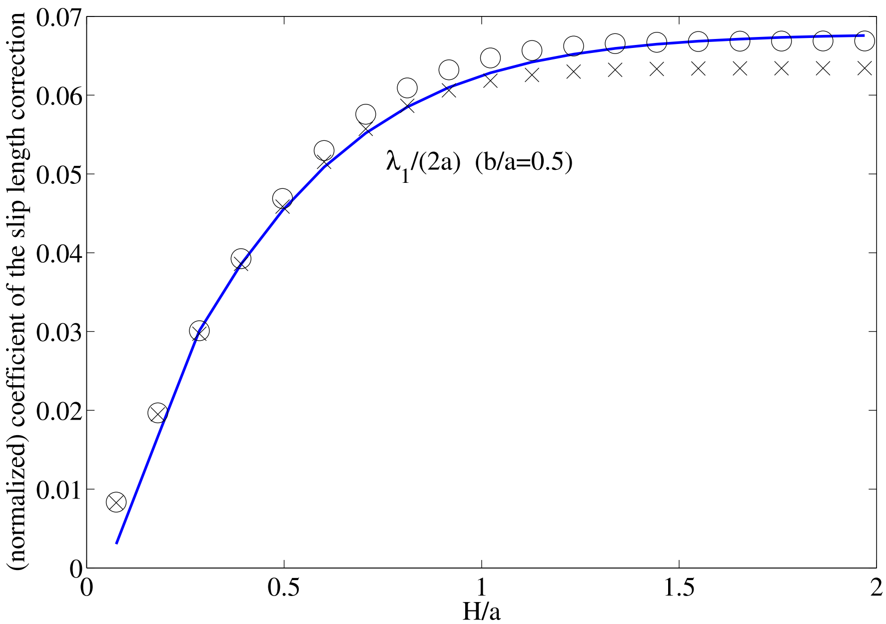


Figure 2: Graph of the normalized slip length correction, for $b/a = 0.5$, as a function of aspect ratio of the groove. Also shown are the results given by expanding, for large $\epsilon$, formulas derived from the models of [4] (dots) and [5] (crosses).

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