1 Derivation of the equation governing the orbit constant distribution

In this supplementary material, we present the derivation of the one-dimensional evolution equation for $f(C)$(equation 2.5 in the paper) starting from the original convection-diffusion equation, on the unit sphere, for the orientation distribution $\Omega(C,\tau)$ (equation 2.1 in the paper). The derivation is accomplished using a multiple scales analysis. There are two time scales in the problem; the first of $O(\dot{\gamma}^{-1})$ corresponds to the Stokesian convection due to the imposed shear and the second corresponds to the slower drift due to fluid inertia, and is of $O(\dot{\gamma}^{-1}Re^{-1})$ (the parameter $RePe_r$ is assumed arbitrary, so that Brownian diffusion occurs on a time scale comparable to the inertial drift).

Equation 2.1 in the paper can be rewritten as :

$$\frac{\partial\Omega}{\partial t} + \boldsymbol{\nabla_p}.[\dot{\boldsymbol{p}}_{jeff}\Omega] = \frac{1}{Pe_r}\nabla^2_{\boldsymbol{p}}\Omega - Re\boldsymbol{\nabla_p}.[\dot{\boldsymbol{p}}_i\Omega]. \tag{1}$$

The first and the second terms on the right-hand side above, capture the effects of Brownian motion and fluid inertia on the orientation distribution. The second term on the left-hand side of (1) denotes the convection due to the Jeffery angular velocity. Defining the non-dimensional fast and slow time scales as t_1 and t_2 respectively, with $t_1 = t$ and $t_2 = Re\,t$, and writing $\Omega = \Omega_0(C,\tau,t_1,t_2) + Re\,\Omega_1(C,\tau,t_1,t_2)$, (1) takes the form:

$$\frac{\partial\Omega_0}{\partial t_1} + \boldsymbol{\nabla_p}.[\dot{\boldsymbol{p}}_{jeff}\Omega_0] = 0, \tag{2}$$

at leading order, and

$$\frac{\partial\Omega_1}{\partial t_1} + \boldsymbol{\nabla_p}.[\dot{\boldsymbol{p}}_{jeff}\Omega_1] = \frac{1}{RePe_r}\nabla^2_{\boldsymbol{p}}\Omega_0 - \boldsymbol{\nabla_p}.[\dot{\boldsymbol{p}}_i\Omega_0] - \frac{\partial\Omega_0}{\partial t_2}, \tag{3}$$

at $O(Re)$. The $O(Re)$ term in the expansion for Ω is the direct effect of inertia which is, of course, small. The $O(1)$ term is the indirect effect, via the alteration of the orbit constant distribution at leading order, for long times.

The rate of change of orientation $\dot{\boldsymbol{p}}_{jeff}$ in (1) takes the form $\frac{h_\tau}{\kappa+1/\kappa}\hat{\boldsymbol{\tau}}$ in the (C,τ) orbital coordinate system. Here, h_τ is the metric factor for the τ coordinate, and $\hat{\boldsymbol{\tau}}$ is the unit vector in the τ direction in the orbital coordinate system (see section A). The divergence operator in the (C,τ) orbital coordinate system given in (56) is used to simplify (2) which gives:

$$\frac{\partial\Omega_0}{\partial t_1} + \frac{1}{h_C h_\tau \sin\alpha}\frac{\partial}{\partial\tau}\left(\frac{\Omega_0 h_c h_\tau \sin\alpha}{\kappa+1/\kappa}\right) = 0, \tag{4}$$

The first-order hyperbolic equation above admits an infinite number of solutions, each corresponding to a particular initial orientation distribution, and with all of them, except one, being time dependent. However, as is shown below, if there is a tiny polydispersity in the particle aspect ratios (as is invariably the case in experiments) all the time dependent solutions approach the steady one at an exponential rate on a time scale inversely proportional to the polydispersity. To see this, we assume that the probability density for the spheroid aspect ratios is given by $h(\kappa;\bar{\kappa},\sigma)$, with mean $\bar{\kappa}$ and standard deviation σ. The assumption of a tiny polydispersity implies that the variance of $h(\kappa;\bar{\kappa},\sigma)$ given by $\sigma^2 = \int (\kappa - \bar{\kappa})^2 h(\kappa; \bar{\kappa}, \sigma) d\kappa$ satisfies $\sigma^2 << \bar{\kappa}^2$. Defining $f_1 = \Omega_0 h_c h_\tau \sin \alpha$, (4) can be rewritten as:

$$\frac{\partial f_1}{\partial t} + \left(\frac{\kappa}{\kappa^2 + 1} \right) \frac{\partial f_1}{\partial \tau} = 0. \tag{5}$$

Note that κ in the above equation is a random variable and therefore f_1, which is the probability density for spheroids of a given aspect ratio κ, is also a random variable. Defining a new variable $\tau_0 = \tau - \frac{\bar{\kappa}}{\bar{\kappa}^2+1} t_1$, where τ_0 denotes the (fictitious) initial phase calculated from the current phase (of a spheroid of aspect ratio κ) using the Jeffery angular velocity of the spheroid with the mean aspect ratio, and rewriting (5) in terms of τ_0 leads to:

$$\frac{\partial f_1}{\partial t_1} + \left(\frac{\kappa}{\kappa^2 + 1} - \frac{\bar{\kappa}}{\bar{\kappa}^2 + 1} \right) \frac{\partial f_1}{\partial \tau_0} = 0. \tag{6}$$

The solution to the equation above is given by $f_1 = g(C, \tau_0 - \left(\frac{\kappa}{\kappa^2+1} - \frac{\bar{\kappa}}{\bar{\kappa}^2+1}\right) t_1) = g(C, \tau_1 + \frac{\bar{\kappa}}{\bar{\kappa}^2+1} t_1)$, where $\tau_1 = \tau_0 - \left(\frac{\kappa}{\kappa^2+1}\right) t_1$ and the function g is specified by the initial condition. For $\kappa = \bar{\kappa}$, the equation reduces to $\frac{\partial f_1}{\partial t_1} = 0$, and the solution is therefore $f_1 = g(C, \tau_0)$; that is if the aspect ratio distribution $h(\kappa; \bar{\kappa}, \sigma) = \delta(\kappa - \bar{\kappa})$, then the solution for f_1 merely reflects the fact that the initial distribution of both orbit constants C and phase angles τ_0 is preserved for all times. However, when $h(\kappa; \bar{\kappa}, \sigma)$ is not a δ function, the initial distribution is not preserved and the measurable distribution would then be the average of f_1 which is defined as:

$$\bar{f}_1 = \int f_1 h(\kappa; \bar{\kappa}, \sigma) d\kappa. \tag{7}$$

The average of (6) gives the governing equation for $\bar{f}_1$:

$$\int \frac{\partial f_1}{\partial t_1} h(\kappa; \bar{\kappa}, \sigma) dk + \int \left(\frac{\kappa}{\kappa^2 + 1} - \frac{\bar{\kappa}}{\bar{\kappa}^2 + 1} \right) \frac{\partial f_1}{\partial \tau_0} h(\kappa; \bar{\kappa}, \sigma) dk = 0. \tag{8}$$

In the analysis that follows, we will show that $\bar{f}_1$, at long times, converges to a time-independent distribution. The first term on the left-hand side of (8) is $\frac{\partial \bar{f}_1}{\partial t_1}$. Expanding the bracketed term on the left-hand side of (8) about $\kappa = \bar{\kappa}$ and defining $A = \frac{(1-\bar{\kappa}^2)}{(1+\bar{\kappa}^2)^2}$ and $B = \frac{\bar{\kappa}^3 - 3\bar{\kappa}}{(1+\bar{\kappa}^2)^3}$ the second integral in (8) becomes:

$$\int_0^1 \left(\frac{\kappa}{\kappa^2 + 1} - \frac{\bar{\kappa}}{\bar{\kappa}^2 + 1} \right) \frac{\partial f_1}{\partial \tau_0} h(\kappa; \bar{\kappa}, \sigma) dk = \int_0^1 (A(\kappa - \bar{\kappa}) + B(\kappa - \bar{\kappa})^2) \left(\left. \frac{\partial f_1}{\partial \tau_0} \right|_{\kappa = \bar{\kappa}} + (\kappa - \bar{\kappa}) \frac{\partial}{\partial \kappa} \left(\frac{\partial f_1}{\partial \tau_0} \right) \bigg|_{\kappa = \bar{\kappa}} \right) h(\kappa; \bar{\kappa}, \sigma) dk. \tag{9}$$

The solution of (6) shows that f_1 is an explicit function of τ_1. Therefore $\frac{\partial f_1}{\partial \tau_0}$ is an explicit function of τ_1. $\frac{\partial}{\partial \kappa}$ can be transformed to $\frac{\partial}{\partial \tau_0}$ as:

$$\frac{\partial}{\partial \kappa} = \frac{\partial \tau_1}{\partial \kappa} \frac{\partial}{\partial \tau_1} = \frac{\kappa^2 - 1}{(1 + \kappa^2)^2} t_1 \frac{\partial}{\partial \tau_1} = \frac{\kappa^2 - 1}{(1 + \kappa^2)^2} t_1 \frac{\partial}{\partial \tau_0}. \tag{10}$$

Substituting (10) in the rhs of (9) one obtains:

$$\int_0^1 \left(\frac{\kappa}{\kappa^2+1} - \frac{\bar\kappa}{\bar\kappa^2+1}\right)\frac{\partial f_1}{\partial\tau_0} h(\kappa;\bar\kappa,\sigma)dk = \int_0^1 (A(\kappa-\bar\kappa)+B(\kappa-\bar\kappa)^2)\left(\left.\frac{\partial f_1}{\partial\tau_0}\right|_{\kappa=\bar\kappa}\right.$$
$$\left. -A\,t_1\,(\kappa-\bar\kappa)\frac{\partial}{\partial\tau_0}\left(\frac{\partial f_1}{\partial\tau_0}\right)\right|_{\kappa=\bar\kappa}\Bigg) h(\kappa;\bar\kappa,\sigma)dk, \quad (11)$$

where we have replaced κ by $\bar\kappa$ in the relevant terms to the required order of approximation. Using $\sigma^2 = \int_0^1(\kappa-\bar\kappa)^2 h(\kappa;\bar\kappa,\sigma)dk$ and neglecting higher order corrections, (11) becomes:

$$\int_0^1 \left(\frac{\kappa}{\kappa^2+1} - \frac{\bar\kappa}{\bar\kappa^2+1}\right)\frac{\partial f_1}{\partial\tau_0} h(\kappa;\bar\kappa,\sigma)dk = \left(-A^2 t_1\sigma^2\frac{\partial}{\partial\tau_0}\left(\frac{\partial\bar f_1}{\partial\tau_0}\right)\Bigg|_{\kappa=\bar\kappa} + B\sigma^2\left.\frac{\partial\bar f_1}{\partial\tau_0}\right|_{\kappa=\bar\kappa}\right). \quad (12)$$

To O(σ^2), the derivatives in (12) may be replaced by the corresponding derivatives of $\bar f_1$, and one obtains the following equation for $\bar f_1$:

$$\frac{\partial\bar f_1}{\partial t_1} + B\sigma^2\frac{\partial\bar f_1}{\partial\tau_0} = A^2\,t_1\,\sigma^2\frac{\partial}{\partial\tau_0}\left(\frac{\partial\bar f_1}{\partial\tau_0}\right). \quad (13)$$

In the equation above, one may express $\bar f_1$ in terms of $\bar\Omega_0$ as $\bar f_1$=$\bar\Omega_0 h_C\,h_\tau\sin\alpha$, which then leads to equation (2.3) in the manuscript. Defining a new variable $\hat\tau_0 = \tau_0$ - $B\sigma^2 t_1$, (13) can be written as

$$\frac{\partial\bar f_1}{\partial t_1} = A^2\,t_1\,\sigma^2\frac{\partial}{\partial\hat\tau_0}\left(\frac{\partial\bar f_1}{\partial\hat\tau_0}\right) \quad (14)$$

$$\Longrightarrow \frac{\partial\bar f_1}{\partial t_1^2} = D\frac{\partial}{\partial\hat\tau_0}\left(\frac{\partial\bar f_1}{\partial\hat\tau_0}\right). \quad (15)$$

The governing equation for $\bar f_1$ is a one-dimensional diffusion equation in a finite domain with the diffusion constant $D = A^2\sigma^2/2 = \frac{(1-\bar\kappa^2)^2}{(1+\bar\kappa^2)^4}\sigma^2/2$.

The normalization condition on $\bar\Omega_0$, which is given by $\int\bar\Omega_0\,h_C h_\tau\sin\alpha\,dCd\tau$=1 , results in the following restriction on $\bar f_1$:

$$\int_0^\infty\int_0^{2\pi}\bar f_1 dCd\tau = 1. \quad (16)$$

Symmetry constraints imply that:

$$\frac{\partial\bar f_1}{\partial C} = 0 \text{ at } C\text{=}0 \text{ and } \infty. \quad (17)$$

The solution for (15) can be found in terms of an eigenfunction expansion using separation of variables:

$$\bar f_1 = \sum_{n=0}^{\infty} e^{-n^2 D t_1^2}(A_n\cos n\hat\tau_0 + B_n\sin n\hat\tau_0). \quad (18)$$

Applying the symmetry constraints and the normalisation condtion, one gets

$$\int_0^\infty A_0 dC = \frac{1}{2\pi} \tag{19}$$

and

$$\frac{\partial A_0}{\partial C} = 0 \ \text{ at } C\text{=}0 \text{ and } \infty. \tag{20}$$

The $n = 0$ term in the summation in (18), together with (19) and (20) gives the time-independent solution to (2) as:

$$\bar{\Omega}_0 = \frac{A_0}{h_C h_\tau \sin\alpha}. \tag{21}$$

In (21), A_0 is a function of C alone and therefore captures the orientation distribution across the Jeffery orbits. The term $h_C h_\tau \sin\alpha dC d\tau$ is the differential area element in the $C - \tau$ coordinate system, and ensures that $\bar{\Omega}_0$ satisfies the normalization condition above. The above expression for $\bar{\Omega}_0$ clearly shows that the polydispersity stabilizes a time independent distribution along the orbit, and the distribution is precisely the inverse of the Jeffery angular velocity.

We note that (3) is an inhomogeneous version of (2), and therefore, to solve (3), one needs to find the Green's function of (2). (15) is the averaged version of (2) accounting for the effects of phase mixing due to polydispersity, and therefore, solving (3) requires the Greens function of (15). This Green's function is the solution of:

$$\frac{\partial \bar{f}_1}{\partial t_1^2} - D\frac{\partial}{\partial \hat{\tau}_0}\left(\frac{\partial \bar{f}_1}{\partial \hat{\tau}_0}\right) = \delta(t_1^2 - t_1'^2)\delta(\hat{\tau}_0 - \hat{\tau}_0'). \tag{22}$$

We derive the Green's function, by comparing the solution for an initial value problem obtained in terms of the Green's function, with that obtained using the separation of variables method above. The solution for the initial value problem defined by:

$$\frac{\partial \bar{f}_1}{\partial t_1^2} - D\frac{\partial}{\partial \hat{\tau}_0}\left(\frac{\partial \bar{f}_1}{\partial \hat{\tau}_0}\right) = \delta(t_1^2) I(\hat{\tau}_0), \tag{23}$$

can be formally written in terms of Green's function as

$$\bar{f}_1 = \int_0^{2\pi} G(\hat{\tau}_0 - \hat{\tau}_0', t_1^2) I(\hat{\tau}_0') d\hat{\tau}_0'. \tag{24}$$

Written in terms of an eigen function expansion using the separation of variables, $\bar{f}_1$ is given by:

$$\bar{f}_1 = \frac{1}{2\pi}\int_0^{2\pi} I(\hat{\tau}_0') d\hat{\tau}_0' + \sum_{n=1}^{\infty} \frac{e^{-n^2 D t_1^2}}{\pi}\left(\int_0^{2\pi} I(\hat{\tau}_0')\cos n\hat{\tau}_0' d\hat{\tau}_0' \ \cos n\hat{\tau}_0 \right.$$
$$\left. + \int_0^{2\pi} I(\hat{\tau}_0')\sin n\hat{\tau}_0' d\hat{\tau}_0' \ \sin n\hat{\tau}_0\right). \tag{25}$$

Comparing (24) and (25), the Green's function for (22) can be written as:

$$G(\hat{\tau}_0 - \hat{\tau}_0', t_1^2 - t_1'^2) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{e^{-n^2 D(t_1^2 - t_1'^2)}}{\pi} (\cos n\hat{\tau}_0' \ \cos n\hat{\tau}_0 + \sin n\hat{\tau}_0' \ \sin n\hat{\tau}_0) \tag{26}$$

or, in the alternative form:

$$G(\hat{\tau}_0 - \hat{\tau}_0', t_1^2 - t_1'^2) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} \frac{e^{-n^2 D(t_1^2 - t_1'^2)}}{\pi} \left(\cos n\left(\tau_0 - \tau_0' - B\sigma^2(t_1 - t_1')\right)\right). \tag{27}$$

To calculate Ω_1, we rewrite (3) as:

$$\frac{\partial \Omega_1}{\partial t_1} + \frac{1}{h_C h_\tau \sin\alpha} \frac{\partial}{\partial \tau}\left(\frac{\Omega_1 h_c h_\tau \sin\alpha}{\kappa + 1/\kappa}\right) = \frac{1}{RePe_r} \nabla_{\boldsymbol{p}}^2 \Omega_0 - \boldsymbol{\nabla}_{\boldsymbol{p}} \boldsymbol{.} [\dot{\boldsymbol{p}}_i \Omega_0] - \frac{\partial \Omega_0}{\partial t_2}, \tag{28}$$

The left-hand side of equation above, can be simplified in a manner similar to that done for the leading order equation in (4). Defining $f_2 = \Omega_1 h_C h_\tau \sin\alpha$, one can rewrite (28) as:

$$\frac{1}{h_C h_\tau \sin\alpha}\left(\frac{\partial f_2}{\partial t_1} + \frac{\partial}{\partial \tau}\left(\frac{f_2}{\kappa + 1/\kappa}\right)\right) = \frac{1}{RePe_r} \nabla_{\boldsymbol{p}}^2 \Omega_0 - \boldsymbol{\nabla}_{\boldsymbol{p}} \boldsymbol{.} [\dot{\boldsymbol{p}}_i \Omega_0] - \frac{\partial \Omega_0}{\partial t_2}, \tag{29}$$

Taking the average of (29), one gets:

$$\int_0^1 \left(\frac{\partial f_2}{\partial t_1} + \frac{\partial}{\partial \tau}\left(\frac{f_2}{\kappa + 1/\kappa}\right)\right) h(\kappa, \bar{\kappa}, \sigma) d\kappa = \int_0^1 h_C h_\tau \sin\alpha \left(\frac{1}{RePe_r} \nabla_{\boldsymbol{p}}^2 \Omega_0 - \boldsymbol{\nabla}_{\boldsymbol{p}} \boldsymbol{.} [\dot{\boldsymbol{p}}_i \Omega_0] - \frac{\partial \Omega_0}{\partial t_2}\right) h(\kappa, \bar{\kappa}, \sigma) d\kappa \tag{30}$$

Further simplification of the equation above gives:

$$\frac{\partial \bar{f}_2}{\partial t_1^2} + \frac{B\sigma^2}{2t_1} \frac{\partial \bar{f}_2}{\partial \tau_0} - \frac{A^2\sigma^2}{2} \frac{\partial}{\partial \tau_0}\left(\frac{\partial \bar{f}_2}{\partial \tau_0}\right) = \frac{h_C h_\tau \sin\alpha}{2t_1} \left(\frac{1}{RePe_r} \nabla_{\boldsymbol{p}}^2 \bar{\Omega}_0 - \boldsymbol{\nabla}_{\boldsymbol{p}} \boldsymbol{.} [\dot{\boldsymbol{p}}_i \bar{\Omega}_0] - \frac{\partial \bar{\Omega}_0}{\partial t_2}\right)\Bigg|_{\kappa = \bar{\kappa}}, \tag{31}$$

where the right-hand side above is evaluated at $\kappa = \bar{\kappa}$. In (31), one may express $\bar{f}_2$ in terms of $\bar{\Omega}_1$ as $\bar{f}_2 = \bar{\Omega}_1 h_C \, h_\tau \sin\alpha$, which then leads to equation (2.4) in the manuscript. The equation (31) can be further simplified to obtain:

$$\frac{\partial \bar{f}_2}{\partial t_1^2} - D\frac{\partial}{\partial \hat{\tau}_0}\left(\frac{\partial \bar{f}_2}{\partial \hat{\tau}_0}\right) = \frac{h_C h_\tau \sin\alpha}{2t_1} \left(\frac{1}{RePe_r} \nabla_{\boldsymbol{p}}^2 \bar{\Omega}_0 - \boldsymbol{\nabla}_{\boldsymbol{p}} \boldsymbol{.} [\dot{\boldsymbol{p}}_i \bar{\Omega}_0] - \frac{\partial \bar{\Omega}_0}{\partial t_2}\right)\Bigg|_{\kappa = \bar{\kappa}}, \tag{32}$$

The Green's function presented in (27) can be used to obtain the solution of (32). The solution for $\bar{\Omega}_1 = \bar{f}_2/(h_C h_\tau \sin\alpha)$ can then be formally written as:

$$\bar{\Omega}_1 = \frac{1}{h_C h_\tau \sin\alpha}\int_0^{t_1^2}\frac{1}{2t_1'}dt_1'^2\int_0^{2\pi} G(\hat{\tau}_0-\hat{\tau}_0', t_1^2-t_1'^2)\left\{\left(\frac{1}{RePe_r}\nabla^2_{\boldsymbol{p}}\bar{\Omega}_0 - \boldsymbol{\nabla_p .}[\dot{\boldsymbol{p}}_i\bar{\Omega}_0] - \frac{\partial\bar{\Omega}_0}{\partial t_2}\right)\right.$$
$$\left. h_C h_\tau \sin\alpha\right\} d\hat{\tau}_0'. \tag{33}$$

For times much longer than $1/D^{1/2}$, neglecting exponentially small corrections, one may obtain the leading order estimate of (33) by using the time-independent form, (21), for $\bar{\Omega}_0$, and further, retaining only the time-independent terms in the Green's function in (33). As a result, $\bar{\Omega}_1$, for long times, is given by:

$$\bar{\Omega}_1 = \frac{1}{h_C h_\tau \sin\alpha}t_1\int_0^{2\pi}\frac{1}{2\pi}\left\{\left(\frac{1}{RePe_r}\nabla^2_{\boldsymbol{p}}\left(\frac{A_0}{h_C h_\tau \sin\alpha}\right) - \boldsymbol{\nabla_p .}\left[\dot{\boldsymbol{p}}_i\left(\frac{A_0}{h_C h_\tau \sin\alpha}\right)\right]\right.\right.$$
$$\left.\left. -\frac{\partial}{\partial t_2}\left(\frac{A_0}{h_C h_\tau \sin\alpha}\right)\right) h_C h_\tau \sin\alpha\right\} d\hat{\tau}_0'. \tag{34}$$

Clearly, $\bar{\Omega}_1$ grows as $O(t_1)$ for long times. In order to avoid this aphysical secular growth, one must have:

$$\int_0^{2\pi}\left\{\left(\frac{1}{RePe_r}\nabla^2_{\boldsymbol{p}}\left(\frac{A_0}{h_C h_\tau \sin\alpha}\right) - \boldsymbol{\nabla_p .}\left[\dot{\boldsymbol{p}}_i\left(\frac{A_0}{h_C h_\tau \sin\alpha}\right)\right]\right.\right.$$
$$\left.\left. -\frac{\partial}{\partial t_2}\left(\frac{A_0}{h_C h_\tau \sin\alpha}\right)\right) h_C h_\tau \sin\alpha\right\} d\hat{\tau}_0' = 0. \tag{35}$$

Denoting A_0 as f, the third term in (35) becomes:

$$\int_0^{2\pi} -\frac{\partial}{\partial t_2}\left(\frac{f}{h_C h_\tau \sin\alpha}\right) h_C h_\tau \sin\alpha d\tau = -2\pi\frac{\partial f}{\partial t_2}. \tag{36}$$

Note that in (36), we have changed the variable $\hat{\tau}_0'$ to τ using the relation $\hat{\tau}_0'$=τ-$B\sigma^2 t_1$-$\frac{\bar{\kappa}^2}{\bar{\kappa}^2+1}t_1$. The fluid inertial correction to the angular velocity is given by $\dot{\boldsymbol{p}}_{\boldsymbol{i}} = u_{CC}\hat{\boldsymbol{C}} + u_{C\tau}\hat{\boldsymbol{\tau}}$. The component along $\hat{\boldsymbol{C}}$, u_{CC} , is responsible for the drift across Jeffery orbits [1]. The divergence operator in (56) is then used to simplify the second term in (35). Noting that the τ derivative integrates to zero over (0 to 2π), the second term simplifies as:

$$\int_0^{2\pi}\left(-\nabla_p .\left(\dot{\boldsymbol{p}}_i\left(\frac{f}{h_C h_\tau \sin\alpha}\right)\right)\right) h_C h_\tau \sin\alpha d\tau = \int_0^{2\pi}\left(-\frac{\partial}{\partial C}\left(h_\tau \sin\alpha\, u_{CC}\left(\frac{f}{h_C h_\tau \sin\alpha}\right)\right)\right) d\tau$$
$$= \int_0^{2\pi} -\frac{\partial}{\partial C}\left(\frac{u_{CC}}{h_C}f\right) d\tau = -\frac{\partial}{\partial C}(2\pi\Delta C f), \tag{37}$$

where $\Delta C_i = \int_0^{2\pi} \frac{u_{CC}}{2\pi h_C} d\tau$ is given in [1].

The first term in (35) is simplified using the Laplacian, derivable from the gradient and divergence operators in section A, and the term reduces to:

$$\int_0^{2\pi} \left(\frac{1}{RePe_r} \nabla_p^2 \left(\frac{f}{h_C h_\tau \sin\alpha} \right) \right) h_C h_\tau \sin\alpha d\tau = \int_0^{2\pi} \frac{1}{RePe_r} \frac{\partial}{\partial C} \left(\left(\frac{1}{h_C \sin^2\alpha} \frac{\partial \left(\frac{f}{h_C h_\tau \sin\alpha} \right)}{\partial C} - \frac{\cot\alpha}{h_\tau \sin\alpha} \frac{\partial \left(\frac{f}{h_C h_\tau \sin\alpha} \right)}{\partial \tau} \right) h_\tau \sin\alpha \right) d\tau = \frac{\pi}{RePe_r} \frac{\partial}{\partial C} \left(\frac{df}{dC} \chi_1 + \frac{f}{C} \chi_2 \right), \tag{38}$$

where $\chi_1(C;\kappa) = \left(\frac{\kappa^2+1}{\kappa^2} + C^2 \left(\frac{7}{2} + \frac{1}{4\kappa^2} + \frac{\kappa^2}{4} \right) + C^4(\kappa^2+1) \right)$ and $\chi_2 = \left(-\frac{\kappa^2+1}{\kappa^2} + C^2 \left(6 - \left(\frac{7}{2} + \frac{1}{4\kappa^2} + \frac{\kappa^2}{4} \right) \right) + 2C^4(\kappa^2+1) \right)$.

The final equation for the evolution of f, the distribution across Jefferys orbits, can be obtained by substituting (36),(37) and (38) in (35) and is given by

$$\frac{\partial f}{\partial t_2} + \frac{\partial}{\partial C}(\Delta C_i f) = \frac{1}{2RePe_r} \frac{\partial}{\partial C} \left(\frac{df}{dC} \chi_1 + \frac{f}{C} \chi_2 \right). \tag{39}$$

2 Calculation of the suspension shear viscosity

In this section we present the calculation of the viscosity, the plots for which have been presented in figures 5 and 6 of the paper. The hydrodynamic contribution to the particulate phase to the averaged suspension stress in the dilute limit may be written as $\langle \sigma_{ij}^p \rangle = n \langle S_{ij} \rangle$. Here, the stresslet, S_{ij}, is that associated with an isolated torque-free spheroid immersed in an ambient simple shear flow. In the nearly athermal limit of interest, the Brownian stress in negligible. Here the angled brackets therefore denote an average over the relevant orientation probability density. $\langle S_{ij} \rangle$ is given by:

$$\langle S_{ij} \rangle = \int \Omega(\boldsymbol{p}) d\boldsymbol{p} \int_{S_p} \frac{1}{2} [\sigma_{ik} x_j n_k + \sigma_{jk} x_i n_k - \frac{2}{3} \delta_{ij} (\sigma_{lk} x_l n_k)] dA, \tag{40}$$

where S_p denotes the surface of the spheroid, and $\Omega(\boldsymbol{p})$ is the orientation distribution of an isolated spheroid in an ambient simple shear flow. The instantaneous stresslet in (40) is a function of $\boldsymbol{p}$ and may be written down from symmetry arguments ([2]) as:

$$S_{ij}(\boldsymbol{p}) = \frac{3}{2} D_1(\xi_0)(E_{kl} p_k p_l)(p_i p_j - \frac{1}{3} \delta_{ij}) + D_2(\xi_0)[(\delta_{ik} - p_i p_k) E_{kl} p_l p_j + (\delta_{jk} - p_j p_k) E_{kl} p_l p_i] + D_3(\xi_0)[(\delta_{ik} - p_i p_k) E_{kl} (\delta_{jl} - p_l p_j) + \frac{1}{2} (E_{kl} p_k p_l)(\delta_{ij} - p_i p_j)], \tag{41}$$

where the coefficients D_1, D_2 and D_3, respectively, denote the aspect-ratio-dependent strength of the stresslet singularities, with ξ_0 being the inverse of the spheroid eccentricity, and $\boldsymbol{E}$ the rate of strain tensor of the simple shear flow. The strength of stress singularities for an oblate spheroid is given by:

$$D_1(\xi_0) = \frac{16\pi}{9\xi_0^3[(3\xi_0^2-2)\cot^{-1}\bar{\xi_0} - 3\bar{\xi_0}]}, \tag{42}$$

$$D_2(\xi_0) = \frac{16\pi\bar{\xi_0}}{[3\xi_0(1-2\xi_0^2)(1-3\xi_0^2+3\xi_0^2\bar{\xi_0}\cot^{-1}\bar{\xi_0})]}, \tag{43}$$

$$D_3(\xi_0) = \frac{32\pi}{3\xi_0(-(2+3\xi_0^2)\bar{\xi_0}+3\xi_0^4\cot^{-1}\bar{\xi_0})}. \tag{44}$$

Above, $\bar{\xi_0}=\sqrt{\xi_0^2-1}$. To evaluate (40), we first calculate the viscosity due to a distribution of the form $\delta(C-C')/(2\pi h_C h_\tau \sin\alpha)$. The viscosity due to the delta function distribution, $\eta_{delta}(C')$, is given by:

$$\eta_{delta}(C') = \frac{(\frac{3}{2}D_1 - 2D_2 + \frac{D_3}{2})\kappa^2\left(C'^2(\kappa^2+1) - 2\sqrt{(C'^2+1)(C'^2\kappa^2+1)}+2\right)}{2(\kappa^2-1)^2\sqrt{(C'^2+1)(C'^2\kappa^2+1)}} + \frac{(D_2-D_3)}{2}\left(1-\frac{1}{\sqrt{(C'^2+1)(C'^2\kappa^2+1)}}\right) + \frac{D_3}{2}. \tag{45}$$

The viscosity for the distribution in section 1 is given by:

$$\eta = \int_0^{2\pi}\int_0^{\infty} \eta_{delta}(C')f(C')\,dC'd\tau. \tag{46}$$

where f is governed by (39).

A The C -τ Coordinate system

The details of the (C,τ) coordinate system are given below. The orbital coordinates (C,τ) are related to the angular coordinates of the orientation vector (θ_j,ϕ_j) in a spherical coordinate system as:

$$C = \frac{\tan\theta_j(\kappa^2\sin^2\phi_j+\cos^2\phi_j)^{1/2}}{\kappa}, \tag{47}$$

$$\tan\tau = \frac{1}{\kappa\tan\phi_j}. \tag{48}$$

Here, κ is the aspect ratio of the spheroid. The leading order angular velocity can be expressed in orbital coordinates as:

$$\frac{dC}{dt} = 0, \tag{49}$$

$$\frac{d\tau}{dt} = \frac{\kappa}{(\kappa^2+1)}. \tag{50}$$

The unit vectors and the metric factors in the orbital coordinate system are given by:

$$\hat{\boldsymbol{C}} = \cos\theta_j \cos\phi_j \mathbf{1}_x + \cos\theta_j \sin\phi_j \mathbf{1}_y - \sin\theta_j \mathbf{1}_z = \hat{\boldsymbol{\theta}}_{\boldsymbol{j}}, \tag{51}$$

$$\hat{\boldsymbol{\tau}} = \frac{\frac{\partial\theta_j}{\partial\tau}}{\sqrt{\left(\frac{\partial\theta_j}{\partial\tau}\right)^2 + \left(\frac{\partial\phi_j}{\partial\tau}\right)^2 \sin^2\theta_j}}\hat{\boldsymbol{\theta}}_{\boldsymbol{j}} + \frac{\frac{\partial\phi_j}{\partial\tau}\sin\theta_j}{\sqrt{\left(\frac{\partial\theta_j}{\partial\tau}\right)^2 + \left(\frac{\partial\phi_j}{\partial\tau}\right)^2 \sin^2\theta_j}}\hat{\boldsymbol{\phi}}_{\boldsymbol{j}}, \tag{52}$$

$$h_C = \frac{\partial\theta_j}{\partial C}, \tag{53}$$

$$h_\tau = \sqrt{\left(\frac{\partial\theta_j}{\partial\tau}\right)^2 + \left(\frac{\partial\phi_j}{\partial\tau}\right)^2 \sin^2\theta_j}, \tag{54}$$

where $\hat{\boldsymbol{\theta}}_{\boldsymbol{j}}$ and $\hat{\boldsymbol{\phi}}_{\boldsymbol{j}} = -\sin(\phi_j)\mathbf{1}_x + \cos(\phi_j)\mathbf{1}_y$ are the polar and azimuthal unit vectors in spherical coordinate system.$\mathbf{1}_x$,$\mathbf{1}_y$ and $\mathbf{1}_z$ are the unit vectors along the flow, gradient and vorticity axes corresponding to a simple shear flow.

The (C, τ) is a non-orthogonal coordinate system and the angle (α) between the unit vectors $\hat{\boldsymbol{C}}$ and $\hat{\boldsymbol{\tau}}$ is given by:

$$\cos\alpha = \frac{\frac{\partial\theta_j}{\partial\tau}}{\sqrt{\left(\frac{\partial\theta_j}{\partial\tau}\right)^2 + \left(\frac{\partial\phi_j}{\partial\tau}\right)^2 \sin^2\theta_j}}. \tag{55}$$

The divergence operator in the (C, τ) coordinate system is given by:

$$\nabla.\boldsymbol{f} = \frac{1}{h_C h_\tau \sin\alpha}\frac{\partial}{\partial C}\left(h_\tau \sin\alpha \boldsymbol{f}.\hat{\boldsymbol{C}}\right) + \frac{1}{h_C h_\tau \sin\alpha}\frac{\partial}{\partial\tau}\left(h_C \sin\alpha \boldsymbol{f}.\hat{\boldsymbol{\tau}}\right). \tag{56}$$

The gradient operator in the (C, τ) coordinate system is given by:

$$\nabla f = \left(\frac{1}{h_C \sin^2\alpha}\frac{\partial f}{\partial C} - \frac{\cot\alpha}{h_\tau \sin\alpha}\frac{\partial f}{\partial\tau}\right)\hat{\boldsymbol{C}} + \left(\frac{1}{h_\tau \sin^2\alpha}\frac{\partial f}{\partial\tau} - \frac{\cot\alpha}{h_C \sin\alpha}\frac{\partial f}{\partial C}\right)\hat{\boldsymbol{\tau}}. \tag{57}$$

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