

# Derivation of the third-order adaptive and Narimanov–Moiseev type nonlinear modal systems

## Supplementary Materials to “Resonant sloshing in an upright annular tank”

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The derivation consists of *five steps* which were described by Faltinsen *et al.* (2003) and Faltinsen & Timokha (2013) for other tank shapes. In the most general case, the third-order adaptive modal system is derived suggesting that the finite liquid depth, the forcing magnitude is equal to  $O(\epsilon)$ , and all generalised coordinates and velocities in (2.8) are of the lowest order  $O(\epsilon^{1/3})$ . Furthermore, the  $o(\epsilon)$ -order quantities can be neglected.

The *first step* suggests the Taylor expansion by  $\zeta$  of  $I_{(Ab)}^{(0)}$ ,  $I_{(Ab)(Mn)}^{(1)}$ , and  $I_{(Ab)(Mn)}^{(2)}$  by (2.13) and (2.14). By definition,  $\zeta = O(\epsilon^{1/3})$ . Analysis shows that  $I_{(Ab)}^{(0)}$  should be expanded up to the third order but  $I_{(Ab)(Mn)}^{(1)}$  and  $I_{(Ab)(Mn)}^{(2)}$  require expansion up to the second order, i.e.

$$I_{(Ab)}^{(0)} = k_{Ab}^{-1} \tanh(k_{Ab}h) + \zeta + \frac{1}{2}\kappa_{Ab}\zeta^2 + \frac{1}{6}k_{Ab}^2\zeta^3 + \dots, \quad (1a)$$

$$I_{(Ab)(Mn)}^{(1)} = O(1) + \zeta + \frac{1}{2}(\kappa_{Ab} + \kappa_{Mn})\zeta^2 + \dots, \quad (1b)$$

$$I_{(Ab)(Mn)}^{(2)} = O(1) + \kappa_{Ab}\kappa_{Mn}\zeta + \frac{1}{2}(k_{Ab}^2\kappa_{Mn} + k_{Mn}^2\kappa_{Ab})\zeta^2 + \dots. \quad (1c)$$

Inserting (1b) and (1c) into (2.13) gives

$$\begin{aligned} \mathcal{G}_{(Ab)(Mn)}^{(1)} &= O(1) + (\mathcal{R}'_{Ab}\mathcal{R}'_{Mn} + \mathcal{R}_{Ab}\mathcal{R}_{Mn}\kappa_{Ab}\kappa_{Mn})\zeta \\ &\quad + \frac{1}{2}[(\kappa_{Ab} + \kappa_{Mn})\mathcal{R}'_{Ab}\mathcal{R}'_{Mn} + \mathcal{R}_{Ab}\mathcal{R}_{Mn}(k_{Ab}^2\kappa_{Mn} + k_{Mn}^2\kappa_{Ab})]\zeta^2, \end{aligned} \quad (2a)$$

$$\mathcal{G}_{(Ab)(Mn)}^{(2)} = O(1) + r^{-2}AM\mathcal{R}_{Ab}\mathcal{R}_{Mn}\zeta + \frac{1}{2}r^{-2}AM(\kappa_{Ab} + \kappa_{Mn})\mathcal{R}_{Ab}\mathcal{R}_{Mn}\zeta^2. \quad (2b)$$

By the *second step*,  $A_{Ab}^P$  and  $A_{ab}^r$  should be expanded up to  $O(\epsilon)$  in terms to the sloshing-related generalised coordinates. For this purpose, (2.8a) is inserted into expressions of (1a) and, thereafter, substituted into the corresponding formulas of (2.12). This gives

$$\begin{aligned} A_{Ab}^P &= \Lambda_{AA}p_{Ab} + \frac{1}{2} \sum_{MN,ij}^{I_\theta, I_r} \chi_{(Mi)(Nj),(Ab)}^{pp} p_{Mi}p_{Nj} + \frac{1}{2} \sum_{mn,ij}^{I_\theta, I_r} \chi_{(mi)(nj),(Ab)}^{rr} r_{mi}r_{nj} \\ &\quad + \frac{1}{3} \sum_{MNK,ijl}^{I_\theta, I_r} \chi_{(Mi)(Nj)(Kl),(Ab)}^{ppp} p_{Mi}p_{Nj}p_{Kl} + \sum_{Mnk,ijl}^{I_\theta, I_r} \chi_{(Mi),(nj)(kl),(Ab)}^{prr} p_{Mi}r_{nj}r_{kl}, \end{aligned} \quad (3a)$$

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$$A_{ab}^r = \Lambda_{,aa} r_{ab} + \sum_{Mn,ij}^{I_\theta, I_r} \chi_{(Mi),(nj),(ab)}^{pr} p_{Mi} r_{nj} + \frac{1}{3} \sum_{mnk,ijl}^{I_\theta, I_r} \chi_{(mi)(nj)(kl),(ab)}^{rrr} r_{mi} r_{nj} r_{kl} \\ + \sum_{MNk,ijl}^{I_\theta, I_r} \chi_{(Mi)(Nj),(kl),(ab)}^{ppr} p_{Mi} p_{Nj} r_{kl}, \quad I_\theta, I_r \rightarrow \infty, \quad (3b)$$

where

$$\begin{aligned} \chi_{(Mi)(Nj),(Ab)}^{pp} &= \kappa_{Ab} \Lambda_{AMN} \lambda_{(Ab)(Mi)(Nj)}, \quad \chi_{(Mi)(nj),(Ab)}^{rr} = \kappa_{Ab} \Lambda_{A,mn} \lambda_{(Ab)(mi)(nj)}, \\ \chi_{(Mi)(Nj)(Kl),(Ab)}^{ppp} &= \frac{1}{2} k_{Ab}^2 \Lambda_{AMNK} \lambda_{(Ab)(Mi)(Nj)(Kl)}, \\ \chi_{(Mi),(nj)(kl),(Ab)}^{prr} &= \frac{1}{2} k_{Ab}^2 \Lambda_{AM,nk} \lambda_{(Ab)(Mi)(nj)(kl)}, \\ \chi_{(Mi),(nj),(ab)}^{pr} &= \kappa_{ab} \Lambda_{M,an} \lambda_{(Mi)(nj)(ab)}, \quad \chi_{(mi)(nj)(kl),(ab)}^{rrr} = \frac{1}{2} k_{ab}^2 \Lambda_{amnk} \lambda_{(mi)(nj)(kl)(ab)}, \\ \chi_{(Mi)(Nj),(kl),(ab)}^{ppr} &= \frac{1}{2} k_{ab}^2 \Lambda_{MN,ak} \lambda_{(Mi)(Nj)(kl)(ab)} \end{aligned}$$

within the  $\Lambda$ -tensor whose elements

$$\Lambda_{M\dots N,i\dots j} = \int_{-\pi}^{\pi} \cos(A\theta) \dots \cos(M\theta) \cdot \sin(i\theta) \dots \sin(j\theta) d\theta. \quad (4)$$

which can be computed by recursive formulas

$$\begin{aligned} \Lambda_{M,i} &= 0, \quad \Lambda_{,ij} = \pi \delta_{ij}, \quad \Lambda_{MN} = \pi \delta_{MN}, \quad M^2 + N^2 \neq 0, \quad \Lambda_{00} = 2\pi, \\ \Lambda_{M\dots NK,i\dots j} &= \frac{1}{2} (\Lambda_{M\dots |N-K|,i\dots j} + \Lambda_{M\dots |N+K|,i\dots j}), \\ \Lambda_{M\dots N,i\dots ljk} &= \frac{1}{2} (\Lambda_{M\dots |j-k|,i\dots l} - \Lambda_{M\dots |j+k|,i\dots l}) \end{aligned}$$

following from the corresponding trigonometrical relations and the  $\lambda$ -tensors defined by the formula

$$\lambda_{(Ab)\dots(Mn)} = \int_{r_1}^1 r \mathcal{R}_{Ab}(r) \dots \mathcal{R}_{Mn}(r) dr. \quad (5)$$

The partial derivatives of (3) by the generalised coordinates take the form

$$\begin{aligned} \frac{\partial A_{Ab}^p}{\partial p_{Df}} &= \Lambda_{AD} \delta_{bf} + \sum_{M,i}^{I_\theta, I_r} \chi_{(Mi)(Df),(Ab)}^{pp} p_{Mi} + \sum_{NK,jl}^{I_\theta, I_r} \chi_{(Df)(Nj)(Kl),(Ab)}^{ppp} p_{Nj} p_{Kl} \\ &\quad + \sum_{nk,jl}^{I_\theta, I_r} \chi_{(Df),(nj)(kl),(Ab)}^{prr} r_{nj} r_{kl}, \quad (6a) \end{aligned}$$

$$\frac{\partial A_{Ab}^p}{\partial r_{df}} = \sum_{m,i}^{I_\theta, I_r} \chi_{(mi)(df),(Ab)}^{rr} r_{mi} + 2 \sum_{Mn,ij}^{I_\theta, I_r} \chi_{(Mi),(nj)(df),(Ab)}^{prr} p_{Mi} r_{nj}, \quad (6b)$$

$$\frac{\partial A_{ab}^r}{\partial p_{Df}} = \sum_{n,j}^{I_\theta, I_r} \chi_{(Df),(nj),(ab)}^{pr} r_{nj} + 2 \sum_{Mn,ij}^{I_\theta, I_r} \chi_{(Mi)(Df),(nj),(ab)}^{ppr} p_{Mi} r_{nj}, \quad (6c)$$

$$\begin{aligned} \frac{\partial A_{ab}^r}{\partial r_{df}} &= \Lambda_{,ad} \delta_{bf} + \sum_{M,i}^{I_\theta, I_r} \chi_{(Mi),(df),(ab)}^{pr} p_{Mi} + \sum_{mn,ij}^{I_\theta, I_r} \chi_{(mi)(nj)(df),(ab)}^{rrr} r_{mi} r_{nj} \\ &\quad + \sum_{MN,ij}^{I_\theta, I_r} \chi_{(Mi)(Nj),(df),(ab)}^{ppr} p_{Mi} p_{Nj}, \quad I_\theta, I_r \rightarrow \infty. \quad (6d) \end{aligned}$$

The *third step* should lead to analogous expressions for  $A_{(Ab)(Mn)}^{pp}$ ,  $A_{(ab)(mn)}^{rr}$  and  $A_{(Ab),(mn)}^{pr}$  but up to the second-order terms,  $O(\epsilon^{2/3})$ . The  $O(1)$ -order term can be taken from the linear modal theory. The result is

$$A_{(Ab)(Mn)}^{pp} = \Lambda_{AM} \delta_{bn} \kappa_{Ab} + \sum_{K,l}^{I_\theta, I_r} \Pi_{(Kl), (Ab)(Mn)}^{p,p} p_{Kl} + \sum_{KC,ld}^{I_\theta, I_r} \Pi_{(Kl)(Cd), (Ab)(Mn)}^{p,pp} p_{Kl} p_{Cd} + \sum_{kc,ld}^{I_\theta, I_r} \Pi_{(kl)(cd), (Ab)(Mn)}^{p,rr} r_{kl} r_{cd}, \quad (7a)$$

$$A_{(ab)(mn)}^{rr} = \Lambda_{am} \delta_{bn} \kappa_{ab} + \sum_{K,l}^{I_\theta, I_r} \Pi_{(Kl), (ab)(mn)}^{r,p} p_{Kl} + \sum_{KC,ld}^{I_\theta, I_r} \Pi_{(Kl)(Cd), (ab)(mn)}^{r,pp} p_{Kl} p_{Cd} + \sum_{kc,ld}^{I_\theta, I_r} \Pi_{(kl)(cd), (ab)(mn)}^{r,rr} r_{kl} r_{cd}, \quad (7b)$$

$$A_{(Ab),(mn)}^{pr} = \sum_{k,l}^{I_\theta, I_r} \Pi_{(kl), (Ab), (mn)}^r r_{kl} + \sum_{KC,ld}^{I_\theta, I_r} \Pi_{(Kl), (cd), (Ab), (mn)}^{pr} p_{Kl} r_{cd}, \quad (7c)$$

where

$$\begin{aligned} \Pi_{(Kl), (Ab)(Mn)}^{p,p} &= \Lambda_{AMK} G_{(Ab)(Mn), (Kl)}^{(11)} + \Lambda_{K,AM} G_{(Ab)(Mn), (Kl)}^{(12)}, \\ \Pi_{(Kl), (ab)(mn)}^{r,p} &= \Lambda_{K,am} G_{(ab)(mn), (Kl)}^{(11)} + \Lambda_{amK} G_{(ab)(mn), (Kl)}^{(12)}, \\ \Pi_{(kl), (Ab), (mn)}^r &= \Lambda_{A,mk} G_{(Ab)(mn), (kl)}^{(11)} - \Lambda_{m,Ak} G_{(Ab)(mn), (kl)}^{(12)}, \\ \Pi_{(Kl)(Cd), (Ab)(Mn)}^{p,pp} &= \Lambda_{AMKC} G_{(Ab)(Mn), (Kl)(Cd)}^{(21)} + \Lambda_{KC,AM} G_{(Ab)(Mn), (Kl)(Cd)}^{(22)}, \\ \Pi_{(kl)(cd), (Ab)(Mn)}^{p,rr} &= \Lambda_{AM,kc} G_{(Ab)(Mn), (kl)(cd)}^{(21)} + \Lambda_{AMkc} G_{(Ab)(Mn), (kl)(cd)}^{(22)}, \\ \Pi_{(Kl)(Cd), (ab)(mn)}^{r,pp} &= \Lambda_{KC,am} G_{(ab)(mn), (Kl)(Cd)}^{(21)} + \Lambda_{KC,am} G_{(ab)(mn), (Kl)(Cd)}^{(22)}, \\ \Pi_{(kl)(cd), (ab)(mn)}^{r,rr} &= \Lambda_{am,kc} G_{(ab)(mn), (kl)(cd)}^{(21)} + \Lambda_{am,kc} G_{(ab)(mn), (kl)(cd)}^{(22)}, \\ \Pi_{(Kl), (cd), (Ab), (mn)}^{pr} &= 2[\Lambda_{AK,mc} G_{(Ab)(mn), (Kl)(cd)}^{(21)} - \Lambda_{Km,Ac} G_{(Ab)(mn), (Kl)(cd)}^{(22)}]; \\ G_{(Ab)(Mn), (Kl)}^{(11)} &= \lambda'_{(Ab)(Mn), (Kl)} + \kappa_{Ab} \kappa_{Mn} \lambda_{(Ab)(Mn)(Kl)}, \\ G_{(Ab)(Mn), (Kl)}^{(12)} &= AM \bar{\lambda}_{(Ab)(Mn)(Kl)}, \\ G_{(Ab)(Mn), (Kl)(Cd)}^{(21)} &= \frac{1}{2} [(\kappa_{Ab} + \kappa_{Mn}) \lambda'_{(Ab)(Mn), (Kl)(Cd)} + (k_{Ab}^2 \kappa_{Mn} + k_{Mn}^2 \kappa_{Ab}) \lambda_{(Ab)(Mn)(Kl)(Cd)}], \\ G_{(Ab)(Mn), (Kl)(Cd)}^{(22)} &= \frac{1}{2} AM (\kappa_{Ab} + \kappa_{Mn}) \bar{\lambda}_{(Ab)(Mn)(Kl)(Cd)} \end{aligned}$$

and

$$\begin{aligned} \lambda'_{(Ab)(Mn), (Cd) \dots (Ef)} &= \int_{r_1}^1 r \mathcal{R}'_{Ab}(r) \mathcal{R}'_{Mn}(r) \dots \mathcal{R}_{Cd}(r) \dots \mathcal{R}_{Ef}(r) dr, \\ \bar{\lambda}_{(Ab) \dots (Mn)} &= \int_{r_1}^1 r^{-1} \mathcal{R}_{Ab}(r) \dots \mathcal{R}_{Mn}(r) dr. \end{aligned} \quad (8)$$

The partial derivatives of (7) by the generalised coordinates are

$$\frac{\partial A_{(Ab)(Cd)}^{pp}}{\partial p_{Ef}} = \Pi_{(Ef), (Ab)(Cd)}^{p,p} + 2 \sum_{M,i}^{I_\theta, I_r} \Pi_{(Mi)(Ef), (Ab)(Cd)}^{p,pp} p_{Mi}, \quad (9a)$$

$$\frac{\partial A_{(Ab)(Cd)}^{pp}}{\partial r_{ef}} = 2 \sum_{m,i}^{I_\theta, I_r} \Pi_{(mi)(ef), (Ab)(Cd)}^{p,rr} r_{mi}, \quad (9b)$$

$$\frac{\partial A_{(ab)(cd)}^{rr}}{\partial p_{Ef}} = \Pi_{(Ef),(ab)(cd)}^{rp} + 2 \sum_{M,i}^{I_\theta, I_r} \Pi_{(Mi)(Ef), (ab)(cd)}^{r,pp} p_{Mi}, \quad (9c)$$

$$\frac{\partial A_{(ab)(cd)}^{rr}}{\partial r_{ef}} = 2 \sum_{m,i}^{I_\theta, I_r} \Pi_{(mi)(ef), (ab)(cd)}^{r,rr} r_{mi}, \quad (9d)$$

$$\frac{\partial A_{(Ab),(cd)}^{pr}}{\partial p_{Ef}} = \sum_{n,j}^{I_\theta, I_r} \Pi_{(Ef),(nj),(Ab)(cd)}^{pr} r_{nj}, \quad (9e)$$

$$\frac{\partial A_{(Ab),(cd)}^{pr}}{\partial r_{ef}} = \Pi_{(ef),(Ab),(cd)}^r + \sum_{M,i}^{I_\theta, I_r} \Pi_{(Mi),(ef),(Ab)(cd)}^{pr} p_{Mi}. \quad (9f)$$

By the *fourth step*, the kinematic equations (2.9) should be resolved with respect to the generalised velocities  $P_{Ab}$  and  $R_{ab}$  by postulating

$$\begin{aligned} P_{Ab} = & \frac{1}{\kappa_{Ab}} \dot{P}_{Ab} + \sum_{MN,ij}^{I_\theta, I_r} V_{(Mi),(Nj),(Ab)}^{pp} \dot{P}_{Mi} p_{Nj} + \sum_{mn,ij}^{I_\theta, I_r} V_{(mi),(nj),(Ab)}^{rr} \dot{r}_{mi} r_{nj} + \sum_{Mnk,ijl}^{I_\theta, I_r} V_{(Mi),(nj),(kl),(Ab)}^{prr} \dot{P}_{Mi} r_{nj} r_{kl} \\ & + \sum_{Mnk,ijl}^{I_\theta, I_r} V_{(Mi),(Nj),(Kl),(Ab)}^{ppp} \dot{P}_{Mi} p_{Nj} p_{Kl} + \sum_{Mnk,ijl}^{I_\theta, I_r} V_{(nj),(Mi),(kl),(Ab)}^{rpr} \dot{r}_{nj} p_{Mi} r_{kl}, \end{aligned} \quad (10a)$$

$$\begin{aligned} R_{ab} = & \frac{1}{\kappa_{ab}} \dot{r}_{ab} + \sum_{Mn,ij}^{I_\theta, I_r} V_{(Mi),(nj),(ab)}^{pr} \dot{P}_{Mi} r_{nj} + \sum_{Mn,ij}^{I_\theta, I_r} V_{(nj),(Mi),(ab)}^{rp} \dot{r}_{nj} p_{Mi} + \sum_{mnk,ijl}^{I_\theta, I_r} V_{(mi),(nj),(kl),(ab)}^{rrr} \dot{r}_{mi} r_{nj} r_{kl} \\ & + \sum_{mnk,ijl}^{I_\theta, I_r} V_{(kl),(Mi),(Nj),(ab)}^{rpp} \dot{r}_{kl} p_{Mi} p_{Nj} + \sum_{mnk,ijl}^{I_\theta, I_r} V_{(Mi),(Nj),(kl),(ab)}^{ppr} \dot{P}_{Mi} p_{Nj} r_{kl}, \end{aligned} \quad (10b)$$

substituting (10) into the kinematic subsystem (2.9), and matching the similar quantities. The procedure derives the  $V$ -coefficients as

$$\begin{aligned} V_{(Mi),(Nj),(Ab)}^{pp} &= \frac{1}{\Lambda_{AA}\kappa_{Ab}} \left[ \chi_{(Nj)(Mi),(Ab)}^{pp} - \frac{\Pi_{(Nj),(Ab)(Mi)}^{p,p}}{\kappa_{Mi}} \right], \\ V_{(mi),(nj),(Ab)}^{rr} &= \frac{1}{\Lambda_{AA}\kappa_{Ab}} \left[ \chi_{(nj)(mi),(Ab)}^{rr} - \frac{\Pi_{(nj),(Ab),(mi)}^r}{\kappa_{mi}} \right], \\ V_{(nj),(Mi),(ab)}^{rp} &= \frac{1}{\Lambda_{aa}\kappa_{ab}} \left[ \chi_{(Mi),(nj),(ab)}^{pr} - \frac{\Pi_{(Mi),(ab)(nj)}^{r,p}}{\kappa_{nj}} \right], \\ V_{(Mi),(nj),(ab)}^{pr} &= \frac{1}{\Lambda_{aa}\kappa_{ab}} \left[ \chi_{(Mi),(nj),(ab)}^{pr} - \frac{\Pi_{(nj),(Mi),(ab)}^r}{\kappa_{Mi}} \right], \\ V_{(Mi),(Nj),(Kl),(Ab)}^{ppp} &= \frac{1}{\Lambda_{AA}\kappa_{Ab}} \left[ \chi_{(Mi)(Nj)(Kl),(Ab)}^{ppp} - \frac{\Pi_{(Nj)(Kl),(Ab)(Mi)}^{p,pp}}{\kappa_{Mi}} \right. \\ &\quad \left. - \sum_{C,d}^{I_\theta, I_r} V_{(Mi),(Nj),(Cd)}^{pp} \Pi_{(Kl),(Ab)(Cd)}^{p,p} \right]; \quad V_{(Mi),(nj),(kl),(Ab)}^{prr} = \frac{1}{\Lambda_{AA}\kappa_{Ab}} \end{aligned}$$

$$\begin{aligned}
 & \times \left[ \chi_{(Mi),(nj),(kl),(Ab)}^{prr} - \frac{\Pi_{(nj)(kl),(Ab)(Mi)}^{P,rr}}{\kappa_{Mi}} - \sum_{c,d}^{I_\theta, I_r} V_{(Mi),(nj),(cd)}^{pr} \Pi_{(kl),(Ab),(cd)}^r \right], \\
 & V_{(mi),(nj),(kl),(ab)}^{rrr} = \frac{1}{\Lambda_{aa}\kappa_{ab}} \left[ \chi_{(nj)(kl)(mi),(ab)}^{rrr} - \frac{\Pi_{(nj)(kl),(ab)(mi)}^{r,rr}}{\kappa_{mi}} \right. \\
 & \quad \left. - \sum_{C,d}^{I_\theta, I_r} V_{(mi),(nj),(Cd)}^{rr} \Pi_{(kl),(Cd),(ab)}^r \right]; \quad V_{(kl),(Mi),(Nj),(ab)}^{rpp} = \frac{1}{\Lambda_{aa}\kappa_{ab}} \\
 & \quad \times \left[ \chi_{(Mi)(Nj),(kl),(ab)}^{ppr} - \frac{\Pi_{(Mi)(Nj),(ab)(kl)}^{P,pp}}{\kappa_{kl}} - \sum_{c,d}^{I_\theta, I_r} V_{(kl),(Mi),(cd)}^{rp} \Pi_{(Nj),(ab),(cd)}^{r,p} \right], \\
 & V_{(Mi),(Nj),(kl),(ab)}^{ppr} = \frac{1}{\Lambda_{aa}\kappa_{ab}} \left[ 2\chi_{(Mi)(Nj),(kl),(ab)}^{ppr} - \frac{\Pi_{(Nj),(kl),(Mi)(ab)}^{pr}}{\kappa_{Mi}} \right. \\
 & \quad \left. - \sum_{C,d}^{I_\theta, I_r} V_{(Mi),(Nj),(Cd)}^{pp} \Pi_{(kl),(Cd),(ab)}^r - \sum_{c,d}^{I_\theta, I_r} V_{(Mi),(kl),(cd)}^{pr} \Pi_{(Nj),(ab),(cd)}^{r,p} \right], \\
 & V_{(nj),(Mi),(kl),(Ab)}^{rpr} = \frac{1}{\Lambda_{AA}\kappa_{Ab}} \left[ 2\chi_{(Mi),(kl),(nj),(Ab)}^{prr} - \frac{\Pi_{(Mi),(kl),(Ab)(nj)}^{pr}}{\kappa_{nj}} \right. \\
 & \quad \left. - \sum_{C,d}^{I_\theta, I_r} V_{(nj),(kl),(Cd)}^{rr} \Pi_{(Mi),(Ab),(Cd)}^{p,p} - \sum_{c,d}^{I_\theta, I_r} V_{(nj),(Mi),(cd)}^{rp} \Pi_{(kl),(Ab),(cd)}^r \right].
 \end{aligned}$$

By the *fifth step*, expressions (6), (9) and (10) are substituted into the dynamic equations (2.10). Excluding the  $o(\epsilon)$ -terms gives the *required adaptive weakly-nonlinear modal equations*

$$\begin{aligned}
 & \sum_{M,i}^{I_\theta, I_r} \ddot{p}_{Mi} \left[ \delta_{ME} \delta_{if} + \sum_{N,j}^{I_\theta, I_r} d_{(Mi),(Nj)}^{pp,(Ef)} p_{Nj} + \sum_{NK,jl}^{I_\theta, I_r} d_{(Mi),(Nj),(Kl)}^{ppp,(Ef)} p_{Nj} p_{Kl} \right. \\
 & \quad \left. + \sum_{nk,jl}^{I_\theta, I_r} d_{(Mi),(nj),(kl)}^{prr,(Ef)} r_{nj} r_{kl} \right] + \sum_{mn,ij}^{I_\theta, I_r} \ddot{r}_{mi} r_{nj} \left[ d_{(mi),(nj)}^{rr,(Ef)} + \sum_{K,l}^{I_\theta, I_r} d_{(mi),(nj),(Kl)}^{rrp,(Ef)} p_{Kl} \right] \\
 & \quad + \sum_{MN,ij}^{I_\theta, I_r} \dot{p}_{Mi} \dot{p}_{Nj} \left[ t_{(Mi),(Nj)}^{pp,(Ef)} + \sum_{K,l}^{I_\theta, I_r} t_{(Mi),(Nj),(Kl)}^{ppp,(Ef)} p_{Kl} \right] + \sum_{Mn,ijl}^{I_\theta, I_r} t_{(Mi),(nj),(kl)}^{prr,(Ef)} \dot{p}_{Mi} \dot{r}_{nj} r_{kl} \\
 & \quad + \sum_{mn,ij}^{I_\theta, I_r} \dot{r}_{mi} \dot{r}_{nj} \left[ t_{(mi),(nj)}^{rr,(Ef)} + \sum_{K,l}^{I_\theta, I_r} t_{(mi),(nj),(Kl)}^{rrp,(Ef)} p_{Kl} \right] + \sigma_{Ef}^2 p_{Ef} \\
 & \quad = -(\ddot{\eta}_1 - g\eta_5 - S_b \ddot{\eta}_5) \delta_{1E} \kappa_{1f} P_f; \quad E = 0, \dots, I_\theta; \quad f = 1, \dots, I_r, \quad (11a)
 \end{aligned}$$

$$\begin{aligned}
 & \sum_{m,i}^{I_\theta, I_r} \ddot{r}_{mi} \left[ \delta_{me} \delta_{ij} + \sum_{N,j}^{I_\theta, I_r} d_{(mi),(Nj)}^{rp,(ef)} p_{Nj} + \sum_{NK,jl}^{I_\theta, I_r} d_{(mi),(Nj),(Kl)}^{rpp,(ef)} p_{Nj} p_{Kl} \right. \\
 & \quad \left. + \sum_{nk,jl}^{I_\theta, I_r} d_{(mi),(nj),(kl)}^{rrr,(ef)} r_{nj} r_{kl} \right] + \sum_{Mn,ij}^{I_\theta, I_r} \ddot{p}_{Mi} r_{nj} \left[ d_{(Mi),(nj)}^{pr,(ef)} + \sum_{K,l}^{I_\theta, I_r} d_{(Mi),(nj),(Kl)}^{prp,(ef)} p_{Kl} \right] \\
 & \quad + \sum_{Mn,ij}^{I_\theta, I_r} \dot{p}_{Mi} \dot{r}_{nj} \left[ t_{(Mi),(nj)}^{pr,(ef)} + \sum_{K,l}^{I_\theta, I_r} t_{(Mi),(nj),(Kl)}^{prp,(ef)} p_{Kl} \right]
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{MNk,ijl}^{I_\theta, I_r} t_{(Mi), (Nj), (kl)}^{ppr, (ef)} \dot{p}_{Mi} \dot{p}_{Nj} r_{kl} + \sum_{mnk,ijl}^{I_\theta, I_r} t_{(mi), (nj), (kl)}^{rrr, (ef)} \dot{r}_{mi} \dot{r}_{nj} r_{kl} + \sigma_{ef}^2 r_{ef} \\
& = -(\ddot{\eta}_2 + g\eta_4 + S_b \ddot{\eta}_4) \delta_{1e} \kappa_{1f} P_f; \quad e = 1, \dots, I_\theta; \quad f = 1, \dots, I_r, \quad (11b)
\end{aligned}$$

where the natural sloshing frequencies  $\sigma_{Ef}$  are defined by (2.5),  $P_f$  and  $S_b$  come from (2.7) and (2.11), respectively, and the hydrodynamic coefficients at the nonlinear quantities can explicitly be computed, as functions of  $h$  and  $r_1$ , by the formulas

$$\begin{aligned}
d_{(Mi), (Nj)}^{pp, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} V_{(Mi), (Nj), (Ef)}^{pp} + \frac{\chi_{(Nj)(Ef), (Mi)}^{pp}}{\kappa_{Mi}} \right], \\
d_{(Mi), (Nj), (Kl)}^{ppp, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} V_{(Mi), (Nj), (Kl), (Ef)}^{ppp} + \frac{\chi_{(Ef)(Nj)(Kl), (Mi)}^{ppp}}{\kappa_{Mi}} \right. \\
&\quad \left. + \sum_{A,b}^{I_\theta, I_r} V_{(Mi), (Nj), (Ab)}^{pp} \chi_{(Kl)(Ef), (Ab)}^{pp} \right], \\
d_{(Mi), (Nj), (kl)}^{prr, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} V_{(Mi), (Nj), (kl), (Ef)}^{prr} + \frac{\chi_{(Ef), (nj)(kl), (Mi)}^{prr}}{\kappa_{Mi}} \right. \\
&\quad \left. + \sum_{a,b}^{I_\theta, I_r} V_{(Mi), (nj), (ab)}^{pr} \chi_{(Ef), (kl), (ab)}^{pr} \right], \\
d_{(mi), (nj)}^{rr, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} V_{(mi), (nj), (Ef)}^{rr} + \frac{\chi_{(Ef), (nj), (mi)}^{rr}}{\kappa_{mi}} \right], \\
d_{(mi), (nj), (Kl)}^{rrp, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} V_{(mi), (Kl), (nj), (Ef)}^{rrp} + \frac{2\chi_{(Kl)(Ef), (nj), (mi)}^{rrp}}{\kappa_{mi}} \right. \\
&\quad \left. + \sum_{a,b}^{I_\theta, I_r} V_{(mi), (Kl), (ab)}^{rp} \chi_{(Ef), (nj), (ab)}^{pr} + \sum_{A,b}^{I_\theta, I_r} V_{(mi), (nj), (Ab)}^{rr} \chi_{(Kl)(Ef), (Ab)}^{pp} \right], \\
t_{(Mi), (Nj)}^{pp, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} V_{(Mi), (Nj), (Ef)}^{pp} + \frac{\Pi_{(Ef), (Mi)(Nj)}^{p,p}}{2\kappa_{Mi}\kappa_{Nj}} \right], \\
t_{(Mi), (Nj), (Kl)}^{ppp, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} \bar{V}_{(Mi), (Nj), (Kl), (Ef)}^{ppp} + \frac{\Pi_{(Kl)(Ef), (Mi)(Nj)}^{p,pp}}{\kappa_{Mi}\kappa_{Nj}} \right. \\
&\quad \left. + \sum_{A,b}^{I_\theta, I_r} V_{(Mi), (Nj), (Ab)}^{pp} \chi_{(Kl)(Ef), (Ab)}^{pp} + \sum_{A,b}^{I_\theta, I_r} \frac{\Pi_{(Ef), (Mi)(Ab)}^{p,p}}{\kappa_{Mi}} V_{(Nj), (Kl), (Ab)}^{pp} \right], \\
t_{(mi), (nj)}^{rr, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} V_{(mi), (nj), (Ef)}^{rr} + \frac{\Pi_{(Ef), (mi)(nj)}^{r,p}}{2\kappa_{mi}\kappa_{nj}} \right], \\
t_{(mi), (nj), (Kl)}^{rrp, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} V_{(mi), (Kl), (nj), (Ef)}^{rrp} + \frac{\Pi_{(Kl)(Ef), (mi)(nj)}^{r,pp}}{\kappa_{mi}\kappa_{nj}} \right. \\
&\quad \left. + \sum_{A,b}^{I_\theta, I_r} V_{(mi), (nj), (Ab)}^{rr} \chi_{(Kl)(Ef), (Ab)}^{pp} + \sum_{a,b}^{I_\theta, I_r} \frac{\Pi_{(Ef), (mi)(ab)}^{r,p}}{\kappa_{mi}} V_{(nj), (Kl), (ab)}^{rp} \right], \\
t_{(Mi), (nj), (kl)}^{prr, (Ef)} &= \frac{\kappa_{Ef}}{\Lambda_{EE}} \left[ \Lambda_{EE} \bar{V}_{(Mi), (nj), (kl), (Ef)}^{prr} + \frac{\Pi_{(Ef), (kl), (Mi)(nj)}^{pr}}{\kappa_{Mi}\kappa_{nj}} \right]
\end{aligned}$$

$$\begin{aligned}
 & + \sum_{a,b}^{I_\theta, I_r} \left( \bar{V}_{(Mi), (n_j), (ab)}^{pr} \chi_{(Ef), (kl), (ab)}^{pr} + \frac{1}{\kappa_{n_j}} V_{(Mi), (kl), (ab)}^{pr} \Pi_{(Ef), (ab), (n_j)}^{r,p} \right) \\
 & \quad + \sum_{A,B}^{I_\theta, I_r} \frac{\Pi_{(Ef), (Mi), (Ab)}^{p,p}}{\kappa_{Mi}} V_{(n_j), (kl), (Ab)}^{rr} \Bigg], \\
 d_{(Mi), (n_j)}^{pr, (ef)} & = \frac{\kappa_{ef}}{\Lambda_{ee}} \left[ \Lambda_{ee} V_{(Mi), (n_j), (ef)}^{pr} + \frac{\chi_{(n_j), (ef), (Mi)}^{rr}}{\kappa_{Mi}} \right], \\
 d_{(Mi), (n_j), (Kl)}^{pp, (ef)} & = \frac{\kappa_{ef}}{\Lambda_{ee}} \left[ \Lambda_{ee} V_{(Mi), (Kl), (n_j), (ef)}^{ppr} + \frac{2\chi_{(Kl), (n_j), (ef), (Mi)}^{prr}}{\kappa_{Mi}} \right. \\
 & \quad \left. + \sum_{A,B}^{I_\theta, I_r} V_{(Mi), (Kl), (Ab)}^{pp} \chi_{(n_j), (ef), (Ab)}^{rr} + \sum_{a,b}^{I_\theta, I_r} V_{(Mi), (n_j), (ab)}^{pr} \chi_{(Kl), (ef), (ab)}^{pr} \right], \\
 d_{(mi), (Nj)}^{rp, (ef)} & = \frac{\kappa_{ef}}{\Lambda_{ee}} \left[ \Lambda_{ee} V_{(mi), (Nj), (ef)}^{rp} + \frac{\chi_{(Nj), (ef), (mi)}^{pr}}{\kappa_{mi}} \right], \\
 d_{(mi), (Nj), (Kl)}^{pp, (ef)} & = \frac{\kappa_{ef}}{\Lambda_{ee}} \left[ \Lambda_{ee} V_{(mi), (Nj), (Kl), (ef)}^{ppr} + \frac{\chi_{(Nj), (Kl), (ef), (mi)}^{ppr}}{\kappa_{mi}} \right. \\
 & \quad \left. + \sum_{a,b}^{I_\theta, I_r} V_{(mi), (Nj), (ab)}^{rp} \chi_{(Kl), (ef), (ab)}^{pr} \right], \\
 d_{(mi), (n_j), (kl)}^{rr, (ef)} & = \frac{\kappa_{ef}}{\Lambda_{ee}} \left[ \Lambda_{ee} V_{(mi), (n_j), (kl), (ef)}^{rr} + \frac{\chi_{(n_j), (kl), (ef), (mi)}^{rrr}}{\kappa_{mi}} \right. \\
 & \quad \left. + \sum_{A,B}^{I_\theta, I_r} V_{(mi), (n_j), (Ab)}^{rr} \chi_{(kl), (ef), (Ab)}^{rr} \right], \\
 t_{(Mi), (n_j)}^{pr, (ef)} & = \frac{\kappa_{ef}}{\Lambda_{ee}} \left[ \Lambda_{ee} \bar{V}_{(Mi), (n_j), (ef)}^{pr} + \frac{\Pi_{(ef), (Mi), (n_j)}^r}{\kappa_{Mi} \kappa_{n_j}} \right], \\
 t_{(Mi), (n_j), (Kl)}^{pp, (ef)} & = \frac{\kappa_{ef}}{\Lambda_{ee}} \left[ \Lambda_{ee} \bar{V}_{(n_j), (Mi), (Kl), (ef)}^{ppr} + \frac{\Pi_{(Kl), (ef), (Mi), (n_j)}^{pr}}{\kappa_{Mi} \kappa_{n_j}} \right. \\
 & \quad \left. + \sum_{a,b}^{I_\theta, I_r} \bar{V}_{(Mi), (n_j), (ab)}^{pr} \chi_{(Kl), (ef), (ab)}^{pr} + \sum_{a,b}^{I_\theta, I_r} \frac{V_{(n_j), (Kl), (ab)}^{rp}}{\kappa_{Mi}} \Pi_{(ef), (Mi), (ab)}^r \right. \\
 & \quad \left. + \sum_{A,B}^{I_\theta, I_r} \frac{V_{(Mi), (Kl), (Ab)}^{pp}}{\kappa_{n_j}} \Pi_{(ef), (Ab), (n_j)}^r \right], \\
 t_{(Mi), (Nj), (kl)}^{pp, (ef)} & = \frac{\kappa_{ef}}{\Lambda_{ee}} \left[ \Lambda_{ee} V_{(Mi), (Nj), (kl), (ef)}^{ppr} + \frac{\Pi_{(kl), (ef), (Mi), (Nj)}^{p, rr}}{\kappa_{Mi} \kappa_{Nj}} \right. \\
 & \quad \left. + \sum_{A,B}^{I_\theta, I_r} V_{(Mi), (Nj), (Ab)}^{pp} \chi_{(kl), (ef), (Ab)}^{rr} + \sum_{a,b}^{I_\theta, I_r} \frac{V_{(Nj), (kl), (ab)}^{pr}}{\kappa_{Mi}} \Pi_{(ef), (Mi), (ab)}^r \right], \\
 t_{(mi), (n_j), (kl)}^{rr, (ef)} & = \frac{\kappa_{ef}}{\Lambda_{ee}} \left[ \Lambda_{ee} \bar{V}_{(mi), (n_j), (kl), (ef)}^{rrr} + \frac{\Pi_{(kl), (ef), (mi), (n_j)}^{r, rr}}{\kappa_{mi} \kappa_{n_j}} \right. \\
 & \quad \left. + \sum_{A,B}^{I_\theta, I_r} V_{(mi), (n_j), (Ab)}^{rr} \chi_{(kl), (ef), (Ab)}^{rr} + \sum_{a,b}^{I_\theta, I_r} \frac{V_{(mi), (kl), (Ab)}^{rr}}{\kappa_{n_j}} \Pi_{(ef), (Ab), (n_j)}^r \right];
 \end{aligned}$$

$$\begin{aligned}
\bar{V}_{(Mi),(Nj),(Kl),(Ab)}^{PPP} &= V_{(Mi),(Nj),(Kl),(Ab)}^{PPP} + V_{(Mi),(Kl),(Nj),(Ab)}^{PPP}, \\
\bar{V}_{(Mi),(n,j),(kl),(Ab)}^{Prr} &= V_{(Mi),(n,j),(kl),(Ab)}^{Prr} + V_{(Mi),(kl),(n,j),(Ab)}^{Prr} + V_{(n,j),(Mi),(kl),(Ab)}^{Prr}, \\
\bar{V}_{(Mi),(n,j),(ab)}^{Pr} &= V_{(Mi),(n,j),(ab)}^{Pr} + V_{(n,j),(Mi),(ab)}^{Pr}, \\
\bar{V}_{(mi),(n,j),(kl),(ab)}^{Rrr} &= V_{(mi),(n,j),(kl),(ab)}^{Rrr} + V_{(mi),(kl),(n,j),(ab)}^{Rrr}, \\
\bar{V}_{(kl),(Mi),(Nj),(ab)}^{Rpp} &= V_{(kl),(Mi),(Nj),(ab)}^{Rpp} + V_{(kl),(Nj),(Mi),(ab)}^{Rpp} + V_{(Mi),(Nj),(kl),(ab)}^{Rpp}.
\end{aligned}$$

The formulas suggest finite  $I_\theta$  and  $I_r$ .

The Narimanov–Moiseev modal equations (2.17)–(2.20) follow from the general third-order equations (11) by using the simplifying asymptotic relations (2.16) neglecting the  $o(\epsilon)$ -terms. The hydrodynamic coefficients of this modal system are computed by the formulas

$$\begin{aligned}
d_1 &= d_{(11),(11),(11)}^{PPP,(11)} = d_{(11),(11),(11)}^{Rrr,(11)} = t_{(11),(11),(11)}^{PPP,(11)} = t_{(11),(11),(11)}^{Rrr,(11)}, \\
d_2 &= d_{(11),(11),(11)}^{Prr,(11)} = d_{(11),(11),(11)}^{Rpp,(11)} = \frac{1}{2}t_{(11),(11),(11)}^{Prr,(11)} = \frac{1}{2}t_{(11),(11),(11)}^{Rpp,(11)}, \\
d_1 - d_2 &= d_{(11),(11),(11)}^{Rrp,(11)} = d_{(11),(11),(11)}^{Prp,(11)}, \quad d_1 - 2d_2 = t_{(11),(11),(11)}^{Rrp,(11)} = t_{(11),(11),(11)}^{Prp,(11)}, \\
d_3^{(j)} &= d_{(11),(2j)}^{PP,(11)} = d_{(11),(2j)}^{Rr,(11)} = d_{(11),(2j)}^{Pr,(11)} = -d_{(11),(2j)}^{Rp,(11)} = t_{(11),(2j)}^{Pr,(11)} = -t_{(2j),(11)}^{Pr,(11)} \\
&\quad = t_{(11),(2j)}^{PP,(11)} + t_{(2j),(11)}^{PP,(11)} = t_{(11),(2j)}^{Rr,(11)} + t_{(2j),(11)}^{Rr,(11)}, \\
d_4^{(j)} &= d_{(2j),(11)}^{PP,(11)} = d_{(2j),(11)}^{Rr,(11)} = -d_{(2j),(11)}^{Pr,(11)} = d_{(2j),(11)}^{Rp,(11)}, \\
d_5^{(j)} &= d_{(11),(0j)}^{PP,(11)} = d_{(11),(0j)}^{Rr,(11)} = t_{(0j),(11)}^{Pr,(11)} = t_{(0j),(11)}^{Rp,(11)} + t_{(11),(0j)}^{Rp,(11)}, \\
d_6^{(j)} &= d_{(0j),(11)}^{PP,(11)} = d_{(0j),(11)}^{Pr,(11)}, \quad d_{7,k} = t_{(11),(11)}^{PP,(2k)} = -t_{(11),(11)}^{Rr,(2k)} = \frac{1}{2}t_{(11),(11)}^{Pr,(2k)}, \\
d_{8,k} &= t_{(11),(11)}^{PP,(0k)} = t_{(11),(11)}^{Rr,(0k)}, \quad d_{10,k} = d_{(11),(11)}^{PP,(0k)} = d_{(11),(11)}^{Rr,(0k)}, \\
d_{9,k} &= d_{(11),(11)}^{PP,(2k)} = -d_{(11),(11)}^{Rr,(2k)} = d_{(11),(11)}^{Pr,(2k)} = d_{(11),(11)}^{Rp,(2k)}, \\
d_{11,k} &= d_{(11),(11),(11)}^{PPP,(3k)} = -d_{(11),(11),(11)}^{Rrr,(3k)} = -\frac{1}{2}d_{(11),(11),(11)}^{Rrp,(3k)}, \\
d_{12,k} &= t_{(11),(11),(11)}^{PPP,(3k)} = -t_{(11),(11),(11)}^{Rrp,(3k)} = -\frac{1}{2}t_{(11),(11),(11)}^{Prr,(3k)} = t_{(11),(11),(11)}^{Prp,(3k)} \\
&\quad = -t_{(11),(11),(11)}^{Rrr,(3k)} = \frac{1}{2}t_{(11),(11),(11)}^{Prp,(3k)}, \\
d_{13,k}^{(j)} &= d_{(11),(2j)}^{PP,(3k)} = -d_{(11),(2j)}^{Rr,(3k)} = d_{(11),(2j)}^{Pr,(3k)} = d_{(11),(2j)}^{Rp,(3k)}, \\
d_{14,k}^{(j)} &= d_{(2j),(11)}^{PP,(3k)} = -d_{(2j),(11)}^{Rr,(3k)} = d_{(2j),(11)}^{Pr,(3k)} = d_{(2j),(11)}^{Rp,(3k)}, \\
d_{15,k}^{(j)} &= t_{(11),(2j)}^{PP,(3k)} + t_{(2j),(11)}^{PP,(3k)} = -t_{(11),(2j)}^{Rr,(3k)} - t_{(2j),(11)}^{Rr,(3k)} = t_{(11),(2j)}^{Pr,(3k)} = t_{(2j),(11)}^{Pr,(3k)}, \\
d_{16,n} &= d_{(11),(11),(11)}^{PPP,(1n)} = d_{(11),(11),(11)}^{Rrr,(1n)}, \quad d_{17,n} = d_{(11),(11),(11)}^{Prr,(1n)} = d_{(11),(11),(11)}^{Rpp,(1n)}, \\
d_{18,n} &= t_{(11),(11),(11)}^{PPP,(1n)} = t_{(11),(11),(11)}^{Rrr,(1n)}, \quad d_{19,n} = t_{(11),(11),(11)}^{Prr,(1n)} = t_{(11),(11),(11)}^{Rpp,(1n)}, \\
d_{16,n} - d_{17,n} &= d_{(11),(11),(11)}^{Rrp,(1n)} = d_{(11),(11),(11)}^{Prp,(1n)}, \quad d_{18,n} - d_{19,n} = t_{(11),(11),(11)}^{Rrp,(1n)} = t_{(11),(11),(11)}^{Prp,(1n)}, \\
d_{20,n}^{(j)} &= d_{(11),(2j)}^{PP,(1n)} = d_{(11),(2j)}^{Rr,(1n)} = d_{(11),(2j)}^{Pr,(1n)} = -d_{(11),(2j)}^{Rp,(1n)}, \\
d_{21,n}^{(j)} &= d_{(2j),(11)}^{PP,(1n)} = d_{(2j),(11)}^{Rr,(1n)} = -d_{(2j),(11)}^{Pr,(1n)} = d_{(2j),(11)}^{Rp,(1n)}, \\
d_{22,n}^{(j)} &= t_{(11),(2j)}^{PP,(1n)} + t_{(2j),(11)}^{PP,(1n)} = t_{(11),(2j)}^{Rr,(1n)} + t_{(2j),(11)}^{Rr,(1n)} = t_{(11),(2j)}^{Pr,(1n)} = -t_{(2j),(11)}^{Pr,(1n)}, \\
d_{23,n}^{(j)} &= d_{(11),(0j)}^{PP,(1n)} = d_{(11),(0j)}^{Rr,(1n)}; \quad d_{24,n}^{(j)} = d_{(0j),(11)}^{PP,(1n)} = d_{(0j),(11)}^{Pr,(1n)}, \\
d_{25,n}^{(j)} &= t_{(11),(0j)}^{PP,(1n)} + t_{(0j),(11)}^{PP,(1n)} = t_{(0j),(11)}^{Pr,(1n)}.
\end{aligned}$$