Derivations of $A_g$, the air gap area between the drop and bath.

Figure 1: (a) Geometry and variables defined. (b) The calculated area of a rigid oblate ellipsoid as it is increasingly submerged grows less than a sphere and more than a disk, as expected.

To begin, we choose a coordinate system shown in figure 1a. With this system the surface area for a surface of revolution is,

$$A = 2\pi \int_{y'}^{b} z(y) \sqrt{1 + (z_y(y))^2} dy. \quad (1)$$

The ellipsoidal geometry defines $z(y)$ as

$$z(y) = a \sqrt{1 - \frac{y^2}{b^2}}, \quad \epsilon^2 = 1 - \frac{b^2}{a^2}, \quad (2)$$

with ellipticity $\epsilon$ and $a$, $b$ and $y'$ defined in figure 1a. Utilizing the relationships in equation 2 we get

$$A = \frac{2\pi \epsilon}{1 - \epsilon^2} \int_{z'}^{b} \sqrt{g^2 + z^2} dz, \quad g = \frac{a(1 - \epsilon^2)}{\epsilon}. \quad (3)$$

Completing the definite integral we arrive at,

$$A_g = \pi a^2 - \frac{\pi}{a^2} \sqrt{(a^2 - z^2)(a^2 - \epsilon^2 z^2)}$$
$$+ \pi a^2 \left( \frac{1 - \epsilon^2}{\epsilon} \right) \ln \frac{a(1 + \epsilon)}{\epsilon \sqrt{a^2 - z^2 + \sqrt{a^2 - \epsilon^2 z^2}}}, \quad (4)$$

which, for $z = a$, reduces to the familiar form for the surface area of a hemi-ellipsoid.

$$A_g = \pi a^2 \left[ 1 + \left( \frac{1 - \epsilon^2}{\epsilon} \right) \tanh^{-1} \epsilon \right]. \quad (5)$$

Equation 4 is the air gap area for an elliptical cap. The plane at $z = z'$ is the plane intersecting the periphery of the air gap.
The gravitational energy for a spherical droplet is,

\[ E_{g0} = \frac{4}{3} \pi \rho g (r^3) r, \]  

(6)

where the vertical coordinate is measured from the lowest point of the droplet. When the droplet flattens we have,

\[ E_g = \frac{4}{3} \pi \rho g (a^2 b) b. \]  

(7)

Since the volume remains constant we have \( r^3 = a^2 b \). In terms of the semi-major (\( a \)) and semi-minor axes (\( b \)), the reduction in gravitational energy is,

\[ \Delta E_g = \frac{4}{3} \pi \rho g a^2 b \left[ a^{2/3} b^{1/3} - b \right]. \]  

(8)

The surface energy of the sphere is,

\[ E_{\sigma} = 4\pi \sigma r^2. \]  

(9)

For the flattened droplet this increases to,

\[ E_{\sigma} = 2\pi \sigma a^2 \left[ 1 + \left( \frac{1 - \epsilon^2}{\epsilon} \right) \tanh^{-1} \epsilon \right], \]  

(10)

where we have used twice the area from equation 5. The increase in surface energy, in terms of \( a \) and \( b \) is then,

\[ \Delta E_{\sigma} = 2\pi \sigma a^2 \left\{ \left[ 1 + \left( \frac{1 - \epsilon^2}{\epsilon} \right) \tanh^{-1} \epsilon \right] - 2b^{2/3}a^{-2/3} \right\}. \]  

(11)

Additional flattening of the drop would incur a larger increase in surface energy than the energy lost from further lowering the center-of-mass of the droplet. Therefore, \( a \) and \( b \) define the shape of the stable oblate spheroid when \( \Delta E_g = \Delta E_{\sigma} \). After some simplification, this equality can be written in a somewhat cleaner form as \( F(a, b) = G(\epsilon) \), where

\[ F(a, b) = \left( \frac{2\rho g}{2\sigma} \right) b^{4/3} \left[ a^{2/3} - b^{2/3} \right] + 2a^{-2/3}b^{2/3}, \]  

(12)

and

\[ G(\epsilon) = 1 + \left( \frac{1 - \epsilon^2}{\epsilon} \right) \tanh^{-1} \epsilon. \]  

(13)

Numerically solving for values of \( a \) and \( b \) for which \( F(a, b) = G(\epsilon) \) allows us to calculate \( \beta = (2/5)(2 + b/D) \) as a function of droplet diameter, \( D \). Doing so for the range of droplet diameters explored gives an inverse relationship as seen in figure.
Figure 2: Numerical (closed circles and measured (open circles) results for $\beta(D)$. 