Supplementary material to 'A model for temperate ice formation'

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1 Introduction

Here we describe in more detail the asymptotic solution of the steady state polythermal ice problem in one dimension, and the direct numerical solution of the same problem using a multiple shooting method. Recall that the polythermal ice problem was cast in the form

$$\mathrm{Pe} u \frac{\mathrm{d}T}{\mathrm{d}z} - \frac{\mathrm{d}^2 T}{\mathrm{d}z^2} = a \tag{1}$$

subject to $T \leq 0$ for the cold subdomain Ω^-, and

$$\mathrm{Pe} u \frac{\mathrm{d}\phi}{\mathrm{d}z} + \phi p_e = a, \tag{2a}$$

$$\mathrm{Pe} u \frac{\mathrm{d}\phi}{\mathrm{d}z} + \frac{\mathrm{d}}{\mathrm{d}z}\left[\kappa \phi^\alpha \left(g + \delta \frac{\mathrm{d}p_e}{\mathrm{d}z}\right)\right] = a. \tag{2b}$$

subject to $\phi > 0$ in the temperate subdomain Ω^+.

As velocity fields, we consider either $u =$ constant, or the divergence-free simple model for an ice divide,

$$\mathbf{u} = (x, 0, -z), \tag{3}$$

with $u = -z$.

At the cold exterior boundary z_c,

$$T = T_0 \tag{4a}$$

An inflow boundary at $z = z_{ct}$ requires

$$T = -\frac{\mathrm{d}T}{\mathrm{d}z} = q = \phi = 0 \qquad \text{at } z = z_{ct}, \tag{4b}$$

with

$$q = \mathrm{Pe} u \phi + \kappa \phi^\alpha \left(g + \delta \frac{\mathrm{d}p_e}{\mathrm{d}z}\right) \tag{4c}$$

along with

$$p_e = N_0. \tag{4d}$$

at the temperate boundary $z = z_t$, while an outflow boundary at z_{ct} has

$$T = 0, \qquad -\frac{\mathrm{d}T}{\mathrm{d}z} = q = \mathrm{Pe}u\phi \qquad \text{at } z = z_{ct}. \tag{4e}$$

along with (4d) and

$$\phi = \phi_0 \tag{4f}$$

The leading order small-δ model is

$$\left(\mathrm{Pe}u + \alpha\kappa\phi^{\alpha-1}g\right)\frac{\mathrm{d}\phi}{\mathrm{d}z} = a, \tag{5}$$

to be integrated from the relevant inflow boundary (either z_t or z_{ct}) at which ϕ is prescribed, and the compaction pressure is

$$p_e = \frac{\alpha\kappa\phi^{\alpha-2}ga}{\mathrm{Pe}u + \alpha\kappa\phi^{\alpha-1}g}, \tag{6}$$

and, if z_{ct} is an outflow boundary, its location at leading order is determined by

$$-\left.\frac{\mathrm{d}T}{\mathrm{d}z}\right|_{z=z_{ct}} = q(z_{ct}) = \mathrm{Pe}u(z_{ct})\phi(z_{ct}) + \kappa\phi(z_{ct})^\alpha g,$$

where ϕ solves (5).

2 Asymptotic approximations

An asymptotic solution that satisfies (5)–(6) is valid at leading order in δ except in boundary layers that can exist near $z = z_t$ or $z = z_{ct}$. We will refer to such a solution as the 'outer' solution. There are a number of different forms the boundary layers can take, depending on the boundary conditions that apply at the boundary in question. The simplest boundary layer forms at the exterior temperate boundary $z = z_t$.

2.1 Boundary layer at the exterior boundary

The outer solution for ϕ depends purely on the imposed boundary condition on ϕ (either (4b) or (4f)), while equation (6) computes the compaction pressure p_e as a function of ϕ only. which will therefore not in general satisfy the boundary condition (4d). A rescaling of (2) captures the boundary layer structure that allows (4d) to be satisfied and which matches with the outer solution in a matching region away from the boundary layer (Holmes, 1995). We define

$$\hat{n} = (z - z_t)/\delta^{1/2}, \qquad \hat{p}_e(\hat{n}) = p_e(z), \qquad \hat{\phi}(\hat{n}) = \phi(z),$$

and expand $\hat{\phi} = \hat{\phi}^{(0)} + \delta^{1/2}\hat{\phi}^{(1)} + O(\delta)$; only the zeroth order solution for $\hat{p}_e$ will be required, and we omit the superscript there. At leading order, we find $\hat{\phi}^{(0)} = \text{constant} = \phi(z_t)$ by matching with the outer solution. At first order,

$$\mathrm{Pe}\hat{u}\frac{\mathrm{d}\hat{\phi}^{(1)}}{\mathrm{d}\hat{n}} + \hat{\phi}^{(0)}\hat{p}_e = a, \qquad \left(\mathrm{Pe}\hat{u} + \alpha\kappa{\phi^{(0)}}^{\alpha-1}g\right)\frac{\mathrm{d}\hat{\phi}^{(1)}}{\mathrm{d}\hat{n}} + \kappa{\phi^{(0)}}^{\alpha}\frac{\mathrm{d}^2\hat{p}_e}{\mathrm{d}\hat{n}^2} = a,$$

where $\hat{u} = u(z_t)$. Combining the last two equations gives

$$\kappa{\phi^{(0)}}^{\alpha}\frac{\mathrm{d}^2\hat{p}_e}{\mathrm{d}\hat{n}^2} - \frac{\mathrm{Pe}\hat{u} + \alpha\kappa{\phi^{(0)}}^{\alpha-1}g}{\mathrm{Pe}\hat{u}}\hat{\phi}^{(0)}\hat{p}_e = -\frac{\alpha\kappa{\phi^{(0)}}^{\alpha-1}g}{\mathrm{Pe}\hat{u}} \tag{7}$$

With $\hat{\phi}^{(0)} = \phi(z_t)$, $\hat{u} = u(z_t)$ and $p_e(z_t)$ defined through (6), the appropriate solution $\hat{p}_e$ that satisfies the boundary condition $\hat{p}_e = N_0$ at $\hat{n} = 0$

$$\hat{p}_e = p_e(z_t) + (N_0 - p_e(z_t))\exp\left(-\lambda|\hat{n}|\right), \quad \text{where} \quad \lambda = \sqrt{\frac{\mathrm{Pe}u(z_t) + \alpha\kappa\phi(z_t)^{\alpha-1}g}{\mathrm{Pe}u(z_t)\kappa\phi(z_t)^{\alpha}}}. \tag{8}$$

This matches the outer solution if and only if λ is finite, non-zero and real. From these constraints, we glean that

$$\mathrm{sgn}[\mathrm{Pe}u(z_t) + \alpha\kappa\phi(z_t)^{\alpha-1}g] = \mathrm{sgn}[u(z_t)], \tag{9}$$

with neither side being zero.

This sign constraint, which will reappear below, has a straightforward interpretation. If we take the one-dimensional version of the time-dependent hyperbolic problem (5), then $\mathrm{Pe}u + \alpha\kappa\phi^{\alpha-1}g$ is the characteristic velocity. The sign constraint implies that advection velocity u and characteristic velocity point in the same direction, at least at the exterior boundary. In other words, if there is a sensible boundary layer structure as derived above, then replacing the advection velocity u with the characteristic velocity $\mathrm{Pe}u + \alpha\kappa\phi^{\alpha-1}g$ also leaves the identification of the boundary as an inflow or outflow boundary unchanged. We show in section 2.2 that an analogous result holds true at the cold-temperate boundary.

We identified inflow boundaries (on which Cauchy data on ϕ are prescribed) in the main paper solely based on the direction of the advection velocity relative to that of the boundary, that is, on $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{n}$. The boundary layer structure above suggests that the hyperbolic problem (5) can only give a viable outer solution (with a well-defined boundary layer structure) if its characteristics lead to the same identification of exterior inflow boundaries. We have only shown this in steady state in one spatial dimension, but the result can be extended to any number of spatial dimensions and to the nonsteady case: boundary layer solutions analogous to the above, with an exponentially decaying pressure that can match the outer solution, in general exist at the temperate exterior boundary only if $(\mathbf{u} - \mathbf{v}) \cdot \mathbf{n}$ has the same sign as $(\mathrm{Pe}\mathbf{u} + \alpha\kappa\phi^{\alpha-1}\mathbf{g} - \mathbf{v}) \cdot \mathbf{n}$.

Clearly, neither $\mathrm{Pe}u(z_t) + \alpha\kappa\phi(z_t)^{\alpha-1}g$ nor $u(z_t)$ can vanish in the boundary layer solution (8). When either of these quantities is close to zero, however, the boundary layer model above no longer applies, and rather specialized alternative boundary layer models are necessary. We consider first the case of $u(z_t)$ being close to zero, which can happen if the velocity field is given by (3). A different boundary layer model then becomes necessary if $z_t = O(\delta)$ and therefore $\hat{u} = O(\delta)$. The relevant rescaling is $\acute{n} = (z - z_t)/\delta$, $\acute{\phi}(\acute{n}) = \phi(z)$, $\acute{p}_e(\acute{n}) = p_e(z)$, and we write $\acute{u} = u(z_t)/\delta = u_0 z_t/(\delta h)$, $\acute{u}_z = \mathrm{d}u/\mathrm{d}z = u_0/h$, with $\acute{u}$ and $\acute{u}_z$ both of $O(1)$. The leading order boundary layer model then becomes, after some straightforward manipulations,

$$\frac{\mathrm{d}\acute{\phi}}{\mathrm{d}\acute{n}} = \frac{a - \acute{\phi}\acute{p}_e}{\mathrm{Pe}(\acute{u} + \acute{u}_z \acute{n})}, \qquad \frac{\mathrm{d}\acute{p}_e}{\mathrm{d}\acute{n}} = \frac{\acute{q}}{\kappa \acute{\phi}^{\alpha}} - g,$$

where matching with the outer problem requires that $\acute{q} = \kappa\phi(z_t)^\alpha g$ is the moisture flux at the boundary in the outer solution.

We consider problems with near-vanishing velocity u at z_t only if z_t is an outflow boundary: this is the physically most realistic configuration of the ice divide problem with the velocity field (3) as given by Robin (1955). The solution must then satisfy $\acute{p}_e = N_0$ at $\acute{n} = 0$ and $\acute{\phi} \to \phi(z_{ct})$ as $\acute{n} \to \infty$ by matching. We solve this boundary layer problem purely numerically by a multiple shooting method, applying the far-field boundary condition at a large but finite value of $\acute{n}$.

Yet another boundary layer structure emerges if $\mathrm{Pe}u(z_t) + \alpha\kappa\phi(z_t)^{\alpha-1}g$ is close to zero. This is the case in which the outer problem only just has a viable solution up to the temperate boundary, or only just fails to have one, as discussed in the main paper.

Assume that z_t is an outflow boundary near which the characteristic velocity $\mathrm{Pe}u(z_t) + \alpha\kappa\phi(z_t)^{\alpha-1}g$ approaches zero, with (5) integrated from a cold-temperate inflow boundary; it is not clear that the reverse case of z_t being an inflow boundary at which the characteristic velocity is small can lead to a steady state solution at all. With this assumption, we can define a point $z = z_f$ that lies close to z_t at which $\mathrm{Pe}u(z_f) + \alpha\kappa\phi(z_f)^{\alpha-1}g = 0$. This is the point at which the outer solution breaks down, or would break down as z_f may lie outside the domain. Defining it relies on extending (5) past the edge of the domain, by continuing the velocity field u and dissipation rate a smoothly outside the domain (which is trivial with the velocity fields u and constant dissipation rates a that we have defined).

The boundary layer structure deduced in (8) breaks down when $z_f - z_t \sim O(\delta^{2/5})$ even if z_f lies outside the domain. A different rescaling becomes necessary, of the form $\grave{n} = (z - z_t)/\delta^{2/5}$, $\grave{\phi}(\grave{n}) = \phi(z)$, $\grave{p}_e(\grave{n}) = \delta^{1/5} p_e(z)$. Let ϕ_c be the critical porosity determined through the outer problem at z_f, which therefore satisfies

$$\mathrm{Pe}\grave{u} + \alpha\kappa g \phi_c^{\alpha-1} = 0, \tag{10}$$

where $\grave{u} = u(z_f)$. Note that this only has a solution when $\grave{u}$ and g have opposite signs, which is necessary for the characteristic velocity to be able to change sign. The boundary layer is intended to capture near-boundary behaviour when ϕ is close to the critical value, so we expand as $\grave{\phi} = \phi_c + \delta^{1/5}\grave{\phi}^{(1)} + \delta^{2/5}\grave{\phi}^{(2)} + \ldots$. Only the leading order term in $\grave{p}_e$ is required, and we omit the superscript as before. At leading order, (2b) is satisfied trivially, while at $O(\delta^{1/5})$, it yields

$$\mathrm{Pe}\grave{u}\frac{\mathrm{d}\grave{\phi}^{(1)}}{\mathrm{d}\grave{n}} + \alpha\kappa g\phi_c^{\alpha-1}\frac{\mathrm{d}\grave{\phi}^{(1)}}{\mathrm{d}\grave{n}} = 0,$$

which again holds trivially by the definition of ϕ_c. At $O(\delta^{2/5})$, we finally obtain

$$\kappa g\alpha(\alpha-1)\phi_c^{\alpha-2}\grave{\phi}^{(1)}\frac{\mathrm{d}\grave{\phi}^{(1)}}{\mathrm{d}\grave{n}} + \frac{\mathrm{d}}{\mathrm{d}\grave{n}}\left(\kappa\phi_c^{\alpha}\frac{\mathrm{d}\grave{p}_e}{\mathrm{d}\grave{n}}\right) + \mathrm{Pe}\grave{u}\frac{\mathrm{d}\grave{\phi}^{(2)}}{\mathrm{d}\grave{n}} + \alpha\kappa\phi_c^{\alpha-1}\frac{\mathrm{d}\grave{\phi}^{(2)}}{\mathrm{d}\grave{n}} = a,$$

or more simply, by integrating,

$$\frac{1}{2}\kappa g\alpha(\alpha-1)\phi_c^{\alpha-2}\ {\grave{\phi}^{(1)}}^{2} + \kappa\phi_c^{\alpha}\frac{\mathrm{d}\grave{p}_e}{\mathrm{d}\grave{n}} = a(\grave{n} - \grave{n}_0), \tag{11}$$

where $\grave{n}_0$ is a constant of integration that can be identified as $\grave{n}_0 = (z_f - z_t)/\delta^{2/5}$ by matching with the outer solution for q. $\grave{n}_0$ is $O(1)$ by our assumption that z_f is sufficiently close to the boundary at z_t.

Equation (2a) similarly yields, expanding to $O(\delta^{1/5})$,

$$\mathrm{Pe}\grave{u}\frac{\mathrm{d}\grave{\phi}^{(1)}}{\mathrm{d}\grave{n}} + \phi_c\grave{p}_e = 0. \tag{12}$$

In the far field, $\grave{p}_e \to 0$ as $\mathrm{sgn}(z_t - z_{ct}) \times \grave{n} \to \infty$, while boundary condition $p_e = N_0$ at $z = z_t$ translates into $\grave{p}_e = 0$ at $\grave{n} = 0$.

We can make the approach to the far field condition more explicit. We expect that, with $\grave{p}_e \to 0$, $\grave{\phi}^{(1)} \sim -\sqrt{2a\grave{n}/[\kappa g\alpha(\alpha-1)\phi_c^{\alpha-2}]}$, and so

$$\grave{p}_e \sim -\frac{\mathrm{Pe}\grave{u}\sqrt{a}}{\sqrt{2\kappa\alpha(\alpha-1)\phi_c^{\alpha}g\grave{n}}} \sim \frac{a}{(\alpha-1)\grave{\phi}^{(1)}},$$

where the assumptions we have made about z_{ct} being an outflow boundary and about a real ϕ_c satisfying the definition (10) ensure that the far field limit $\mathrm{sgn}(z_t - z_{ct}) \times \grave{n} \to \infty$ also corresponds to the limit $g\grave{n} \to \infty$, so the expression on the right-hand side has a real solution. We solve the boundary layer problem using a multiple shooting method, applying the far-field condition above at the relevant end of a long but finite domain.

2.2 Boundary layer at cold-temperate outflow boundaries

Different boundary layers occur at the cold-temperate boundary, and their form depends on whether this is an inflow or outflow boundary. Take the outflow case first. The outer solution in general does not satisfy the zero Darcy flux boundary condition, which at leading order in δ reads $\kappa\phi^{\alpha}g = 0$. To remedy this, we need a rescaling that allows a leading-order contribution to be made to the Darcy flux by pressure gradients,

$$\tilde{n} = (z - z_{ct})/\delta^{1/2}, \qquad \tilde{p}_e(\tilde{n}) = \delta^{1/2} p_e(z), \qquad \tilde{\phi}(\tilde{n}) = \phi(z). \tag{13}$$

At leading order, (2) then becomes

$$\mathrm{Pe}\tilde{u}\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tilde{n}} + \tilde{p}_e\tilde{\phi} = 0, \qquad \mathrm{Pe}\tilde{u}\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tilde{n}} + \frac{\mathrm{d}}{\mathrm{d}\tilde{n}}\left[\kappa\tilde{\phi}^{\alpha}\left(g + \frac{\mathrm{d}\tilde{p}_e}{\mathrm{d}\tilde{n}}\right)\right] = 0, \tag{14}$$

where $\tilde{u} = u(z_{ct})$. Integrating the second of these equations with respect to $\tilde{n}$ gives

$$\mathrm{Pe}\tilde{u}\tilde{\phi} + \kappa\tilde{\phi}^{\alpha}\left(g + \frac{\mathrm{d}\tilde{p}_e}{\mathrm{d}\tilde{n}}\right) = \tilde{q},$$

where $\tilde{q}$ is a constant of integration that we can identify by matching with the outer solution as the total hyperbolic moisture flux in the outer solution at z_{ct},

$$\tilde{q} = \mathrm{Pe}\tilde{u}\phi(z_{ct}) + \kappa\phi(z_{ct})^{\alpha} g.$$

The fact that this total moisture flux remains constant throughout the boundary layer at leading order is the basis for using the outer solution for $q = \mathrm{Pe}\tilde{u}\phi(z_{ct}) + \kappa\phi(z_{ct})^{\alpha} g$ locate the cold-temperate boundary location z_{ct} through $(4\mathrm{e})_2$, $-\mathrm{d}T/\mathrm{d}z = q = \tilde{q}$ at $z = z_{ct}$.

Rearranging yields

$$\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tilde{n}} = -\frac{\tilde{p}_e\tilde{\phi}}{\mathrm{Pe}\tilde{u}}, \qquad \frac{\mathrm{d}\tilde{p}_e}{\mathrm{d}\tilde{n}} = \frac{\tilde{q}}{\kappa\tilde{\phi}^{\alpha}} - \frac{\mathrm{Pe}\tilde{u}}{\kappa\tilde{\phi}^{\alpha-1}} - g \tag{15}$$

In the far field, $\mathrm{sgn}(z_t - z_{ct}) \times \tilde{n} \to \infty$, matching with the outer solution further requires that $(\tilde{\phi}, \tilde{p}_e) \to (\phi(z_{ct}), 0)$, which is a fixed point of the dynamical system (15) given the definition of $\tilde{q}$. At the cold-temperate boundary itself, $\tilde{n} = 0$ and we must have zero Darcy flux $\kappa\tilde{\phi}^{\alpha}(g + \mathrm{d}\tilde{p}_e/\mathrm{d}\tilde{n}) = 0$, or equivalently, $\tilde{\phi} = \tilde{q}/(\mathrm{Pe}u)$.

We therefore require an orbit of (15) that connects the fixed point to the line $\tilde{\phi} = \tilde{q}/(\mathrm{Pe}\tilde{u})$ in the $(\tilde{\phi}, \tilde{p}_e)$ plane. This turns out to require a constraint analogous to (9) to be satisfied. Note that the substitution $\tilde{\psi} = \log(\tilde{\phi})$ renders (15) in the form of a Hamiltonian dynamical system, and the fixed point must therefore generally be a saddle point in order for the required orbit to exist. Linearizing about the fixed point, we find the Jacobian of the dynamical system as

$$J = \begin{pmatrix} 0 & -\frac{\phi(z_{ct})}{Pe\,\tilde{u}} \\ \frac{-\alpha\tilde{q} + (\alpha-1)Pe\,\tilde{u}\phi(z_{ct})}{\kappa\phi(z_{ct})^{\alpha+1}} & 0 \end{pmatrix}$$

The eigenvalues of J take the form of a pair $\pm\lambda$ that is opposite in sign but may be either real or pure imaginary. λ is real and non-zero (so that the fixed point is a saddle) if and only if $-\alpha\tilde{q} + (\alpha - 1)Pe\,u\phi(z_{ct})$ and $\mathrm{Pe}u$ are non-zero and of the same sign. Using the definition of $\tilde{q}$, that constraint can be written analogously to (9) as

$$\mathrm{sgn}[\mathrm{Pe}u(z_{ct}) + \alpha\kappa\phi(z_{ct})^{\alpha-1} g] = \mathrm{sgn}[u(z_{ct})], \tag{16}$$

We conclude that the constraint (16) must generally apply at cold-temperate boundaries as well as at exterior boundaries. (We have only dealt with cold-temperate outflow boundaries here, but the boundary condition $\phi = 0$ at inflow boundaries means that the constraint is satisfied trivially.) The relevant orbit of (15) is then straightforward to compute numerically, which we use to construct the composite solutions shown in the main paper.

There is a caveat to this. again associated with the characteristic velocity $\mathrm{Pe}u(z_{ct}) + \alpha\kappa\phi(z_{ct})^{\alpha-1}$ nearly vanishing. Technically, if we set the characteristic velocity to zero in (15), we obtain a non-hyperbolic fixed point (a degenerate saddle) with $\lambda = 0$, but which still has an orbit connected to it along a centre manifold. However, even though this still furnishes a viable solution to the boundary layer problem, the boundary layer description no longer applies when the characteristic velocity is near zero at the boundary, and matching with the outer solution instead involves a second boundary layer similar to the one described by (11)–(12). We do not explore this in detail here.

2.3 Boundary layer at cold-temperate inflow boundaries

It is not immediately obvious whether a boundary layer is required when the cold-temperate boundary is an inflow boundary. If we solve (5)–(6) near such a boundary with a zero porosity boundary condition at $z = z_{ct}$, we obtain

$$\phi \sim \frac{a}{Pe\tilde{u}}(z - z_{ct}) - \frac{\kappa g}{\mathrm{Pe}\tilde{u}}\left(\frac{a(z - z_{ct})}{\mathrm{Pe}\tilde{u}}\right)^{\alpha} + \dots \tag{17a}$$

$$p_e \sim \frac{\alpha a \kappa g}{\mathrm{Pe}\tilde{u}}\left(\frac{a(z - z_{ct})}{\mathrm{Pe}\tilde{u}}\right)^{\alpha-2} - 2\alpha(\alpha-1)\frac{\kappa^2 g^2 a}{\mathrm{Pe}\tilde{u}}\left(\frac{a(z - z_{ct})}{\mathrm{Pe}\tilde{u}}\right)^{2\alpha-3} + \dots \tag{17b}$$

where $\tilde{u} = u(z_{ct})$ in the notation of the previous subsection. For $\alpha > 3/2$, this leading order solution also satsifies the zero Darcy flux boundary condition to $O(\delta)$: we have not only $\kappa\phi^\alpha g = 0$ at $z = z_{ct}$, but also $\kappa\phi^\alpha \mathrm{d}p_e/\mathrm{d}z \to 0$ as $z \to z_{ct}$. As in the remainder of the paper, we will focus on $\alpha \geq 2$ below.

To determine whether a boundary layer is needed, we therefore need to go further and look at the $O(\delta)$ corrections $\delta\phi^{(1)}$ and $\delta {p_e}^{(1)}$ to the zeroth order solution ϕ, p_e computed from (5)–(6), from which we just derived (17b) (we omit superscripts $^{(0)}$ for consistency with the notation of (5)–(6)). The main issue at stake is whether these first order corrections remain small compared with the zeroth order solutions, which themselves approach zero near the boundary for $\alpha > 2$.

The first order corrections satisfy

$$\mathrm{Pe}u\frac{\mathrm{d}\phi^{(1)}}{\mathrm{d}z} + \phi^{(1)}p_e + \phi {p_e}^{(1)} = 0, \qquad (\mathrm{Pe}u + \alpha\kappa\phi^\alpha g)\frac{\mathrm{d}\phi^{(1)}}{\mathrm{d}z} + \frac{\mathrm{d}}{\mathrm{d}z}\left(\kappa\phi^\alpha\frac{\mathrm{d}p_e}{\mathrm{d}z}\right),$$

where ϕ and p_e without superscripts denote the zeroth order solution as above.

With $\phi^{(1)} = 0$ at $z = z_{ct}$, the solutions near $z = z_{ct}$ behave as

$$\phi^{(1)} \sim \frac{\alpha(\alpha-2)\kappa^2 g}{\mathrm{Pe}\tilde{u}} \left(\frac{a}{\mathrm{Pe}\tilde{u}}\right)^3 \left(\frac{a(z-z_{ct})}{\mathrm{Pe}\tilde{u}}\right)^{2\alpha-4} + \ldots \tag{18a}$$

$$p_e{}^{(1)} \sim \alpha(\alpha-2)\kappa^2 g \left(\frac{a}{\mathrm{Pe}\tilde{u}}\right)^4 \left(\frac{a(z-z_{ct})}{\mathrm{Pe}\tilde{u}}\right)^{2\alpha-5} + \ldots \tag{18b}$$

If we restrict our attention to $\alpha \geq 2$, it follows from the exponents on $(z - z_{ct})$ that the first order correction $\delta p_e{}^{(1)}$ is larger than or similar in size to the zeroth order solution p_e in (17b) itself when $2 < \alpha < 3$ and $|z - z_{ct}| \lesssim \delta^{1/(3-\alpha)}$, indicating that a boundary layer is required. For $\alpha = 2$, the leading order (in $z - z_{ct}$) term displayed vanishes due to the coefficient $\alpha - 2$. By including higher order terms in $(z - z_{ct})$, it can be shown that $p_e{}^{(1)} \sim (z - z_{ct})$ near the boundary in that case, while $p_e \sim \alpha a\kappa g/(\mathrm{Pe}\tilde{u})$, so that the asymptotic expansion remains well-ordered and no boundary layer is needed.

The relevant rescaling that captures the boundary layer for $2 < \alpha < 3$ is $\breve{n} = \delta^{-1/(3-\alpha)}(z - z_{ct})$, $\breve{\phi}(\breve{n}) = \delta^{-1/(3-\alpha)}\phi(z)$, and $\breve{p}_e(\breve{n}) = \delta^{(\alpha-2)/(3-\alpha)}p_e(z)$. The leading order rescaled version of (2) is

$$\mathrm{Pe}\tilde{u}\frac{\partial\breve{\phi}}{\partial\breve{n}} = a, \tag{19a}$$

$$\frac{\partial}{\partial\breve{n}}\left[\breve{\phi}^\alpha\left(g + \frac{\partial\breve{p}_e}{\partial\breve{n}}\right)\right] - \breve{\phi}\breve{p}_e = 0. \tag{19b}$$

Combined with $\breve{\phi} = 0$ at $\breve{n} = 0$, the porosity solution $\breve{\phi} = a/(\mathrm{Pe}\tilde{u})\breve{n}$ is unchanged from (17a), while $\breve{p}_e$ satisfies a linear elliptic problem

$$\frac{\partial}{\partial\breve{n}}\left[\kappa\left(\frac{a}{\mathrm{Pe}\tilde{u}}\breve{n}\right)^\alpha \frac{\partial\breve{p}_e}{\partial\breve{n}}\right] - \frac{a}{Pe\tilde{u}}\breve{n}\breve{p}_e = -\frac{\alpha a\kappa g}{\mathrm{Pe}\tilde{u}}\left(\frac{a\breve{n}}{\mathrm{Pe}\tilde{u}}\right)^{\alpha-1}. \tag{20}$$

To match, we require that $\breve{p}_e$ satisfies

$$\breve{p}_e \sim \frac{\alpha a\kappa g}{\mathrm{Pe}\tilde{u}}\left(\frac{a\breve{n}}{\mathrm{Pe}\tilde{u}}\right)^{\alpha-2} \tag{21}$$

in the far field, and in addition we require that $\breve{\phi}^\alpha(g + \mathrm{d}\breve{p}_e/\mathrm{d}\breve{n})$ vanishes as $\breve{n} \to 0$.

With $\alpha > 2$, (20) has an irregular singular point at $\breve{n} = 0$, and classical approaches such as a Frobenius expansion will not work. Numerically, we handle the problem analogously to the inflow boundary in the shooting method solution to the full steady state problem (section 3), by transforming to the variables $\Upsilon = \kappa\breve{\phi}^{\alpha-1}(g + \mathrm{d}\breve{p}_e/\mathrm{d}\breve{n})$, $\Psi = \breve{\phi}^{\alpha-2}\breve{p}_e$, $\mathrm{d}\zeta/\mathrm{d}\breve{n} = \breve{\phi}^{-1}$, which leads to the dynamical system

$$\frac{\mathrm{d}\breve{\phi}}{\mathrm{d}\zeta} = \frac{a}{\mathrm{Pe}\tilde{u}}\breve{\phi}, \qquad \frac{\mathrm{d}\Upsilon}{\mathrm{d}\zeta} = -\frac{a}{\mathrm{Pe}\tilde{u}}\Upsilon + \breve{\phi}^{3-\alpha}\Psi, \qquad \frac{\mathrm{d}\Psi}{\mathrm{d}\zeta} = \frac{1}{\kappa}\Upsilon + \frac{a(\alpha-2)}{\mathrm{Pe}u}\Psi - g\breve{\phi}^{\alpha-1},$$

with a fixed point at $(\breve{\phi}, \Psi, \Upsilon) = (0, 0, 0)$, corresponding to the cold-temperate boundary, approached as $\mathrm{sgn}(\tilde{u}) \times \zeta \to -\infty$ in the transformed coordinates. As in the corresponding dynamical system in section 3 for $2 < \alpha < 3$, we find that the Jacobian evaluated at the fixed point has two eigenvalues of the opposite sign to $\mathrm{sgn}\,\tilde{u}$, indicating that there is a one-parameter family of orbits into the fixed point. As in the corresponding case in section 3, there is a caveat in that the fixed point is on the boundary of the domain on which the dynamical system is defined, and the Jacobian is one-sided; however we still expect to be able to find a single orbit that matches the far field condition $\Psi \sim \alpha\kappa g a/(\mathrm{Pe}\tilde{u})\breve{\phi}^{2\alpha-4}$ as $\mathrm{sgn}(\tilde{u}) \times \zeta \to +\infty$ and connects to the fixed point. We compute an approximation to that orbit numerically using a multiple shooting method over a large but finite interval in ζ as in section 3, applying the far field condition at one end of the interval with large $\breve{\phi}$ (while still allowing $\breve{\phi}$ to become very small at the other end of the interval), and putting $\Upsilon/\breve{\phi} = 0$ at the other in order to ensure that the orbit lies on the stable manifold to a linear approximation in $\breve{\phi}$.

3 Steady state solution

Here we describe the multiple shooting method used to solve the one-dimensional temperate ice problem directly. There are two parts to this: the first is a nearly closed-form solution of the heat equation (1), and the second is the solution of the temperate ice problem (2), reformulated as a first-order system.

3.1 The heat equation

The steady state heat equation (1) is straightforward to solve through the use of integrating factors. For constant u, we have

$$T(z) = \frac{az}{\mathrm{Pe}u} + A\exp(\mathrm{Pe}uz) + B. \tag{22}$$

while for u given by (3) with $u_0 < 0$,

$$T(z) = -a\frac{2h}{\mathrm{Pe}|u_0|}F_+\left(\sqrt{\frac{\mathrm{Pe}|u_0|}{2h}}z\right) + A\exp\left(\frac{Pe\ |u_0|}{2h}z^2\right)D_+\left(\sqrt{\frac{\mathrm{Pe}|u_0|}{2h}}z\right) + B \tag{23}$$

where $D_\pm$ is Dawson's integral

$$D_\pm(x) = \exp(\mp x^2)\int_0^x \exp(\pm x'^2)\mathrm{d}x'$$

and F the anti-derivative of D,$F_\pm(x) = \int_0^x D_\pm(x')\mathrm{d}x'$. For the somewhat less realistic case of $u_0 > 0$, we obtain the analogous

$$T(z) = -a\frac{2h}{\mathrm{Pe}|u_0|}F_-\left(\sqrt{\frac{\mathrm{Pe}|u_0|}{2h}}z\right) + A\exp\left(\frac{Pe\ |u_0|}{2h}z^2\right)D_-\left(\sqrt{\frac{\mathrm{Pe}|u_0|}{2h}}z\right) + B \tag{24}$$

This reduces the heat flow problem (1) into the algebraic problem of finding the constants of integration A and B.

3.2 The shooting method for the temperate ice problem

A closed-form solution of the temperate ice problem (2) is not possible, but we can render it in a form suitable for a shooting method approach. Straightforward manipulations allow the one-dimensional problem (2) for temperate ice to be reformulated as a first order system

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} = \frac{a - \phi p_e}{\mathrm{Pe} u}, \tag{25a}$$

$$\frac{\mathrm{d}p_e}{\mathrm{d}z} = \delta^{-1}\left(\frac{q}{\kappa\phi^{\alpha}} - \frac{\mathrm{Pe}u}{\kappa\phi^{\alpha-1}} - g\right), \tag{25b}$$

$$\frac{\mathrm{d}q}{\mathrm{d}z} = a + \mathrm{Pe}\phi\frac{\mathrm{d}u}{\mathrm{d}z}. \tag{25c}$$

We apply a multiple shooting method, integrating (25) as an initial value problem across N partitions of the temperate domain, each of the form $(\min(z_t, z_{ct}) + \theta_i|z_t - z_{ct}|, \min(z_t, z_{ct}) + \theta_{i+1}|z_t - z_{ct}|)$ with either $0 = \theta_1 < \theta_1 < \ldots < \theta_{N+1} = 1$ or $1 = \theta_1 > \ldots > \theta_{N+1} = 0$. There is an exception to this when the cold-temperate boundary z_{ct} is an inflow boundary, and (25b) becomes singular due to the vanishing porosity at the boundary. In that case, we transform (25) in the partition closest to z_{ct} as described in section 3.3 below.

Except where that transformation is applied, we integrate over each partition, using initial conditions $\phi = \phi_i$, $p_e = p_{e,i}$, $q = q_i$ at θ_i and obtaining final values $\phi = \Phi_{i+1}$, $p_e = P_{e,i+1}$, $q = Q_{i+1}$ at θ_{i+1}, which are functions of ϕ_i, $p_{e,i}$, q_i and, through the length of the partition, of z_{ct}. The shooting method then enforces constraints on the ϕ_i, $p_{e,i}$, q_i and z_{ct} to ensure that all the relevant boundary conditions are satisfied at z_t and z_{ct}, along with continuity of the solution at interior partition end points in the form $\phi_i = \Phi_i$, $p_{e,i} = P_{e,i}$, $q_i = Q_i$ for $i = 2, \ldots, N$ if the equations (25) are solved on each partition of the temperate domain, or at one fewer interior partition end point (so $i = 3, \ldots, N$ or $i = 2, \ldots, N-1$) if the transformed version of the equations is solved on either the first or the last partition.

The implementation of boundary conditions at the boundaries of the temperate domain is straightforward, except in the transformed case. By way of example, assume that we have an outflow boundary at $z = z_{ct}$ with $z_{ct} < z_t$, and have chosen a partition with $\theta_1 = 0$, $\theta_{N+1} = 1$. Then θ_1 corresponds to the cold-temperate boundary z_{ct}, at which we impose (4e) through

$$q_1 = q_{ct}(A, B, z_{ct}), \qquad \mathrm{Pe}u(z_{ct})\phi_1 = q_{ct}(A, B, z_{ct})$$

where $q_{ct}(A, B, z_{ct})$ is defined as a function of z_{ct} and the constants of integration in the solutions for T through

$$q_{ct}(A, B, z_{ct}) = -\left.\frac{\mathrm{d}T}{\mathrm{d}z}\right|_{z=z_{ct}}$$

At the temperate exterior boundary, we impose (4d) and (4f) in the form

$$P_{e,N+1}(\phi_N, p_{e,N}, q_N, z_{ct}) = N_0, \qquad \Phi_{N+1}(\phi_N, p_{e,N}, q_N, z_{ct}) = \phi_0,$$

and the problem is closed by further constraining A,B and z_{ct} through the Dirichlet conditions on T, (4a) and (4b)$_1$,

$$T|_{z_c} = T_0, \qquad T|_{z=z_{ct}} = 0.$$

The problem is solved using Newton's method, with the asymptotic solution described above generally providing the initial guess. The computation of the Jacobian involves the derivative of Φ_{i+1}, $P_{e,i+1}$ and Q_{i+1} with respect to ϕ_i, $p_{e,i}$, q_i and z_{ct}; this is computed in the standard way by solving for that derivative using the same integration routine for ordinary differential equations that is also used to compute Φ_{i+1}, $P_{e,i+1}$ and Q_{i+1}. Other configurations with different orientations of the partitions of the temperate domain follow the same scheme as above, provided the cold-temperate boundary is an outflow boundary.

3.3 Change of variables near cold-temperate inflow boundaries

Inflow boundaries at z_{ct} are handled slightly differently. We can decouple the heat equation entirely from the temperate ice problem because the boundary conditions on T no longer couple to the flux q; this allows us to compute A, B and z_{ct} without reference to the other unknowns, and to pose the temperate ice equations (25) on a known domain.

Second, to avoid the difficulties associated with the singularity in (25b) at $\phi = 0$, we transform in the partition closest to z_{ct} to a new set of independent and dependent variables such that the boundary $z = z_{ct}$ becomes a fixed point of a dynamical system. As we already did in section 2.3 for the boundary layer that forms at a cold-temperate inflow boundary, we consider only the case of $\alpha \geq 2$. The transformation we apply depends on α: for $2 \leq \alpha \leq 3$, we define

$$\Psi = \phi^{\alpha-2} p_e, \qquad \chi = (z - z_{ct})/\phi, \qquad \omega = (q - a(z - z_{ct}))/\phi, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}n} = 1/\phi, \tag{26}$$

which leads to the system

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \frac{a}{\mathrm{Pe}u}\phi - \frac{1}{\mathrm{Pe}u}\phi^{4-\alpha}\Psi, \tag{27a}$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\zeta} = \frac{a}{\kappa\delta}\chi + \frac{1}{\kappa\delta}\omega - \frac{\mathrm{Pe}u}{\kappa\delta} - \frac{g}{\delta}\phi^{\alpha-1} - \frac{(\alpha-2)}{\mathrm{Pe}u}\phi^{3-\alpha}\Psi^2 + \frac{(\alpha-2)a}{\mathrm{Pe}u}\Psi, \tag{27b}$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}\zeta} = 1 - \frac{a}{\mathrm{Pe}u}\chi + \frac{1}{\mathrm{Pe}u}\phi^{3-\alpha}\chi\Psi, \tag{27c}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}\zeta} = \mathrm{Pe}u'\phi - \frac{a}{\mathrm{Pe}u}\omega + \frac{1}{\mathrm{Pe}u}\phi^{3-\alpha}\Psi\omega. \tag{27d}$$

where $u' = \mathrm{d}u/\mathrm{d}z$, and u must be treated as a function of $z - z_{ct} = \phi\chi$, specifically $u = u(z_{ct}) + u'\phi\chi$.

For $\alpha > 3$, we use instead

$$\Psi = \phi^{(\alpha-1)/2} p_e, \qquad \chi = (z - z_{ct})/\phi, \qquad \omega = (q - a(z - z_{ct}))/\phi, \qquad \frac{\mathrm{d}\zeta}{\mathrm{d}n} = 1/\phi^{(\alpha-1)/2}. \tag{28}$$

Then

$$\frac{\mathrm{d}\phi}{\mathrm{d}\zeta} = \frac{a}{\mathrm{Pe}u}\phi^{(\alpha-1)/2} - \frac{1}{\mathrm{Pe}u}\phi\Psi, \tag{29a}$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\zeta} = \frac{a}{\kappa\delta}\chi + \frac{1}{\kappa\delta}\omega - \frac{\mathrm{Pe}u}{\kappa\delta} - \frac{g}{\delta}\phi^{\alpha-1} - \frac{\alpha-1}{2\mathrm{Pe}u}\Psi^2 + \frac{(\alpha-1)a}{2\mathrm{Pe}u}\phi^{(\alpha-3)/2}\Psi, \tag{29b}$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}\zeta} = \phi^{(\alpha-3)/2}\left(1 - \frac{a}{\mathrm{Pe}u}\chi\right) + \frac{1}{\mathrm{Pe}u}\chi\Psi \tag{29c}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}\zeta} = \mathrm{Pe}u'\phi^{(\alpha-1)/2} - \frac{a}{\mathrm{Pe}u}\phi^{(\alpha-3)/2}\omega + \frac{1}{\mathrm{Pe}u}\Psi\omega. \tag{29d}$$

with u a function of $\phi\chi$ as before. Note that for $\alpha = 3$, the two dynamical systems (27) and (29) are identical.

The transformed equations appear far more complicated than the original system (25). The point here is that the boundary $z = z_{ct}$ at which $\phi = 0$ corresponds to a fixed point of the dynamical system reached as $\mathrm{sgn}(u(z_{ct})) \times \zeta \to -\infty$, and we can investigate the local behaviour of orbits into the fixed point by constructing the invariant manifolds connected to the fixed points, which allows solutions to be computed numerically.

Still restricting ourselves to $\alpha \geq 2$, there are three distinct cases to consider, $\alpha = 2$, $2 < \alpha \leq 3$, and $\alpha > 3$. In each case, it is important that we have a degree of freedom in choosing the orbit into the fixed point (meaning, we would like a one-parameter family of orbits into the fixed point if the fixed point is unique, or a one-parameter family of fixed points, each with a unique orbit into it). Otherwise, if there were a unique orbit into a unique fixed point, there would be a unique relationship between the dependent variables $(\phi, \Psi, \chi, \omega)$, which would in turn imply a unique relationship between ϕ, p_e, q and z and therefore no scope for satisfying a boundary condition on p_e at the exterior boundary z_t as required by the formulation of the steady state problem. Conversely, if we have more than a single degree of freedom in choosing the orbit into the fixed point — say, a two-parameter family of orbits into a unique fixed point — the problem will be underdetermined, requiring more than the single boundary condition at the exterior boundary to furnish a unique solution. We are unable to give a complete analysis of the problem for all $\alpha \geq 2$ (in particular, for non-integer α), but sketch a number of relevant observations below, indicating that we are likely to have a single degree of freedom in choosing the orbit into the fixed point in each case.

For $\alpha = 2$, the fixed point is at $(\phi, \Psi, \chi, \omega) = (0, \Psi_0, \mathrm{Pe}u/a, 0)$ with Ψ_0 arbitrary. The fixed point has a one-dimensional centre manifold (which simply reflects the fact that there is in fact a one-parameter family of fixed points, so there is no motion at all along the centre manifold), a one-dimensional stable manifold (stable in the sense of the fixed point being an attractor in the limit $\mathrm{sgn}(u(z_{ct})) \times \zeta \to -\infty$) and a two-dimensional unstable manifold. The fixed point must therefore be approached along the stable manifold, which is unique, but the fact that the fixed point is not unique gives us the required degree of freedom.

For $2 < \alpha \leq 3$, the fixed point is at $(\phi, \Psi, \chi, \omega) = (0, 0, \mathrm{Pe}u/a, 0)$ and appears to have a two-dimensional unstable manifold as well as a two-dimensional stable manifold, giving us the required degree of freedom. We make this statement based on the eigenvalues of the Jacobian of the dynamical system, which masks several complications. For non-integer α, the dynamical system cannot be extended smoothly to $\phi < 0$, so the fixed point intrinsically sits on the boundary of the domain of definition of the dynamical system. Moreover, while we can still somewhat naïvely calculate a one-sided Jacobian, defined as a matrix of partial derivatives computed at the fixed point, these partial derivatives are not in fact continuous near the fixed point if $2 < \alpha < 3$. As a consequence, standard results about the existence and uniqueness of invariant manifolds do not necessarily apply. Still, taking a naïve linearization about the fixed point at face value, we find a one-parameter family of orbits into the fixed point from within the domain of definition in the linearized system; for $\alpha = 3$, the dynamical system (27) can be extended to $\phi = 0$ and is infinitely differentiable at the fixed point, so we can be certain of a two-dimensional unstable and stable manifold, and can therefore definitively establish the existence of a one-parameter family of orbits into the fixed point.

For $\alpha > 3$, the fixed point remains at $(\phi, \Psi, \chi, \omega) = (0, 0, \mathrm{Pe}u/a, 0)$ and the Jacobian of the dynamical system at the fixed point (which is again one-sided if α is not an odd integer, and the partial derivatives making up the Jacobian are again not continuous if $\alpha < 5$) has a negative, a positive and two zero eigenvalues. Matters again become complicated at this stage, and we only sketch a few observations. Requiring that z must not lie on the wrong side of z_{ct} (which constrains the sign of χ) can actually be used to show that the stable manifold is not physically viable, and we expect the physically relevant orbit to approach the fixed point along the centre manifold. Although the centre manifold is two-dimensional, the fixed point is a degenerate saddle on the centre manifold, suggesting a single orbit into the fixed point for a given centre manifold. A one-parameter family of orbits into the fixed point is instead likely to arise because the centre manifold can be non-unique when there is motion on the centre manifold towards the fixed point, and there is simultaneously a stable manifold (essentially, exponentially small terms near the fixed point cause different centre manifolds to diverge away from the fixed point, see Wiggins, 2003). This could potentially be made more rigorous in the case where α is an odd integer. The dynamical system can then be continued smoothly to negative values of ϕ and standard results on the uniqueness of centre manifolds apply (Carr, 1981; Sijbrand, 1985), but we do not pursue the possibility further here.

From the computational perspective, the important aspect of the local behaviour of the dynamical systems (27) and (29) near their fixed points is that in all parameter regimes with $\alpha \geq 2$, the local behaviour of the variables χ and ω on orbits into the fixed point must be of the form

$$\omega \sim \frac{Pe^2 uu'}{2a}\phi + o(\phi), \qquad \chi = \frac{\mathrm{Pe}u}{a} + \frac{Pe^2 uu'}{2a^2}\phi + o(\phi),$$

Moreover, ω and χ departing from these local forms correspond to motion away from the fixed point (either in the sense of motion away from the fixed point along a centre manifold, or of the growth of an unstable mode in the linearized version of the dynamical systems). We can therefore use the local forms above to constrain approximate orbits that approach the fixed point.

Specifically, we pick a large interval of values of ζ over which we integrate (27) and (29). For instance, for (27), we expect ϕ to decay exponentially in ζ with an e-folding length $\mathrm{Pe}u/a$,. We therefore make the interval much larger than that e-folding length and confirm numerically that the result is insensitive to the exact choice of the length of the interval of integration.

Over that integration interval, we apply the same basic multiple shooting approach as above, splitting the interval into partitions and ensuring continuity at the end points of those partitions. At the end point of the whole interval that corresponds to the smallest value of $\mathrm{sgn}(u)\zeta$, we enforce the conditions

$$\frac{\omega - Pe^2 uu'\phi/(2a)}{\phi} = \frac{\chi - \mathrm{Pe}u/a - Pe^2 uu'\phi/(2a^2)}{\phi} = 0.$$

At the other end of the ζ-interval, we enforce continuity with the the solution to the untransformed system (25) on the adjacent partition of the temperate subdomain (that is, the interval between θ_2 and θ_3 or the interval between θ_{N-1} and θ_N in the notation used before), using the definitions (26) or (28) to relate the two sets of variables. This allows us to formulate the temperate ice equations as a root-finding problem, which is again solved using Newton's method.

4 Local behaviour near a cold-temperate inflow boundary

The discussion above has indicated that there is a variety of possible local behaviours near cold-temperate inflow boundaries. Although the details of the near-boundary behaviour can in principle be deduced from the transformed dynamical systems (27) and (29), it is in fact much easier to comprehend the relevant behaviour if we simply look for power-law type local solutions.

Let $n = \mathrm{sgn}(u) \times (z - z_{ct})$ be distance into the temperate domain from the cold-temperate boundary, and similarly let $u_n = |u|$ and $g_n = \mathrm{sgn}(u) \times g$ be the inward-pointing 'components' of velocity and gravity at the boundary. We can look for approximate solutions solutions such that $\phi \sim c_1 n^\beta$ and $p_e \sim p_0 + c_2 n^\gamma$. It is then useful to re-write the temperate ice problem (2) in the alternative form

$$\mathrm{Pe}u_n\frac{\mathrm{d}\phi}{\mathrm{d}n} + \phi N = a, \tag{30a}$$

$$\kappa\frac{\mathrm{d}}{\mathrm{d}n}\left[\phi^\alpha g_n + \delta\phi^\alpha\frac{\mathrm{d}p_e}{\mathrm{d}n}\right] - \phi N = 0, \tag{30b}$$

subject to $\phi \to 0$, $\phi^\alpha \mathrm{d}p_e/\mathrm{d}n \to 0$ as $n \to 0$. These boundary conditions require $\beta > 0$ and $\alpha\beta + \gamma - 1 > 0$, or $\gamma > 1 - \alpha\beta$. We also have the constraint that $\phi \geq 0$ in the temeprate region, which here requires that $c_1 > 0$. Substituting the assumed local power laws, we obtain

$$\beta\mathrm{Pe}u_n c_1 z^{\beta-1} + c_1 p_0 z^\beta + c_1 c_2 z^{\beta+\gamma} \sim a \tag{31a}$$

$$\alpha\beta\kappa g_n c_1^\alpha z^{\alpha\beta-1} + \delta\kappa c_1^\alpha c_2\gamma(\alpha\beta+\gamma-1)z^{\alpha\beta+\gamma-2} - c_1 p_0 z^\beta - c_1 c_2 z^{\beta+\gamma} \sim 0 \tag{31b}$$

The next task is to identify all possible ways of balancing leading order terms (through the choice of β and γ) that also allow us to solve for c_1 and c_2, with c_1 positive. This leads to the following possible local forms, depending on the value of α: We always find linear growth of porosity ($\beta = 1$) near the boundary due to melting as

$$\phi \sim \frac{a}{\mathrm{Pe}u_n}n,$$

while the behaviour of compaction pressure must take one of the following forms, valid only for permeability exponents α satisfying the inequalities or equalities indicated

$$p_e \sim \frac{2\kappa g_n a}{\mathrm{Pe}u_n} + o(n), \qquad \alpha = 2, \tag{32a}$$

$$p_e \sim p_0 - \left(\frac{g_n}{\delta} - \frac{p_0\mathrm{Pe}u_n}{2a}\right)n, \qquad \alpha = 2, \tag{32b}$$

$$p_e \sim p_0 + \frac{p_0(\mathrm{Pe}u_n)^{\alpha-1}}{2a^{\alpha-1}\delta\kappa(3-\alpha)}n^{3-\alpha}, \qquad 2 < \alpha < 3, \tag{32c}$$

$$p_e \sim -\frac{g_n}{\delta}n, \qquad 2 < \alpha < 3, \tag{32d}$$

$$p_e \sim \frac{3\kappa g_n a^2}{(\mathrm{Pe}u_n)^2 + 3\delta\kappa a^2}n, \qquad \alpha = 3, \tag{32e}$$

$$p_e \sim \frac{\alpha\kappa g_n a^{\alpha-1}}{(\mathrm{Pe}u_n)^{\alpha-1}}n^{\alpha-2}, \qquad \alpha > 3, \tag{32f}$$

where p_0 is arbitrary. For values of α that permit more than one of the forms above, the near-wall behaviour cannot be determined by a local analysis alone. All of the local behaviours above are equivalent to orbits into the relevant fixed points of the dynamical systems (27) and (29), but demonstrating this is a lengthy and ultimately not very illuminating task. In the limit of small δ, we also find that the power law behaviour above is consistent with the near-boundary behaviour deduced in section 2.3 for the exponents α considered there.

References

CARR, J. 1981 *Applications of centre manifold theory*. New York: Springer.

HOLMES, M.H. 1995 *Introduction to Perturbation Methods*, *Texts in Applied Mathematics*, vol. 20. New York: Springer-Verlag.

ROBIN, G. Q. 1955 Ice movement and temperature distribution in glaciers and ice sheets. *J. Glaciol.* **2**, 523–532.

SIJBRAND, J. 1985 Properties of center manifolds. *Trans. Amer. Math. Soc.* **289** (2), 431–469.

WIGGINS, S. 2003 *Introduction to Applied Nonlinear Dynamical Systems*. New York: Springer.