

Supplementary Material on

“Uniform electric-field-induced lateral migration of a sedimenting drop”

Aditya Bandopadhyay¹, Shubhadeep Mandal², N. K. Kishore³ and Suman Chakraborty^{1,2†}

¹Advanced Technology Development Center, Indian Institute of Technology Kharagpur, West Bengal- 721302, India

²Department of Mechanical Engineering, Indian Institute of Technology Kharagpur, West Bengal- 721302, India

³Department of Electrical Engineering, Indian Institute of Technology Kharagpur, West Bengal- 721302

A. Determination of unknown solid harmonics present in $O(Re_E)$ velocity and pressure fields

The velocity and pressure fields at $O(Re_E)$ are represented in terms of solid spherical harmonics $p^{(Re_E)}$, $\Phi^{(Re_E)}$ and $\chi^{(Re_E)}$ which contain the coefficients $A^{(Re_E)}$, $B^{(Re_E)}$ and $C^{(Re_E)}$. These unknown coefficients can be determined by applying the boundary conditions given in equation (2.15). One thing to note here is that it is difficult to deal with the present form of the boundary conditions because some of the boundary conditions contain the derivative of solid spherical harmonics which makes the use of orthogonality property of surface harmonics involved. This complexity can be avoided by presenting the boundary conditions in the following form (Haber & Hetsroni 1971; Happel & Brenner 1981; Brenner 1964)

$$\left. \begin{array}{l} \mathbf{u}_i^{(Re_E)} \Big|_{r=1} \cdot \mathbf{e}_r = 0, \\ \mathbf{u}_e^{(Re_E)} \Big|_{r=1} \cdot \mathbf{e}_r = 0, \\ \nabla \cdot \mathbf{u}_i^{(Re_E)} \Big|_{r=1} = \nabla \cdot \mathbf{u}_e^{(Re_E)} \Big|_{r=1}, \\ \mathbf{e}_r \cdot \nabla \times \mathbf{u}_i^{(Re_E)} \Big|_{r=1} = \mathbf{e}_r \cdot \nabla \times \mathbf{u}_e^{(Re_E)} \Big|_{r=1}, \\ \mathbf{e}_r \cdot \nabla \times \left\{ \mathbf{e}_r \times \left(\mathbf{T}_i^{H(Re_E)} + M \mathbf{T}_i^{E(Re_E)} \right) \Big|_{r=1} \right\} = \mathbf{e}_r \cdot \nabla \times \left\{ \mathbf{e}_r \times \left(\mathbf{T}_e^{H(Re_E)} + M \mathbf{T}_e^{E(Re_E)} \right) \Big|_{r=1} \right\}, \\ \mathbf{e}_r \cdot \nabla \times \left(\mathbf{T}_i^{H(Re_E)} + M \mathbf{T}_i^{E(Re_E)} \right) \Big|_{r=1} = \mathbf{e}_r \cdot \nabla \times \left(\mathbf{T}_e^{H(Re_E)} + M \mathbf{T}_e^{E(Re_E)} \right) \Big|_{r=1}, \end{array} \right\} \quad (\text{A1})$$

where $[\xi]_{r=1}$ represents the evaluation of a generic variable, ξ , at the interface of the spherical drop ($r=1$). In equation (A1), the term $\mathbf{T}_{i,e}^{H(Re_E)}$ is the hydrodynamic traction vector at $O(Re_E)$ of the following form (Leal 2007)

$$\begin{aligned}\mathbf{T}_{i,e}^{H(Re_E)} &= \lambda \left[-p_i^{(Re_E)} \mathbf{I} + \nabla \mathbf{u}_i^{(Re_E)} + (\nabla \mathbf{u}_i^{(Re_E)})^T \right] \cdot \mathbf{e}_r, \\ \mathbf{T}_e^{H(Re_E)} &= \left[-p_e^{(Re_E)} \mathbf{I} + \nabla \mathbf{u}_e^{(Re_E)} + (\nabla \mathbf{u}_e^{(Re_E)})^T \right] \cdot \mathbf{e}_r,\end{aligned}\quad (\text{A2})$$

whereas $\mathbf{T}_{i,e}^{E(Re_E)}$ represents the electrical traction vector at $O(Re_E)$ of the following form (represented in terms of components in spherical coordinates) (Xu & Homsy 2006)

$$\begin{aligned}\mathbf{T}_i^{E(Re_E)} &= S \begin{bmatrix} E_{i,r}^{(0)} E_{i,r}^{(Re_E)} - \frac{1}{2} (2E_{i,r}^{(0)} E_{i,r}^{(Re_E)} + 2E_{i,\theta}^{(0)} E_{i,\theta}^{(Re_E)} + 2E_{i,\phi}^{(0)} E_{i,\phi}^{(Re_E)}) \\ (E_{i,r}^{(0)} E_{i,\theta}^{(Re_E)} + E_{i,r}^{(Re_E)} E_{i,\theta}^{(0)}) \\ (E_{i,r}^{(0)} E_{i,\phi}^{(Re_E)} + E_{i,r}^{(Re_E)} E_{i,\phi}^{(0)}) \end{bmatrix}, \\ \mathbf{T}_e^{E(Re_E)} &= \begin{bmatrix} E_{e,r}^{(0)} E_{e,r}^{(Re_E)} - \frac{1}{2} (2E_{e,r}^{(0)} E_{e,r}^{(Re_E)} + 2E_{e,\theta}^{(0)} E_{e,\theta}^{(Re_E)} + 2E_{e,\phi}^{(0)} E_{e,\phi}^{(Re_E)}) \\ (E_{e,r}^{(0)} E_{e,\theta}^{(Re_E)} + E_{e,r}^{(Re_E)} E_{e,\theta}^{(0)}) \\ (E_{e,r}^{(0)} E_{e,\phi}^{(1)} + E_{e,r}^{(1)} E_{e,\phi}^{(0)}) \end{bmatrix}.\end{aligned}\quad (\text{A3})$$

In the above representation, we have decomposed the electric field, \mathbf{E} , in three orthogonal components using spherical coordinates as $\mathbf{E} = E_r \mathbf{e}_r + E_\theta \mathbf{e}_\theta + E_\phi \mathbf{e}_\phi$. To apply the boundary conditions given in equation (A1), at first one has to express $\mathbf{u}_i^{(Re_E)} \Big|_{r=1} \cdot \mathbf{e}_r$, $\mathbf{u}_e^{(Re_E)} \Big|_{r=1} \cdot \mathbf{e}_r$, $\nabla \cdot \mathbf{u}_i^{(Re_E)} \Big|_{r=1}$, $\nabla \cdot \mathbf{u}_e^{(Re_E)} \Big|_{r=1}$, $\mathbf{e}_r \cdot \nabla \times \mathbf{u}_i^{(Re_E)} \Big|_{r=1}$ and $\mathbf{e}_r \cdot \nabla \times \mathbf{u}_e^{(Re_E)} \Big|_{r=1}$ in terms of solid spherical harmonics in the following form (Happel & Brenner 1981; Hetsroni & Haber 1970)

$$\begin{aligned}\mathbf{u}_i^{(Re_E)} \Big|_{r=1} \cdot \mathbf{e}_r &= \left[\mathbf{u}_i^{(Re_E)} \cdot \mathbf{e}_r \right] \Big|_{r=1} = \sum_{n=1}^{\infty} \left[\frac{n}{2\lambda(2n+3)} p_n^{(Re_E)} + n \Phi_n^{(Re_E)} \right], \\ \mathbf{u}_e^{(Re_E)} \Big|_{r=1} \cdot \mathbf{e}_r &= \left[\mathbf{u}_e^{(Re_E)} \cdot \mathbf{e}_r \right] \Big|_{r=1} = \sum_{n=1}^{\infty} \left[\frac{n+1}{2(2n-3)} p_{-n-1}^{(Re_E)} - (n+1) \Phi_{-n-1}^{(Re_E)} \right], \\ \nabla \cdot \mathbf{u}_i^{(Re_E)} \Big|_{r=1} &= - \left[\frac{\partial}{\partial r} (\mathbf{u}_i^{(Re_E)} \cdot \mathbf{e}_r) \right] \Big|_{r=1} = - \sum_{n=1}^{\infty} \left[\frac{n(n+1)}{2\lambda(2n+3)} p_n^{(Re_E)} + n(n-1) \Phi_n^{(Re_E)} \right], \\ \nabla \cdot \mathbf{u}_e^{(Re_E)} \Big|_{r=1} &= - \left[\frac{\partial}{\partial r} (\mathbf{u}_e^{(Re_E)} \cdot \mathbf{e}_r) \right] \Big|_{r=1} = \sum_{n=1}^{\infty} \left[\frac{n(n+1)}{2(2n-1)} p_{-n-1}^{(Re_E)} - (n+1)(n+2) \Phi_{-n-1}^{(Re_E)} \right], \\ \mathbf{e}_r \cdot \nabla \times \mathbf{u}_i^{(Re_E)} \Big|_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \mathbf{u}_i^{(Re_E)} \right] \Big|_{r=1} = \sum_{n=1}^{\infty} n(n+1) \chi_n^{(Re_E)}, \\ \mathbf{e}_r \cdot \nabla \times \mathbf{u}_e^{(Re_E)} \Big|_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \mathbf{u}_e^{(Re_E)} \right] \Big|_{r=1} = \sum_{n=1}^{\infty} n(n+1) \chi_{-n-1}^{(Re_E)}.\end{aligned}\quad (\text{A4})$$

To satisfy the shear stress balance, we have to obtain $\mathbf{e}_r \cdot \nabla \times \left\{ \mathbf{e}_r \times \left(\mathbf{T}^{H(Re_E)} + M \mathbf{T}^{E(Re_E)} \right) \right\}_{r=1}$ and $\mathbf{e}_r \cdot \nabla \times \left(\mathbf{T}^{H(Re_E)} + M \mathbf{T}^{E(Re_E)} \right)_{r=1}$. Among these terms the hydrodynamic stress components can be obtained as (Hetsroni & Haber 1970)

$$\left. \begin{aligned} \mathbf{e}_r \cdot \nabla \times \left\{ \mathbf{e}_r \times \mathbf{T}_i^{H(Re_E)} \right\}_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \left\{ \mathbf{e}_r \times \mathbf{T}_i^{H(Re_E)} \right\} \right]_{r=1} = -\lambda \sum_{n=1}^{\infty} \left[\frac{2(n-1)n(n+1)\Phi_n^{(Re_E)}}{n^2(n+2)} p_n^{(Re_E)} \right], \\ \mathbf{e}_r \cdot \nabla \times \left\{ \mathbf{r} \times \mathbf{T}_e^{H(Re_E)} \right\}_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \left\{ \mathbf{r} \times \mathbf{T}_e^{H(Re_E)} \right\} \right]_{r=1} = -\sum_{n=1}^{\infty} \left[\frac{(n+1)^2(n-1)}{(2n-1)} p_{-n-1}^{(Re_E)} \right], \\ \mathbf{e}_r \cdot \nabla \times \mathbf{T}_i^{H(Re_E)} \Big|_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \mathbf{T}_i^{H(Re_E)} \right]_{r=1} = \lambda \sum_{n=1}^{\infty} (n-1)n(n+1)\chi_n^{(Re_E)}, \\ \mathbf{e}_r \cdot \nabla \times \mathbf{T}_e^{H(Re_E)} \Big|_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \mathbf{T}_e^{H(Re_E)} \right]_{r=1} = -\sum_{n=1}^{\infty} (n+2)n(n+1)\chi_{-n-1}^{(Re_E)}. \end{aligned} \right\} \quad (\text{A5})$$

The electrical stress terms are also represented in terms of surface harmonics as

$$\left. \begin{aligned} \mathbf{e}_r \cdot \nabla \times \left(\mathbf{e}_r \times \mathbf{T}_i^{E(Re_E)} \right)_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \left(\mathbf{e}_r \times \mathbf{T}_i^{E(Re_E)} \right) \right]_{r=1} = \sum_{n=0}^{\infty} \left[g_{n,m}^{i(Re_E)} \cos(m\phi) + \hat{g}_{n,m}^{i(Re_E)} \sin(m\phi) \right] P_{n,m}, \\ \mathbf{e}_r \cdot \nabla \times \left(\mathbf{r} \times \mathbf{T}_e^{E(Re_E)} \right)_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \left(\mathbf{r} \times \mathbf{T}_e^{E(Re_E)} \right) \right]_{r=1} = \sum_{n=0}^{\infty} \left[g_{n,m}^{e(Re_E)} \cos(m\phi) + \hat{g}_{n,m}^{e(Re_E)} \sin(m\phi) \right] P_{n,m}, \\ \mathbf{e}_r \cdot \nabla \times \mathbf{T}_i^{E(Re_E)} \Big|_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \mathbf{T}_i^{E(Re_E)} \right]_{r=1} = \sum_{n=0}^{\infty} \left[h_{n,m}^{e(Re_E)} \cos(m\phi) + \hat{h}_{n,m}^{e(Re_E)} \sin(m\phi) \right] P_{n,m}, \\ \mathbf{e}_r \cdot \nabla \times \mathbf{T}_e^{E(Re_E)} \Big|_{r=1} &= \left[\mathbf{e}_r \cdot \nabla \times \mathbf{T}_e^{E(Re_E)} \right]_{r=1} = \sum_{n=0}^{\infty} \left[h_{n,m}^{e(Re_E)} \cos(m\phi) + \hat{h}_{n,m}^{e(Re_E)} \sin(m\phi) \right] P_{n,m}, \end{aligned} \right\} \quad (\text{A6})$$

where $g_{n,m}^{i(Re_E)}$, $\hat{g}_{n,m}^{i(Re_E)}$, $h_{n,m}^{i(Re_E)}$, $\hat{h}_{n,m}^{i(Re_E)}$, $g_{n,m}^{e(Re_E)}$, $\hat{g}_{n,m}^{e(Re_E)}$, $h_{n,m}^{e(Re_E)}$ and $\hat{h}_{n,m}^{e(Re_E)}$ are the known coefficients after solving $O(Re_E)$ electric potential distribution.

Now, substituting equations (A4)-(A6) in equation (A1), we obtain the unknown coefficients of the solid spherical harmonics at $O(Re_E)$ in following form

$$\left. \begin{aligned} A_{n,m}^{(Re_E)} &= \frac{(2n+3)}{n(2n+1)(\lambda+1)} \left[(2n+1)(2n-1) \beta_{n,m}^{(Re_E)} + M \left(g_{n,m}^{i(Re_E)} - g_{n,m}^{e(Re_E)} \right) \right], \\ B_{n,m}^{(Re_E)} &= -\frac{A_{n,m}^{(Re_E)}}{2(2n+3)}, \\ C_{n,m}^{(Re_E)} &= \frac{M \left(h_{n,m}^{e(Re_E)} - h_{n,m}^{i(Re_E)} \right)}{n(n+1)(\lambda(n-1)+(n+2))}, \end{aligned} \right\} \quad (\text{A7})$$

$$\left. \begin{aligned} A_{-n-1,m}^{(Re_E)} &= -\frac{\left[(4n^2-1)\{(2n+1)\lambda+2\}\beta_{n,m}^{(Re_E)} + M(1-2n)\left(g_{n,m}^{i(Re_E)} - g_{n,m}^{e(Re_E)}\right) \right]}{(n+1)(2n+1)(\lambda+1)}, \\ B_{-n-1,m}^{(Re_E)} &= -\frac{(4n^2-1)\lambda\beta_{n,m}^{(Re_E)} + M\left(g_{n,m}^{e(Re_E)} - g_{n,m}^{i(Re_E)}\right)}{2(2n+1)(n+1)(\lambda+1)}, \\ C_{-n-1,m}^{(Re_E)} &= C_{n,m}^{(Re_E)}. \end{aligned} \right\} \quad (\text{A8})$$

Similarly, the coefficients $\hat{A}_{n,m}^{(Re_E)}$, $\hat{B}_{n,m}^{(Re_E)}$, $\hat{C}_{n,m}^{(Re_E)}$, $\hat{A}_{-n-1,m}^{(Re_E)}$, $\hat{B}_{-n-1,m}^{(Re_E)}$ and $\hat{C}_{-n-1,m}^{(Re_E)}$ can be obtained by replacing $\beta_{n,m}^{(Re_E)}$, $g_{n,m}^{i(Re_E)}$, $h_{n,m}^{i(Re_E)}$, $g_{n,m}^{e(Re_E)}$ and $h_{n,m}^{e(Re_E)}$ by $\hat{\beta}_{n,m}^{(Re_E)}$, $\hat{g}_{n,m}^{i(Re_E)}$, $\hat{h}_{n,m}^{i(Re_E)}$, $\hat{g}_{n,m}^{e(Re_E)}$ and $\hat{h}_{n,m}^{e(Re_E)}$, respectively in equations (A7)-(A8). Equations (A7)-(A8) contain $\beta_{n,m}^{(Re_E)}$ and $\hat{\beta}_{n,m}^{(Re_E)}$ of the following form

$$\left. \begin{aligned} \beta_{n,m}^{(Re_E)} &= \begin{cases} \beta_{1,0}^{(Re_E)} = -U_{dz}^{(Re_E)}, \beta_{1,1}^{(Re_E)} = -U_{dx}^{(Re_E)} \\ 0 \quad \forall n \geq 2 \end{cases} \\ \hat{\beta}_{n,m}^{(Re_E)} &= \begin{cases} \hat{\beta}_{1,1}^{(Re_E)} = -U_{dy}^{(Re_E)} \\ 0 \quad \forall n \geq 2. \end{cases} \end{aligned} \right\} \quad (\text{A8})$$

B. Determination of unknown solid harmonics present in $O(Ca)$ velocity and pressure fields

By following a similar method as outlined for the $O(Re_E)$ flow field, the $O(Ca)$ velocity and pressure fields can be determined. Towards this, we first present the boundary conditions in the following form (Haber & Hetsroni 1971; Brenner 1964)

$$\left. \begin{aligned} & \left[\mathbf{u}_i \Big|_{r=1+Caf^{(Ca)}} \cdot \mathbf{n} \right]^{(Ca)} = 0, \\ & \left[\mathbf{u}_e \Big|_{r=1+Caf^{(Ca)}} \cdot \mathbf{n} \right]^{(Ca)} = 0, \\ & \left[\nabla \mathbf{u}_i \Big|_{r=1+Caf^{(Ca)}} \right]^{(Ca)} = \left[\nabla \mathbf{u}_e \Big|_{r=1+Caf^{(Ca)}} \right]^{(Ca)}, \\ & \left[\mathbf{n} \cdot \nabla \times \mathbf{u}_i \Big|_{r=1+Caf^{(Ca)}} \right]^{(Ca)} = \left[\mathbf{n} \cdot \nabla \times \mathbf{u}_e \Big|_{r=1+Caf^{(Ca)}} \right]^{(Ca)}, \\ & \left[\mathbf{n} \cdot \nabla \times \left\{ \mathbf{n} \times (\mathbf{T}_i^H + M\mathbf{T}_i^E) \Big|_{r=1+Caf^{(Ca)}} \right\} \right]^{(Ca)} = \left[\mathbf{n} \cdot \nabla \times \left\{ \mathbf{n} \times (\mathbf{T}_e^H + M\mathbf{T}_e^E) \Big|_{r=1+Caf^{(Ca)}} \right\} \right]^{(Ca)}, \\ & \left[\mathbf{n} \cdot \nabla \times (\mathbf{T}_i^H + M\mathbf{T}_i^E) \Big|_{r=1+Caf^{(Ca)}} \right]^{(Ca)} = \left[\mathbf{n} \cdot \nabla \times (\mathbf{T}_e^H + M\mathbf{T}_e^E) \Big|_{r=1+Caf^{(Ca)}} \right]^{(Ca)}, \end{aligned} \right\} \quad (B1)$$

where \mathbf{n} is the unit normal given in equation (2.6). In the above representation $\left[\xi \Big|_{r=1+Caf^{(Ca)}} \right]^{(Ca)}$ is used to denote the $O(Ca)$ contribution of the quantity ξ evaluated at $r=1+Caf^{(Ca)}$, where $f^{(Ca)}$ is the correction in drop shape obtained from leading order solution. An important thing to note here is that all the boundary conditions are to be evaluated at the deformed drop interface, which makes the implementation of boundary conditions difficult. The standard method is to use Taylor series expansion about $r=1$ and collect the terms of $O(Ca)$ in the following form (Haber & Hetsroni 1971; Ajayi 1978; Xu & Homsy 2006)

$$\left. \begin{aligned} \xi \Big|_{r=1+Caf^{(Ca)}} &= \xi^{(0)} \Big|_{r=1} + Re_E \xi^{(Re_E)} \Big|_{r=1} + Ca \left(\xi^{(Ca)} \Big|_{r=1} + f^{(Ca)} \frac{\partial \xi^{(0)}}{\partial r} \Big|_{r=1} \right) + \dots, \\ \Rightarrow \left[\xi \Big|_{r=1+Caf^{(Ca)}} \right]^{(Ca)} &= \left(\xi^{(Ca)} \Big|_{r=1} + f^{(Ca)} \frac{\partial \xi^{(0)}}{\partial r} \Big|_{r=1} \right). \end{aligned} \right\} \quad (B2)$$

Following this method we obtain the $O(Ca)$ boundary conditions in the following form (Haber & Hetsroni 1971; Brenner 1964)

$$\left. \begin{aligned} & \left[\mathbf{u}_i^{(Ca)} \cdot \mathbf{e}_r \right]_{r=1} = T_1, \\ & \left[\mathbf{u}_e^{(Ca)} \cdot \mathbf{e}_r \right]_{r=1} = T_2, \\ & \left[\nabla \cdot (\mathbf{u}_i^{(Ca)} - \mathbf{u}_e^{(Ca)}) \right]_{r=1} = T_3, \\ & \left[\mathbf{e}_r \cdot \nabla \times (\mathbf{u}_e^{(Ca)} - \mathbf{u}_i^{(Ca)}) \right]_{r=1} = T_5, \\ & \left[\mathbf{e}_r \cdot \nabla \times \left\{ (\mathbf{T}_e^{H(Ca)} + M\mathbf{T}_e^{E(Ca)}) - (\mathbf{T}_i^{H(Ca)} + M\mathbf{T}_i^{E(Ca)}) \right\} \right]_{r=1} = T_4, \\ & \left[\mathbf{e}_r \cdot \nabla \times \left\{ (\mathbf{T}_e^{H(Ca)} + M\mathbf{T}_e^{E(Ca)}) - (\mathbf{T}_i^{H(Ca)} + M\mathbf{T}_i^{E(Ca)}) \right\} \right]_{r=1} = T_6, \end{aligned} \right\} \quad (B3)$$

where the terms $T_1 - T_6$ are known from the leading order solution in the following form

$$\left. \begin{aligned} T_1 &= \sum_{n=1}^4 \left(T_{1,n,m}^{(Ca)} \cos m\phi + \hat{T}_{1,n,m}^{(Ca)} \sin m\phi \right) P_{n,m} = \mathbf{u}_i^{(0)} \Big|_{r=1} \cdot \tilde{\nabla} f^{(Ca)} - f^{(Ca)} \left(\frac{\partial \mathbf{u}_i^{(0)}}{\partial r} \right) \Big|_{r=1} \cdot \mathbf{e}_r, \\ T_2 &= \sum_{n=1}^4 \left(T_{2,n,m}^{(Ca)} \cos m\phi + \hat{T}_{2,n,m}^{(Ca)} \sin m\phi \right) P_{n,m} = \mathbf{u}_e^{(0)} \Big|_{r=1} \cdot \tilde{\nabla} f^{(Ca)} - f^{(Ca)} \left(\frac{\partial \mathbf{u}_e^{(0)}}{\partial r} \right) \Big|_{r=1} \cdot \mathbf{e}_r, \\ T_3 &= \sum_{n=1}^4 \left(T_{3,n,m}^{(Ca)} \cos m\phi + \hat{T}_{3,n,m}^{(Ca)} \sin m\phi \right) P_{n,m} = \nabla \cdot \left(f^{(Ca)} \left(\frac{\partial \mathbf{u}_e^{(0)}}{\partial r} - \frac{\partial \mathbf{u}_i^{(0)}}{\partial r} \right) \right)_{r=1}, \\ T_4 &= \sum_{n=1}^4 \left(T_{4,n,m}^{(Ca)} \cos m\phi + \hat{T}_{4,n,m}^{(Ca)} \sin m\phi \right) P_{n,m} = \left[\begin{aligned} & + \mathbf{e}_r \cdot \nabla \times \left\{ \mathbf{e}_r \times \left(f^{(Ca)} \left(\frac{\partial \mathbf{T}_i^{(0)}}{\partial r} - \frac{\partial \mathbf{T}_e^{(0)}}{\partial r} \right) \right) \right\} \\ & + \mathbf{e}_r \cdot \nabla \times \left\{ \tilde{\nabla} f^{(Ca)} \times \left(\mathbf{T}_e^{(0)} - \mathbf{T}_i^{(0)} \right) \right\} \\ & + \tilde{\nabla} f^{(Ca)} \cdot \nabla \times \left\{ \mathbf{e}_r \times \left(\mathbf{T}_e^{(0)} - \mathbf{T}_i^{(0)} \right) \right\} \end{aligned} \right], \end{aligned} \right\} \quad (B4)$$

$$\begin{aligned}
T_5 &= \sum_{n=1}^4 \left(T_{5,n,m}^{(Ca)} \cos m\phi + \hat{T}_{5,n,m}^{(Ca)} \sin m\phi \right) P_{n,m} = \left[\begin{array}{l} \mathbf{e}_r \cdot \nabla \times \left(f^{(Ca)} \left(\frac{\partial \mathbf{u}_i^{(0)}}{\partial r} - \frac{\partial \mathbf{u}_e^{(0)}}{\partial r} \right) \Big|_{r=1} \right) \\ + \tilde{\nabla} f^{(Ca)} \cdot \nabla \times \left(\mathbf{u}_e^{(0)} - \mathbf{u}_i^{(0)} \right) \Big|_{r=1} \end{array} \right], \\
T_6 &= \sum_{n=1}^4 \left(T_{6,n,m}^{(Ca)} \cos m\phi + \hat{T}_{6,n,m}^{(Ca)} \sin m\phi \right) P_{n,m} = \left[\begin{array}{l} \tilde{\nabla} f^{(Ca)} \cdot \nabla \times \left(\mathbf{T}_e^{(0)} - \mathbf{T}_i^{(0)} \right) \Big|_{r=1} \\ + \mathbf{e}_r \cdot \nabla \times \left((\boldsymbol{\tau}_e - \boldsymbol{\tau}_i) \Big|_{r=1} \cdot \tilde{\nabla} f^{(Ca)} \right) \\ - \mathbf{e}_r \cdot \nabla \times \left(f^{(Ca)} \left(\frac{\partial \mathbf{T}_e^{(0)}}{\partial r} - \frac{\partial \mathbf{T}_i^{(0)}}{\partial r} \right) \Big|_{r=1} \right) \end{array} \right] \}, \\
\end{aligned} \tag{B5}$$

where the operator $\tilde{\nabla}$ is defined as $\tilde{\nabla} \equiv \mathbf{e}_\theta \frac{\partial}{\partial \theta} + \mathbf{e}_\phi \frac{1}{\sin \theta} \frac{\partial}{\partial \phi}$. In the above expressions $\boldsymbol{\tau}_{i,e}^{(0)}$ and $\mathbf{T}_{i,e}^{(0)}$ are the stress tensors and the traction vector, respectively of the following form

$$\boldsymbol{\tau}_{i,e}^{(0)} = \boldsymbol{\tau}_{i,e}^{H(0)} + M \boldsymbol{\tau}_{i,e}^{E(0)}, \quad \mathbf{T}_{i,e}^{(0)} = \boldsymbol{\tau}_{i,e}^{(0)} \cdot \mathbf{e}_r, \tag{B6}$$

where $\boldsymbol{\tau}_{i,e}^{H(0)}$ is the hydrodynamic stress tensor and $\boldsymbol{\tau}_{i,e}^{E(0)}$ is the Maxwell stress tensor at the leading order of the following form

$$\left. \begin{array}{l} \boldsymbol{\tau}_i^{H(0)} = \lambda \left[-p_i^{(0)} \mathbf{I} + \nabla \mathbf{u}_i^{(0)} + (\nabla \mathbf{u}_i^{(0)})^T \right], \\ \boldsymbol{\tau}_e^{H(0)} = \left[-p_e^{(0)} \mathbf{I} + \nabla \mathbf{u}_e^{(0)} + (\nabla \mathbf{u}_e^{(0)})^T \right], \\ \boldsymbol{\tau}_i^{E(0)} = S \left[\mathbf{E}_i^{(0)} (\mathbf{E}_i^{(0)})^T - \frac{1}{2} |\mathbf{E}_i^{(0)}|^2 \mathbf{I} \right], \\ \boldsymbol{\tau}_e^{E(0)} = \left[\mathbf{E}_e^{(0)} (\mathbf{E}_e^{(0)})^T - \frac{1}{2} |\mathbf{E}_e^{(0)}|^2 \mathbf{I} \right]. \end{array} \right\} \tag{B7}$$

Thus, after expressing the boundary conditions at the deformed surface in terms of spherical drop surface (equation (B3)), we have to represent the expressions present at the left side of equation (B3) in terms of solid harmonics. This representation is similar to the case of $O(Re_E)$ as given in equation (A4)-(A6); only difference is that we have to replace the superscript Re_E by Ca . Now, substituting those form in equation (B3) we obtain the solid coefficients $A^{(Ca)}$, $B^{(Ca)}$ and $C^{(Ca)}$ in the following form

$$\left. \begin{aligned}
A_{n,m}^{(Ca)} &= -\frac{(2n+3) \left[\begin{array}{l} -(4n^2-1)\beta_{n,m}^{(Ca)} + \{(n-1)(2n(\lambda+1))+(2\lambda+1)\}T_{1,n,m} \\ + (n+2)T_{2,n,m} + (2n+1)T_{3,n,m} - T_{4,n,m} + M(g_{n,m}^{e(Ca)} - g_{n,m}^{i(Ca)}) \end{array} \right]}{n(2n+1)(\lambda+1)}, \\
B_{n,m}^{(Ca)} &= -\frac{A_{n,m}^{(Ca)}}{2(2n+3)} + \frac{T_{1,n,m}}{n}, \\
C_{n,m}^{(Ca)} &= \frac{M(h_{n,m}^{e(Ca)} - h_{n,m}^{i(Ca)}) - (n+2)T_{5,n,m} - T_{6,n,m}}{n(n+1)(\lambda(n-1)+(n+2))}, \\
A_{-n-1,m}^{(Ca)} &= -\frac{\left[\begin{array}{l} (4n^2-1)(2+\lambda(2n+1))\beta_{n,m}^{(Ca)} + \lambda(n-1)(2n-1)T_{1,n,m} \\ - (n+2)(2n-1)(2n+\lambda(2n+1))T_{2,n,m} - \lambda(4n^2-1)T_{3,n,m} \\ - (2n-1)T_{4,n,m} + M(2n-1)(g_{n,m}^{e(Ca)} - g_{n,m}^{i(Ca)}) \end{array} \right]}{(\lambda+1)(n+1)(2n+1)}, \\
B_{-n-1,m}^{(Ca)} &= -\frac{\left[\begin{array}{l} \lambda(4n^2-1)\beta_{n,m}^{(Ca)} + \lambda(n-1)T_{1,n,m} - \lambda(2n+1)T_{3,n,m} - T_{4,n,m} \\ + \{2(1-n^2) - \lambda n(2n+1)\}T_{2,n,m} + M(g_{n,m}^{e(Ca)} - g_{n,m}^{i(Ca)}) \end{array} \right]}{2(\lambda+1)(2n+1)(n+1)}, \\
C_{-n-1,m}^{(Ca)} &= \frac{M(h_{n,m}^{e(Ca)} - h_{n,m}^{i(Ca)}) + \lambda(n-1)T_{5,n,m} - T_{6,n,m}}{n\{n^2 + 3n + 2 + \lambda(n^2-1)\}}.
\end{aligned} \right\} \quad (B8)$$

Similarly, the coefficients $\hat{A}_{n,m}^{(Ca)}, \hat{B}_{n,m}^{(Ca)}, \hat{C}_{n,m}^{(Ca)}, \hat{A}_{-n-1,m}^{(Ca)}, \hat{B}_{-n-1,m}^{(Ca)}$ and $\hat{C}_{-n-1,m}^{(Ca)}$ can be obtained by replacing $\beta_{n,m}^{(Ca)}, g_{n,m}^{i(Ca)}, h_{n,m}^{i(Ca)}, g_{n,m}^{e(Ca)}, h_{n,m}^{e(Ca)}$ and $T_{1,n,m} - T_{6,n,m}$ by $\hat{\beta}_{n,m}^{(Ca)}, \hat{g}_{n,m}^{i(Ca)}, \hat{h}_{n,m}^{i(Ca)}, \hat{g}_{n,m}^{e(Ca)}, \hat{h}_{n,m}^{e(Ca)}$ and $\hat{T}_{1,n,m} - \hat{T}_{6,n,m}$, respectively in equation (B8). Equation (B8) contains $\beta_{n,m}^{(Ca)}$ and $\hat{\beta}_{n,m}^{(Ca)}$ of the following form

$$\left. \begin{aligned}
\beta_{n,m}^{(Ca)} &= \begin{cases} \beta_{1,0}^{(Ca)} = -U_{dz}^{(Ca)}, \beta_{1,1}^{(Ca)} = -U_{dx}^{(Ca)} \\ 0 \quad \forall n \geq 2 \end{cases} \\
\hat{\beta}_{n,m}^{(Ca)} &= \begin{cases} \hat{\beta}_{1,1}^{(Ca)} = -U_{dy}^{(Ca)} \\ 0 \quad \forall n \geq 2. \end{cases}
\end{aligned} \right\} \quad (B9)$$

The expression of $g_{n,m}^{e(Ca)}, \hat{g}_{n,m}^{e(Ca)}, h_{n,m}^{e(Ca)}$ and $\hat{h}_{n,m}^{e(Ca)}$ present in equation (B8) can be obtained in the following form:

$$\left. \begin{aligned} & \sum_{n=0}^4 \left[g_{n,m}^{i(Ca)} \cos(m\phi) + \hat{g}_{n,m}^{i(Ca)} \sin(m\phi) \right] P_{n,m} = \left(\mathbf{e}_r \cdot \nabla \times \{ \mathbf{r} \times \mathbf{T}_i^{E(Ca)} \} \right) \Big|_{r=1}, \\ & \sum_{n=0}^4 \left[g_{n,m}^{e(Ca)} \cos(m\phi) + \hat{g}_{n,m}^{e(Ca)} \sin(m\phi) \right] P_{n,m} = \left(\mathbf{e}_r \cdot \nabla \times \{ \mathbf{r} \times \mathbf{T}_e^{E(Ca)} \} \right) \Big|_{r=1}, \\ & \sum_{n=0}^4 \left[h_{n,m}^{i(Ca)} \cos(m\phi) + \hat{h}_{n,m}^{i(Ca)} \sin(m\phi) \right] P_{n,m} = \left(\mathbf{e}_r \cdot \nabla \times \mathbf{T}_i^{E(Ca)} \right) \Big|_{r=1}, \\ & \sum_{n=0}^4 \left[h_{n,m}^{e(Ca)} \cos(m\phi) + \hat{h}_{n,m}^{e(Ca)} \sin(m\phi) \right] P_{n,m} = \left(\mathbf{e}_r \cdot \nabla \times \mathbf{T}_e^{E(Ca)} \right) \Big|_{r=1}, \end{aligned} \right\} \quad (\text{B10})$$

where $\mathbf{T}_{i,e}^{E(Ca)}$ is the electric traction vector at $O(Ca)$ and have the following form

$$\mathbf{T}_i^{E(Ca)} = S \begin{bmatrix} \left(E_{i,r}^{(Ca)} \right)^2 - \frac{1}{2} |\mathbf{E}_i^{(Ca)}|^2 \\ E_{i,r}^{(Ca)} E_{i,\theta}^{(Ca)} \\ E_{i,r}^{(Ca)} E_{i,\phi}^{(Ca)} \end{bmatrix}, \quad \mathbf{T}_e^{E(Ca)} = \begin{bmatrix} \left(E_{e,r}^{(Ca)} \right)^2 - \frac{1}{2} |\mathbf{E}_e^{(Ca)}|^2 \\ E_{e,r}^{(Ca)} E_{e,\theta}^{(Ca)} \\ E_{e,r}^{(Ca)} E_{e,\phi}^{(Ca)} \end{bmatrix}. \quad (\text{B11})$$

REFERENCES

- AJAYI, O.O. 1978 A Note on Taylor's Electrohydrodynamic Theory. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **364**(1719), 499–507.
- BRENNER, H. 1964 The Stokes resistance of a slightly deformed sphere. *Chemical Engineering Science* **19**(8), 519–539.
- HABER, S. & HETSRONI, G. 1971 The dynamics of a deformable drop suspended in an unbounded Stokes flow. *Journal of Fluid Mechanics* **49**(02), 257–277.
- HAPPEL, J. & BRENNER, H. 1981 *Low Reynolds number hydrodynamics*, Dordrecht: Springer Netherlands.
- HETSRONI, G. & HABER, S. 1970 The flow in and around a droplet or bubble submerged in an unbound arbitrary velocity field. *Rheologica Acta* **9**(4), 488–496.
- LEAL, L.G. 2007 *Advanced Transport Phenomena*, Cambridge: Cambridge University Press.
- XU, X. & HOMSY, G.M. 2006 The settling velocity and shape distortion of drops in a uniform electric field. *Journal of Fluid Mechanics* **564**, 395.