Supplementary material: Steady free-surface flow over spatially periodic topography

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Appendix A. Background to the fKdV equation (1.1)

We give brief details behind the derivation of the steady form of the fKdV equation (1.1). Let L be a charactertistic horizontal lengthscale, and H be a much shorter representative vertical length scale, such as the mean water depth. Using an asterisk to denote a dimensional variable, we introduce the scalings,

$$x^*/H = \delta^{-1/2}x, \qquad y^*/H = y, \qquad u^*/(gH)^{1/2} = u, \qquad v^*/(gH)^{1/2} = \delta^{1/2}v \tag{A 1}$$

where $\delta = H/L \ll 1$, g is the gravitational acceleration, and u^*, v^* are the velocity components in the horizontal, streamwise x^* direction and the vertical y^* direction respectively. The bottom topography is located at $y^* = \sigma^*(x^*)$. Assuming small amplitude topography, we introduce the scalings

$$\sigma^*/H = \delta^2\sigma(x), \qquad \eta^*/H = \delta\eta, \tag{A 2}$$

where $\eta^*(x^*)$ is the vertical displacement of the free surface from the constant level H. The variables x, y, u, v, η, and σ are all assumed to be $\mathcal{O}(1)$. The Froude number $F = U/(gH)^{1/2}$, where U is a representative streamwise velocity, is scaled by writing

$$F = 1 + \delta\mu, \tag{A 3}$$

where $\mu = \mathcal{O}(1)$. Under these conditions we derive the fKdV equation (Akylas 1984)

$$\eta_{xxx} + 9\eta\eta_x - 6\mu\eta_x = -3\sigma_x. \tag{A 4}$$

Integrating once with respect to x, we obtain that

$$\eta_{xx} + \frac{9}{2}\eta^2 - 6\mu\eta = -3\sigma + A, \tag{A 5}$$

where A is a constant of integration. In fact the integration constant may be set to zero without loss of generality. To see this, we make the change of dependent variable $\eta = \tilde{\eta} + c$, for constant c, and substitute into (A 5) to obtain

$$\tilde{\eta}_{xx} + \frac{9}{2}\tilde{\eta}^2 - 6\tilde{\mu}\tilde{\eta} = -3\sigma + \tilde{A}, \tag{A 6}$$

where $\tilde{\mu} = \mu - 3c/2$ and $\tilde{A} = A + 6\mu c - 9c^2/2$. We choose the constant c so that $\tilde{A} = 0$, setting

$$c = \frac{2}{3}\left(\mu - \sqrt{\mu^2 + A/2}\right). \tag{A 7}$$

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Then, (A 6) becomes

$$\tilde{\eta}_{xx} + \frac{9}{2}\tilde{\eta}^2 - 6\tilde{\mu}\tilde{\eta} = -3\sigma\,, \tag{A 8}$$

i.e., equation (1.1) in the main text. Thus by analysing (A 8) we effectively analyse the apparently more general (A 5), whose solutions differ only by a uniformly added height level. The integration constant can therefore be set to zero with no loss of generality.

Appendix B. Numerical scheme

In order to obtain numerical solutions to the fKdV equation, we consider a finite domain, on the interval $[-x_N, x_N]$, where $x_N = \pi M/k$ and $M \in \mathbb{N}$, subject to the boundary conditions

$$\begin{aligned} \eta(x_N) = \eta(-x_N), \quad \eta_x(x_N) = \eta_x(-x_N), \quad \eta_{xx}(x_N) = \eta_{xx}(-x_N) \\ \sigma(x_N) = \sigma(-x_N). \end{aligned} \tag{B 1}$$

The domain, $-x_N \leqslant x \leqslant x_N$, is discretised by $N \in \mathbb{N}$ equally-spaced mesh points

$$x_i = \left(-\frac{N}{2} + i - \frac{1}{2}\right)\Delta \quad \text{for} \quad i = 1, \ldots, N, \tag{B 2}$$

where $\Delta = \frac{x_N}{N-1}$ is the mesh spacing. The corresponding unknowns are

$$\eta_i = \eta(x_i) \quad \text{for} \quad i = 1, \ldots, N, \tag{B 3}$$

with forcing

$$\sigma_i = \sigma(x_i) \quad \text{for} \quad i = 1, \ldots, N. \tag{B 4}$$

Using the central difference formula for the second-order derivative in equation (1.1) gives $N-2$ algebraic equations

$$\frac{\eta_{i+1} - 2\eta_i + \eta_{i-1}}{\Delta^2} + \frac{9}{2}\eta_i^2 - 6\mu\eta_i = -3\sigma_i, \tag{B 5}$$

at the mesh points $i = 2, \ldots, N-1$. The periodic boundary condition (B 1), and using the central difference for the second-order derivative of η, provides two additional equations

$$\eta_1 = \eta_N \tag{B 6}$$

and

$$\frac{\eta_2 - 2\eta_1 + \eta_{N-1}}{\Delta^2} + \frac{9}{2}\eta_1^2 - 6\mu\eta_1 = -3\sigma_1. \tag{B 7}$$

Note that the condition (B 1) for σ is automatically satisfied by equation (1.2) at the endpoints of the domain. The system of equations (B 5)–(B 7) yield N nonlinear algebraic equations with N unknowns that can be solved using a Newton iterative method for given values of μ, wavenumber of forcing k, and amplitude of forcing ϵ.

Numerical solutions for the free-surface elevation, $\eta(x)$, are obtained for given values of μ, and wavenumber of forcing, k. This is done either by fixing the amplitude of forcing, ϵ, and allowing the elevation, $\eta(0)$, to be found as part of the solution, or by fixing $\eta(0)$ and allowing ϵ to be found as part of the solution (e.g. the solid curves in figure 2). Using continuation in the $(\epsilon, \eta(0))$ plane, we compute the solid curves shown in figure 3, which (primarily) follow branches that emanate from the three unforced solutions I–III ($\epsilon = 0$, figure 1) until they: (i) intersect at a turning point in the $(\epsilon, \eta(0))$ plane with a solution branch different to one being traversed, (ii) return to the unforced solution along the same solution branch, or (iii) terminates at a point where we were unable to obtain a converged numerical solution.

Appendix C. Non-autonomous theory for solution types I and II

The equations we present for type I and II solutions in the non-autonomous situation arise from an adaptation of the theory presented in Balasuriya & Binder (2014) to the spatially periodic situation. For a detailed discussion on the technicalities of the non-autonomous theory, we refer the reader to that article. Here, we give a brief derivation of how we recover the type I solution as given in (2.1), the type-II solutions as expressed in (2.2)–(2.5), and the peak value of the type II solution as given in (2.6).

Balasuriya & Binder (2014) consider the (integrated) Korteweg de-Vries equation given by (1.1), namely,

$$\eta_{xx} + \frac{9}{2}\eta^2 - 6\mu\eta = -3\sigma(x)\,. \tag{C 1}$$

The non-autonomousness arising from $\sigma(x)$ requires the examination of (C 1) in the appended (η, η_x, x) phase-space. Using this approach, it is shown in equation (11) of Balasuriya & Binder (2014) that there exists a special trajectory given by

$$\eta^I(x) = \frac{3}{2\sqrt{6\mu}}\int_0^\infty [\sigma(x-\tau) + \sigma(x+\tau)]\, e^{-\sqrt{6\mu}\tau}\,\mathrm{d}\tau + \mathcal{O}(\epsilon^2)\,. \tag{C 2}$$

The trajectory (C 2) is a 'hyperbolic trajectory' of the system (C 1). This means that it possesses both stable and unstable manifolds, just as the saddle fixed point at the origin in figure 1b does. The non-autonomous theory outlined in Balasuriya & Binder (2014) shows that with the inclusion of ϵ, the saddle fixed point perturbs to (C 2) which retains the presence of stable and unstable manifolds. The difference, however, is that (C 2) varies with x, unlike the saddle fixed point in the autonomous situation. Moreover, the theory Balasuriya & Binder (2014) shows that there is *only one* trajectory which remains $\mathcal{O}(\epsilon)$-close to the saddle point, which is (C 2). This justifies referring to (C 2) as the 'near-uniform solution,' since it is the *only* solution to (C 1) which remains $\mathcal{O}(\epsilon)$ over the unbounded domain of x.

Substituting $\sigma(x) = \epsilon\cos(kx)$ as given in (1.2) into (C 2) and performing integrations leads to (2.1). This is thus the only near-uniform solution for spatially periodic topography over an unbounded domain. While this expression is also derivable from a purely formal ϵ expansion subsituted into (1.1), the non-autonomous theory provides a justification for this: the hyperbolic trajectory is the only possible trajectory which can remain $\mathcal{O}(\epsilon)$-close to the origin in both backward and forward x, legitimising an expansion for $x \in \mathbb{R}$ (Balasuriya & Padberg-Gehle 2013, 2014).

Next, the near-solitary wave is analysed. The hyperbolicity of (C 2) is associated with the fact that it possesses both stable and unstable manifolds with respect to the (η, η_x, x) phase-space. Points on the unstable manifold asymptote to (C 2) as $x \to -\infty$, while those on the stable manifold do so as $x \to \infty$. If there are solutions which asymptote to (C 2) in *both* these limits, they must lie on *both* the stable and unstable manifold. When there is no forcing, these two manifolds coincide, as shown in figure 1b, to form the homoclinic trajectory. (In this case, since the system is autonomous, there is no necessity for including the additional x-greendimension in the phase-space.) Trajectories on this *homoclinic* manifold all asymptote to $\eta = 0$ as $x \to \pm\infty$, and these solutions correspond to the solitary wave solution shown in figure 1a and represented by (1.3). Actually, figure 1a and (1.3) only show *one* type II solution whereas there are infinitely many of them, obtained by shifting this one solution. The freedom of performing this shift is equivalent to the freedom of choosing an initial condition along the homoclinic manifold in figure 1b.

Now when $\sigma \neq 0$, the stable and unstable manifolds of the hyperbolic trajectory

(C 2) no longer need to coincide. To find near-solitary wave solutions, however, we require trajectories lying on both the stable and unstable manifolds, that is, we need to seek intersections between these manifolds. To do so, consider an x-slice in the (η, η_x, x) phase-space. There is a one-dimensional stable manifold emanating from the hyperbolic point $\left(\eta^I(x), \eta_x^I(x)\right)$, which near this point is somewhat close to the homoclinic trajectory visible in figure 1b by standard perturbation arguments. Similarly, there is a one-dimensional unstable manifold which also near this point is close to the homoclinic trajectory. To investigate intersections, we measure the signed distance between them at where these two curves intersect the η-axis, which is near the value 2μ. It is shown in equation (12) of Balasuriya & Binder (2014) that this distance, scaled by a nonzero factor, is given by

$$M(x) = \int_{-\infty}^{\infty} \operatorname{sech}^2\left(\sqrt{\frac{3\mu}{2}}\,\tau\right) \tanh\left(\sqrt{\frac{3\mu}{2}}\,\tau\right) \sigma(x+\tau)\,\mathrm{d}\tau\,. \tag{C 3}$$

This is an example of a *Melnikov function* (Guckenheimer & Holmes (1983); Rom-Kedar et al. (1990); Balasuriya (2005); Balasuriya & Finn (2012); Grimshaw & Tian (1994)), and the x represents the fact that we have taken a slice at a general x-value (the distance measurement will be different in each x-slice). Putting in $\sigma = \epsilon \cos(kx)$, and evaluating the integrals using contour integration [not shown] eventually leads to the Melnikov function

$$M(x) = -\frac{k^2\sqrt{2}}{(3\mu)^{3/2}} \operatorname{cosech}\left(\frac{\pi k}{\sqrt{6\mu}}\right) \sin(kx) \tag{C 4}$$

This has a simple zero at $x = 0$, implying that there is a trajectory $\eta(x)$ which takes a value of $\eta_x = 0$ when $x = 0$, and which asymptotes to the uniform stream (2.1) in the limits $x \to \pm\infty$ since it lies on both the stable and the unstable manifolds of the hyperbolic trajectory (2.1). This is therefore a homoclinic trajectory (Guckenheimer & Holmes 1983; Rom-Kedar et al. 1990), which is a near-solitary wave (type II) solution. Incidentally, the presence of this zero gives us for free a proof—consonant with the damped fKdV results of Grimshaw & Tian (1994)—that there are nearby chaotic trajectories in this instance. Our focus here is not on those chaotic solutions, but the bifurcation behaviour associated with the homoclinic trajectories which are the 'governers' of the chaotic region. Since $M(x)$ is $(2\pi/k)$-periodic, there are an infinite number of homoclinic solutions. This infinitude is associated with two basic families of near-solitary waves: those situated with their global maximum above $x = 2n\pi/k$ or $x = 2(n+1)\pi/k$ for $n \in \mathbb{Z}$. If $\epsilon > 0$, the first family corresponds to homoclinic trajectories which are centred at a crest of the topography σ, while the second family is centred at troughs. If $\epsilon < 0$, the families interchange. Representatives from these two families are $x = 0$ and $x = \pi/k$; shifts of these solutions by multiples of $2\pi/k$ generate the other members in each family. If $\epsilon < 0$, the $x = 0$ family is trough-centred and the $x = \pi/k$ is crest-centred.

Now, the point is that should there be an intersection between the stable and unstable manifolds along the η-axis, in a general x-slice, then $M(x)$ needs to be zero. Actually, to guarantee an intersection for small ϵ we require a little more than that: we need $M(x)$ to have a *simple* zero at that x-value (Guckenheimer & Holmes (1983)). This means that $M(x)$ needs to cross zero at that point, and this guarantees that this intersection is preserved under the small perturbations resulting from higher-order in ϵ terms which have been neglected in the analysis. Now, the Melnikov function as computed from the spatially periodic forcing, (2.2) is easily seen to satisfy this at all the values at which zeroes occur, since effectively the Melnikov function takes the form $M(x) = A\sin(kx)$ for a constant A. The infinitely many zeroes of M each correspond to an intersection of stable and unstable manifolds, and trajectories passing through these intersections will asymptote to the hyperbolic trajectory (C 2) as $x \to \pm\infty$. Therefore, there are countably many homoclinic trajectories in the spatially periodic non-autonomous system. This is in fact 'less' than there were in the autonomous system which possessed *uncountably* many such solutions, obtained by arbitrarily sliding the solution given in (1.3). In the spatially periodic instance, such a sliding can only be done to a discrete set of x-values.

The presence of simple zeroes in the Melnikov function has other interesting implications, in this situation in which the entire system is *periodic* in x, with period $2\pi/k$. In this case, one can define a *Poincaré map* which maps points from a fixed x_0-plane in the (η, η_x, x) phase-space, to where they go to in the plane $x_0 + 2\pi/k$. The point $(\eta, \eta_x) = \left(\eta^I(x_0), \eta_x^I((x_0)\right)$—the intersection of the hyperbolic trajectory in the initial x-slice—is a fixed point of this Poincaré map, which possesses stable and unstable manifolds. The presence of a simple zero in the Melnikov function implies that there is a transverse intersection between the stable and unstable manifolds, and indeed there are infinitely many because of the periodicity of the Melnikov function. This creates 'lobe' regions between these manifolds. Upon repeated application of the Poincaré map, these lobes get elongated as they get closer to the fixed point, influenced its hyperbolicity which results in exponential stretching in the unstable direction. Moreover, the lobes get folded as they approach the fixed point. This stretching and folding behaviour results in the presence of 'Smale horseshoes' in the system; the classical Smale-Birkhoff Theorem (Guckenheimer & Holmes (1983)) tells us that the system in chaotic. Solutions lying in the lobe regions display seemingly random behaviour as x progresses towards $\pm\infty$. The presence of chaos in spatially periodic systems is already well-known due to the work by Grimshaw & Tian (1994). Here, we manage to extract the 'ordered' type-II from within among the chaotic solutions.

The next step is to determine the profiles of these type-II solutions, corresponding to the x-variation of points which lie on both the stable and unstable manifold. This was determined theoretically, and given in their equations (14) through (17), by Balasuriya & Binder (2014) for general σ. For each zero of the Melnikov function (C 3) occurring at $x = \bar{x}$, the wave-profile was quantified. Here, we simplify these expressions, by focussing on $\epsilon \cos(kx)$ for σ. The resulting Melnikov function (C 4) clearly has a zero at $\bar{x} = 0$, and we shall first focus on the homoclinic solution corresponding to this. By adapting the formulæ of Balasuriya & Binder (2014), and the symmetries associated with the autonomous solitary wave (1.3), the type-II (near solitary wave) solution can be represented by

$$\eta^{II}(x) = \bar{\eta}(x) + \epsilon\left[\eta^n(x) + \eta^t(x)\right] + \mathcal{O}(\epsilon^2) \tag{C 5}$$

in which $\bar{\eta}$ is the fundamental solitary wave solution (1.3), and η^n and η^t are respectively given by

$$\eta^n(x) = -\frac{3\bar{\eta}_{xx}(x)}{\bar{\eta}_x(x)^2 + \bar{\eta}_{xx}(x)^2}\int_{|x|}^{\infty} \bar{\eta}_x(\tau)\cos(k\tau)\,\mathrm{d}\tau \tag{C 6}$$

and

$$\eta^t(x) = -3\bar{\eta}_x(|x|)\int_0^{|x|} \frac{\bar{\eta}_{xx}(\tau)\cos(k\tau) - \Omega(\tau)\int_{|x|}^{\infty}\bar{\eta}_x(\lambda)\cos\left[k\left(\lambda + \tau - x\right)\right]\mathrm{d}\lambda}{\bar{\eta}_x(\tau)^2 + \bar{\eta}_{xx}(\tau)^2}\,\mathrm{d}\tau\,, \tag{C 7}$$

where

$$\Omega(\tau) := \frac{\left[6(1+\mu) - 9\bar{\eta}(\tau) - 5\right]\left[\bar{\eta}_x(\tau)^2 - \bar{\eta}_{xx}(\tau)^2\right]}{\bar{\eta}_x(\tau)^2 + \bar{\eta}_{xx}(\tau)^2}\,. \tag{C 8}$$

These expressions are included in equations (2.2)–(2.5) in the main text. What is of interest is that, in contrast to the homoclinics associated with general σ as outlined in

Balasuriya & Binder (2014), the homoclinic solution of (C 5) is symmetric about $x = 0$. Now, if $\epsilon > 0$, there is a crest of σ at $x = 0$, and thus the solution derived above is a *crest-centred near-solitary wave.* Moreover, the fact that the translation $x \to x + 2n\pi/k$ for $n \in \mathbb{Z}$ leaves the governing equation invariant indicates that this solution at $x = 0$ can be translated to a countable number of other near-solitary wave solutions. This takes care of the other zeroes of the Melnikov function (C 4) of the form $\bar{x} = 2n\pi/k$.

Next, consider a candidate from the trough-centred solutions when $\epsilon > 0$, the solution with $\bar{x} = \pi/k$. Setting a non-symmetric value of $\bar{x}$ complicates the expressions in Balasuriya & Binder (2014) substantially, and therefore is best avoided. However, the crest-centred solution derived in (C 5) can be 'flipped' to a trough-centred solution by the simple strategem of setting $\epsilon < 0$. The resulting trough-centred solution centred at $\bar{x} = 0$ can be then simply translated to $\bar{x} = \pi/k$, or indeed to any $\bar{x} = (2n+1)\pi/k$, where $n \in \mathbb{N}$. Thus, trough-centred solutions can be obtained from (C 5) by flipping the sign of ϵ and translating, relieving us of the necessity of providing additional formulæ. Therefore, *all* zeroes of the Melnikov function (C 4)—all single-humped homoclinic solutions—are also encapsulated within the framework of equation (C 5).

If $\epsilon < 0$, the solution given in (C 5) corresponds to a trough-centred near-solitary wave, and one can similarly determine all the members of this family, as well as the crest-centred waves which are exactly 'out of phase.'

Next, the value of $\eta^{II}(0)$ as shown in (2.6) is derived. Notice that the η^t-term disappears at this value. Using the facts that $\bar{\eta}(0) = 2\mu$, $\bar{\eta}_x(0) = 0$ and $\bar{\eta}_{xx}(0) = -6\mu^2$, we then get

$$\eta^{II}(0) = 2\mu + \frac{\epsilon}{2\mu} \int_0^\infty \bar{\eta}'(\tau) \cos(k\tau) \, \mathrm{d}\tau \, .$$

Integrating by parts, and then putting in $\bar{\eta}$ from (1.3) and simplifying, leads to equation (2.6), the free-surface elevation at the peak of the type-II solution.

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