Weakly nonlinear stability analysis of polymer fiber spinning

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Supplementary material

1 Calculation of the Landau constant

As the Landau constant is direct representation of the nature of bifurcation, the primary objective of our study is to obtain it following the methodology explained next. The substitution of equation (21), in main text, into governing equations forms infinite problem sets of sequential non-homogeneous problems based on values of harmonic index k and order n of the form (Chokshi and Kumaran, 2008; Shukla and Alam, 2011).

$$\mathcal{L}^{(k,n)}[\tilde{\phi}^{(k,n)}] = \mathcal{I}_2^{(k,n)} + \mathcal{I}_3^{(k,n)}. \tag{1}$$

Here, $\mathcal{L}^{(k,n)}$ is the differential operator operating on the unknown eigenfunction $\tilde{\phi}^{(k,n)}$, and $\mathcal{I}_2^{(k,n)}$ and $\mathcal{I}_3^{(k,n)}$ represent the quadratic and the cubic nonlinear inhomogeneities respectively, that arise due to the presence of nonlinearities in the governing equations. In general, the problem at order (k,n) contains inhomogeneous terms of order (j,m), where $m < n$ and $j + m < k + n$. So, the problem sets are arranged as follows. The problem with $(k,n) = (1,1)$ represents the linear stability problem providing the solution for the fundamental disturbance mode $\tilde{\phi}^{(1,1)}$. For higher order

problem with $(k, n) = (1, 2)$, it is found that equation (1) is same as that for problem (1,1). Hence, without loss of generality, $\tilde{\phi}^{(1,2)} = 0$. Similarly, all the hierarchical problems utilizing solution of (1,2) eigenfunction would vanish, and subsequently it could be proven that $\tilde{\phi}^{(k,n)} = 0$ for $k + n$=odd. Hence, only problems with non-trivial solution are with (k, n) equal to $(0, 2)$, $(2, 2)$ and $(1, 3)$ which remain to be solved in a sequential manner. Finally, the Landau constant, l, is obtained at the order $(k, n) = (1, 3)$. The governing equations for each of the problems leading to calculation of the Landau constant are discussed next.

2 Formulation of weakly nonlinear stability analysis

In weakly nonlinear analysis, various problems are solved to obtain the eigenfunctions $\tilde{\phi}^{(k,n)}$ for harmonic index k and amplitude order n. The nonlinearities in the disturbance governing equations involve self-interaction of the fundamental mode $\tilde{\phi}^{(1,1)}$, and also its interaction with higher harmonics $\tilde{\phi}^{(k,n)}$. First, the linear growth rate s and fundamental eigenfunction $\tilde{\phi}^{(1,1)}$ are obtained as solution of the linear stability problem governed by equations (9)-(14). To calculate the nonlinear correction to growth rate, l, the Landau constant, following hierarchy of problems are solved in a sequential manner. These higher amplitude order problems are inhomogeneous, represented by equation (26).

2.1 Problem (0,2)

The solution is understood as second order correction or distortion of the base-state flow. This arises by interaction of the fundamental mode $\tilde{\phi}^{(1,1)}$ with its complex conjugate $\tilde{\phi}^{\dagger(1,1)}$ producing $O(A^2)$ distortion of the mean flow. The governing equa-

tions for mass, momentum and constitutive relation at this order are:

$$\tilde{v}^{(0,2)}\frac{d\bar{a}}{dz} + \tilde{a}^{(0,2)}\frac{d\bar{v}}{dz} = -\frac{1}{2}\left[\tilde{v}^{\dagger(1,1)}\frac{d\tilde{a}^{(1,1)}}{dz} + \tilde{a}^{\dagger(1,1)}\frac{d\tilde{v}^{(1,1)}}{dz}+\right.$$
$$\left.\tilde{v}^{(1,1)}\frac{d\tilde{a}^{\dagger(1,1)}}{dz} + \tilde{a}^{(1,1)}\frac{d\tilde{v}^{\dagger(1,1)}}{dz}\right], \quad (2)$$

$$\tilde{\tau}_{zz}^{(0,2)}\frac{d\bar{a}}{dz} - \tilde{\tau}_{rr}^{(0,2)}\frac{d\bar{a}}{dz} + \tilde{a}^{(0,2)}\frac{d\bar{\tau}_{zz}}{dz} - \tilde{a}^{(0,2)}\frac{d\bar{\tau}_{rr}}{dz} = -\frac{1}{2}\left[\tilde{\tau}_{zz}^{\dagger(1,1)}\frac{d\tilde{a}^{(1,1)}}{dz}-\right.$$
$$\tilde{\tau}_{rr}^{\dagger(1,1)}\frac{d\tilde{a}^{(1,1)}}{dz} + \tilde{a}^{\dagger(1,1)}\frac{d\tilde{\tau}_{zz}^{(1,1)}}{dz} - \tilde{a}^{\dagger(1,1)}\frac{d\tilde{\tau}_{rr}^{(1,1)}}{dz} + \tilde{\tau}_{zz}^{(1,1)}\frac{d\tilde{a}^{\dagger(1,1)}}{dz}-$$
$$\left.\tilde{\tau}_{rr}^{(1,1)}\frac{d\tilde{a}^{\dagger(1,1)}}{dz} + \tilde{a}^{(1,1)}\frac{d\tilde{\tau}_{zz}^{\dagger(1,1)}}{dz} - \tilde{a}^{(1,1)}\frac{d\tilde{\tau}_{rr}^{\dagger(1,1)}}{dz}\right], \quad (3)$$

$$\frac{1}{De}\left(\tilde{F}^{(0,2)} + \bar{F}\,\tilde{\tau}_{zz}^{(0,2)} + \tilde{F}^{(0,2)}\,\bar{\tau}_{zz}\right) - 2\,\tilde{\tau}_{zz}^{(0,2)}\frac{d\bar{v}}{dz} + \tilde{v}^{(0,2)}\frac{d\bar{\tau}_{zz}}{dz} =$$
$$-\frac{1}{2}\left[\frac{1}{De}\left(\tilde{F}^{\dagger(1,1)}\,\tilde{\tau}_{zz}^{(1,1)} + \tilde{F}^{(1,1)}\,\tilde{\tau}_{zz}^{\dagger(1,1)}\right) - 2\,\tilde{\tau}_{zz}^{\dagger(1,1)}\frac{d\tilde{v}^{(1,1)}}{dz} + \tilde{v}^{\dagger(1,1)}\frac{d\tilde{\tau}_{zz}^{(1,1)}}{dz}\right.$$
$$\left.- 2\,\tilde{\tau}_{zz}^{(1,1)}\frac{d\tilde{v}^{\dagger(1,1)}}{dz} + \tilde{v}^{(1,1)}\frac{d\tilde{\tau}_{zz}^{\dagger(1,1)}}{dz}\right], \quad (4)$$

$$\frac{1}{De}\left(\tilde{F}^{(0,2)} + \bar{F}\,\tilde{\tau}_{rr}^{(0,2)} + \tilde{F}^{(0,2)}\,\bar{\tau}_{rr}\right) + \tilde{\tau}_{rr}^{(0,2)}\frac{d\bar{v}}{dz} + \tilde{v}^{(0,2)}\frac{d\bar{\tau}_{rr}}{dz} =$$
$$-\frac{1}{2}\left[\frac{1}{De}\left(\tilde{F}^{\dagger(1,1)}\,\tilde{\tau}_{rr}^{(1,1)} + \tilde{F}^{(1,1)}\,\tilde{\tau}_{rr}^{\dagger(1,1)}\right) + \tilde{\tau}_{rr}^{\dagger(1,1)}\frac{d\tilde{v}^{(1,1)}}{dz} + \tilde{v}^{\dagger(1,1)}\frac{d\tilde{\tau}_{rr}^{(1,1)}}{dz}+\right.$$
$$\left.\tilde{\tau}_{rr}^{(1,1)}\frac{d\tilde{v}^{\dagger(1,1)}}{dz} + \tilde{v}^{(1,1)}\frac{d\tilde{\tau}_{rr}^{\dagger(1,1)}}{dz}\right], \quad (5)$$

$$\tilde{F}^{(0,2)} = \frac{1}{\bar{\Lambda}^3\,q^2}\left(-2\,\tilde{\Lambda}^{(0,2)}\,(q^2 + 2\,e^{\frac{\bar{\Lambda}}{q}}\,\bar{\Lambda}^2\,q\,r - 2\,e^{\frac{\bar{\Lambda}}{q}}\,\bar{\Lambda}^3\,q\,r - e^{\frac{\bar{\Lambda}}{q}}\,\bar{\Lambda}\,q^2\,r)\right) +$$
$$\frac{\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{\dagger(1,1)}}{2}\left[\frac{6}{\bar{\Lambda}^4} + 4\,r\,e^{-\frac{2}{q}+\frac{2\bar{\Lambda}}{q}}\left(-\frac{1}{\bar{\Lambda}^3} + \frac{2}{q^2} - \frac{2}{\bar{\Lambda}\,q^2} + \frac{2}{\bar{\Lambda}^2\,q}\right)\right], \quad (6)$$

$$\tilde{\Lambda}^{(0,2)} = \frac{\tilde{\tau}_{zz}^{(0,2)}}{6\,\bar{\Lambda}^3} + \frac{\tilde{\tau}_{rr}^{(0,2)}}{3\,\bar{\Lambda}^3} + \frac{\tilde{\tau}_{zz}^{(0,2)}\,\bar{\tau}_{zz}}{18\,\bar{\Lambda}^3} + \frac{\tilde{\tau}_{rr}^{(0,2)}\,\bar{\tau}_{zz}}{9\,\bar{\Lambda}^3} + \frac{\tilde{\tau}_{zz}^{(0,2)}\,\bar{\tau}_{rr}}{9\,\bar{\Lambda}^3} + \frac{2\,\tilde{\tau}_{rr}^{(0,2)}\,\bar{\tau}_{rr}}{9\,\bar{\Lambda}^3}+$$
$$\frac{1}{2}\left(-\frac{\tilde{\tau}_{zz}^{(1,1)}\,\tilde{\tau}_{zz}^{\dagger(1,1)}}{36\,\bar{\Lambda}^3} - \frac{\tilde{\tau}_{rr}^{(1,1)}\,\tilde{\tau}_{zz}^{\dagger(1,1)}}{18\,\bar{\Lambda}^3} - \frac{\tilde{\tau}_{zz}^{(1,1)}\,\tilde{\tau}_{rr}^{\dagger(1,1)}}{18\,\bar{\Lambda}^3} - \frac{\tilde{\tau}_{rr}^{(1,1)}\,\tilde{\tau}_{rr}^{\dagger(1,1)}}{9\,\bar{\Lambda}^3}\right). \quad (7)$$

2.2   Problem (2,2)

This problem arises from interaction of fundamental with itself, and it represents the higher harmonic of the fundamental wave. The disturbance governing equations are as given below:

$$2\,s\,\tilde{a}^{(2,2)} + \tilde{v}^{(2,2)}\frac{d\bar{a}}{dz} + \tilde{a}^{(2,2)}\frac{d\bar{v}}{dz} = \tilde{v}^{(1,1)}\frac{d\tilde{a}^{(1,1)}}{dz} + \tilde{a}^{(1,1)}\frac{d\tilde{v}^{(1,1)}}{dz}, \tag{8}$$

$$\tilde{\tau}_{zz}^{(2,2)}\frac{d\bar{a}}{dz} - \tilde{\tau}_{rr}^{(2,2)}\frac{d\bar{a}}{dz} + \tilde{a}^{(2,2)}\frac{d\bar{\tau}_{zz}}{dz} - \tilde{a}^{(2,2)}\frac{d\bar{\tau}_{rr}}{dz} = \tilde{\tau}_{zz}^{(1,1)}\frac{d\tilde{a}^{(1,1)}}{dz} - \tilde{\tau}_{rr}^{(1,1)}\frac{d\tilde{a}^{(1,1)}}{dz} +$$
$$\tilde{a}^{(1,1)}\frac{d\tilde{\tau}_{zz}^{(1,1)}}{dz} - \tilde{a}^{(1,1)}\frac{d\tilde{\tau}_{rr}^{(1,1)}}{dz}, \tag{9}$$

$$\frac{1}{De}\left(\tilde{F}^{(2,2)} + \bar{F}\,\tilde{\tau}_{zz}^{(2,2)} + \tilde{F}^{(2,2)}\,\bar{\tau}_{zz}\right) + 2\,s\,\tilde{\tau}_{zz}^{(2,2)} - 2\,\tilde{\tau}_{zz}^{(2,2)}\frac{d\bar{v}}{dz} + \tilde{v}^{(2,2)}\frac{d\bar{\tau}_{zz}}{dz} =$$
$$\frac{1}{De}\tilde{F}^{(1,1)}\,\tilde{\tau}_{zz}^{(1,1)} - 2\,\tilde{\tau}_{zz}^{(1,1)}\frac{d\tilde{v}^{(1,1)}}{dz} + \tilde{v}^{(1,1)}\frac{d\tilde{\tau}_{zz}^{(1,1)}}{dz}, \tag{10}$$

$$\frac{1}{De}\left(\tilde{F}^{(2,2)} + \bar{F}\,\tilde{\tau}_{rr}^{(2,2)} + \tilde{F}^{(2,2)}\,\bar{\tau}_{rr}\right) + 2\,s\,\tilde{\tau}_{rr}^{(2,2)} + \tilde{\tau}_{rr}^{(2,2)}\frac{d\bar{v}}{dz} + \tilde{v}^{(2,2)}\frac{d\bar{\tau}_{rr}}{dz} = \frac{1}{De}\tilde{F}^{(1,1)}\,\tilde{\tau}_{rr}^{(1,1)} +$$
$$\tilde{\tau}_{rr}^{(1,1)}\frac{d\tilde{v}^{(1,1)}}{dz} + \tilde{v}^{(1,1)}\frac{d\tilde{\tau}_{rr}^{(1,1)}}{dz}, \tag{11}$$

$$\tilde{F}^{(2,2)} = \frac{1}{\bar{\Lambda}^3}\left(2\,\tilde{\Lambda}^{(2,2)}\left(-1 + e^{\frac{2\,(-1+\bar{\Lambda})}{q}}\,\bar{\Lambda}\left(2\left(-1+\bar{\Lambda}\right)\bar{\Lambda} + q\right)\frac{r}{q}\right)\right) +$$
$$\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{(1,1)}\left[\frac{3}{\bar{\Lambda}^4} + 2\,r\,e^{\frac{-2}{q}+\frac{2\bar{\Lambda}}{q}}\left(-\frac{1}{\bar{\Lambda}^3} + \frac{2}{q^2} - \frac{2}{\bar{\Lambda}\,q^2} + \frac{2}{\bar{\Lambda}^2\,q}\right)\right], \tag{12}$$

$$\tilde{\Lambda}^{(2,2)} = \frac{\tilde{\tau}_{zz}^{(2,2)} + 2\,\tilde{\tau}_{rr}^{(2,2)}}{6\,\bar{\Lambda}} - \frac{\tilde{\tau}_{zz}^{(1,1)}\,\tilde{tzz}^{(1,1)}}{72\,\bar{\Lambda}^3} - \frac{\tilde{\tau}_{zz}^{(1,1)}\,\tilde{\tau}_{rr}^{(1,1)}}{18\,\bar{\Lambda}^3} - \frac{\tilde{\tau}_{rr}^{(1,1)}\,\tilde{trr}^{(1,1)}}{18\,\bar{\Lambda}^3}. \tag{13}$$

2.3   Problem (1,3)

The solution is understood as correction or distortion of the fundamental mode, due to interaction of fundamental with all second order contributions. The inhomogeneities are due to interaction of $\tilde{\phi}^{(1,1)}$ with $\tilde{\phi}^{(0,2)}$ and $\tilde{\phi}^{\dagger(1,1)}$ with $\tilde{\phi}^{(2,2)}$. In

addition, there are cubic nonlinearities made up of $\tilde{\phi}^{(1,1)}$ and its complex conjugate. The Landau constant, l, appears at this order. The disturbance governing equations are:

$$s\,\tilde{a}^{(1,3)} + \tilde{v}^{(1,3)}\frac{d\bar{a}}{dz} + \tilde{a}^{(1,3)}\frac{d\bar{v}}{dz} = \left(l + \frac{ds_r}{dDR}\right)\tilde{a}^{(1,1)} + 2\frac{d\tilde{v}^{(0,2)}}{dz}\tilde{a}^{(1,1)}$$
$$+ 2\frac{d\tilde{a}^{(0,2)}}{dz}\tilde{v}^{(1,1)} + 2\,\tilde{v}^{(0,2)}\frac{d\tilde{a}^{(1,1)}}{dz} + 2\,\tilde{a}^{(0,2)}\frac{d\tilde{v}^{(1,1)}}{dz} + \frac{d\tilde{v}^{\dagger(1,1)}}{dz}\tilde{a}^{(2,2)} \qquad (14)$$
$$+ \frac{d\tilde{a}^{\dagger(1,1)}}{dz}\tilde{v}^{(2,2)} + \tilde{v}^{\dagger(1,1)}\frac{d\tilde{a}^{(2,2)}}{dz} + \tilde{a}^{\dagger(1,1)}\frac{d\tilde{v}^{(2,2)}}{dz},$$

$$\tilde{\tau}_{zz}^{(1,3)}\frac{d\bar{a}}{dz} - \tilde{\tau}_{rr}^{(1,3)}\frac{d\bar{a}}{dz} + \tilde{a}^{(1,3)}\frac{d\bar{\tau}_{zz}}{dz} - \tilde{a}^{(1,3)}\frac{d\bar{\tau}_{rr}}{dz} = 2\frac{d\tilde{\tau}_{zz}^{(0,2)}}{dz}\tilde{a}^{(1,1)}$$
$$- 2\frac{d\tilde{\tau}_{rr}^{(0,2)}}{dz}\tilde{a}^{(1,1)} + 2\frac{d\tilde{a}^{(0,2)}}{dz}\tilde{\tau}_{zz}^{(1,1)} - 2\frac{d\tilde{a}^{(0,2)}}{dz}\tilde{\tau}_{rr}^{(1,1)} + 2\,\tilde{\tau}_{zz}^{(0,2)}\frac{d\tilde{a}^{(1,1)}}{dz}$$
$$- 2\,\tilde{\tau}_{rr}^{(0,2)}\frac{d\tilde{a}^{(1,1)}}{dz} + 2\,\tilde{a}^{(0,2)}\frac{d\tilde{\tau}_{zz}^{(1,1)}}{dz} - 2\,\tilde{a}^{(0,2)}\frac{d\tilde{\tau}_{rr}^{(1,1)}}{dz} + \frac{d\tilde{\tau}_{zz}^{\dagger(1,1)}}{dz}\tilde{a}^{(2,2)} \qquad (15)$$
$$- \frac{d\tilde{\tau}_{rr}^{\dagger(1,1)}}{dz}\tilde{a}^{(2,2)} + \frac{d\tilde{a}^{\dagger(1,1)}}{dz}\tilde{\tau}_{zz}^{(2,2)} - \frac{d\tilde{a}^{\dagger(1,1)}}{dz}\tilde{\tau}_{rr}^{(2,2)} + \tilde{\tau}_{zz}^{\dagger(1,1)}\frac{d\tilde{a}^{(2,2)}}{dz}$$
$$- \tilde{\tau}_{rr}^{\dagger(1,1)}\frac{d\tilde{a}^{(2,2)}}{dz} + \tilde{a}^{\dagger(1,1)}\frac{d\tilde{\tau}_{zz}^{(2,2)}}{dz} - \tilde{a}^{\dagger(1,1)}\frac{d\tilde{\tau}_{rr}^{(2,2)}}{dz},$$

$$\frac{1}{De}\left(\tilde{F}^{(1,3)} + \bar{F}\,\tilde{\tau}_{zz}^{(1,3)} + \tilde{F}^{(1,3)}\,\bar{\tau}_{zz}\right) + s\,\tilde{\tau}_{zz}^{(1,3)} - 2\,\tilde{\tau}_{zz}^{(1,3)}\frac{d\bar{v}}{dz} + \tilde{v}^{(1,3)}\frac{d\bar{\tau}_{zz}}{dz} =$$
$$\frac{1}{De}\left(\tilde{F}^{(1,1)}\,2\,\tilde{\tau}_{zz}^{(0,2)} + 2\,\tilde{F}^{(0,2)}\,\tilde{\tau}_{zz}^{(1,1)} + \tilde{F}^{(2,2)}\,\tilde{\tau}_{zz}^{\dagger(1,1)} + \tilde{F}^{\dagger(1,1)}\,\tilde{\tau}_{zz}^{(2,2)}\right)$$
$$+ 2\frac{d\tilde{\tau}_{zz}^{(0,2)}}{dz}\tilde{v}^{(1,1)} + \left(l + \frac{ds_r}{dDR}\right)\tilde{\tau}_{zz}^{(1,1)} - 4\frac{d\tilde{v}^{(0,2)}}{dz}\tilde{\tau}_{zz}^{(1,1)} \qquad (16)$$
$$- 4\,\tilde{\tau}_{zz}^{(0,2)}\frac{d\tilde{v}^{(1,1)}}{dz} + 2\,\tilde{v}^{(0,2)}\frac{d\tilde{\tau}_{zz}^{(1,1)}}{dz} + \frac{d\tilde{\tau}_{zz}^{\dagger(1,1)}}{dz}\tilde{v}^{(2,2)}$$
$$- 2\frac{d\tilde{v}^{\dagger(1,1)}}{dz}\tilde{\tau}_{zz}^{(2,2)} - 2\,\tilde{\tau}_{zz}^{\dagger(1,1)}\frac{d\tilde{v}^{(2,2)}}{dz} + \tilde{v}^{\dagger(1,1)}\frac{d\tilde{\tau}_{zz}^{(2,2)}}{dz},$$

$$\frac{1}{De}\left(\tilde{F}^{(1,3)} + \bar{F}\,\tilde{\tau}_{rr}^{(1,3)} + \tilde{F}^{(1,3)}\,\bar{\tau}_{rr}\right) + s\,\tilde{\tau}_{rr}^{(1,3)} + \tilde{\tau}_{rr}^{(1,3)}\frac{d\bar{v}}{dz} + \tilde{v}^{(1,3)}\frac{d\bar{\tau}_{rr}}{dz} =$$
$$\frac{1}{De}\left(\tilde{F}^{(1,1)}\,2\,\tilde{\tau}_{rr}^{(0,2)} + 2\,\tilde{F}^{(0,2)}\,\tilde{\tau}_{rr}^{(1,1)} + \tilde{F}^{(2,2)}\,\tilde{\tau}_{rr}^{\dagger(1,1)} + \tilde{F}^{\dagger(1,1)}\,\tilde{\tau}_{rr}^{(2,2)}\right)$$
$$+ 2\frac{d\tilde{\tau}_{rr}^{(0,2)}}{dz}\tilde{v}^{(1,1)} + \left(l + \frac{ds_r}{dDR}\right)\tilde{\tau}_{rr}^{(1,1)} + 2\frac{d\tilde{v}^{(0,2)}}{dz}\tilde{\tau}_{rr}^{(1,1)} \qquad (17)$$
$$+ 2\,\tilde{\tau}_{rr}^{(0,2)}\frac{d\tilde{v}^{(1,1)}}{dz} + 2\,\tilde{v}^{(0,2)}\frac{d\tilde{\tau}_{rr}^{(1,1)}}{dz} + \frac{d\tilde{\tau}_{rr}^{\dagger(1,1)}}{dz}\tilde{v}^{(2,2)}$$
$$+ \frac{d\tilde{v}^{\dagger(1,1)}}{dz}\tilde{\tau}_{rr}^{(2,2)} + \tilde{\tau}_{rr}^{\dagger(1,1)}\frac{d\tilde{v}^{(2,2)}}{dz} + \tilde{v}^{\dagger(1,1)}\frac{d\tilde{\tau}_{rr}^{(2,2)}}{dz},$$

$$\tilde{F}^{(1,3)} = \frac{1}{\bar{\Lambda}^3}\left(2\,\tilde{\Lambda}^{(1,3)}\left(-1 + e^{\frac{2\,(-1+\bar{\Lambda})}{q}}\,\bar{\Lambda}\left(2\left(-1+\bar{\Lambda}\right)\bar{\Lambda}+q\right)\frac{r}{q}\right)\right)$$
$$+\ (2\,(3\,e^{\frac{2}{q}}\,(\bar{\Lambda}\,2\,\tilde{\Lambda}^{(0,2)}\,\tilde{\Lambda}^{(1,1)} - 2\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{\dagger(1,1)}$$
$$+\ \bar{\Lambda}\,\tilde{\Lambda}^{\dagger(1,1)}\,\tilde{\Lambda}^{(2,2)})\,q^3 + e^{\frac{2\bar{\Lambda}}{q}}\,\bar{\Lambda}\,(3\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{\dagger(1,1)}\,q^3$$
$$-\ 4\,\bar{\Lambda}^3\,(\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{\dagger(1,1)} + 2\,\tilde{\Lambda}^{(0,2)}\,\tilde{\Lambda}^{(1,1)}\,q + \tilde{\Lambda}^{\dagger(1,1)}\,\tilde{\Lambda}^{(2,2)}\,q)$$
$$+\ 4\,\bar{\Lambda}^4\,(\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{\dagger(1,1)} + 2\,\tilde{\Lambda}^{(0,2)}\,\tilde{\Lambda}^{(1,1)}\,q + \tilde{\Lambda}^{\dagger(1,1)}\,\tilde{\Lambda}^{(2,2)}\,q) \tag{18}$$
$$-\ 2\,\bar{\Lambda}\,q^2\,(3\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{\dagger(1,1)} + 2\,\tilde{\Lambda}^{(0,2)}\,\tilde{\Lambda}^{(1,1)}\,q + \tilde{\Lambda}^{\dagger(1,1)}\,\tilde{\Lambda}^{(2,2)}\,q)$$
$$+\ 2\,\bar{\Lambda}^2\,q\,(3\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{(1,1)}\,\tilde{\Lambda}^{\dagger(1,1)} + 4\,\tilde{\Lambda}^{(0,2)}\,\tilde{\Lambda}^{(1,1)}\,q$$
$$+\ 2\,\tilde{\Lambda}^{\dagger(1,1)}\,\tilde{\Lambda}^{(2,2)}\,q))\,r))/(e^{\frac{2}{q}}\,\bar{\Lambda}^5\,q^3),$$

$$\tilde{\Lambda}^{(1,3)} = \frac{\tilde{\tau}_{zz}^{(1,3)} + 2\,\tilde{\tau}_{rr}^{(1,3)}}{6\,\bar{\Lambda}} - \frac{2\,\tilde{\tau}_{zz}^{(0,2)}\,\tilde{\tau}_{zz}^{(1,1)}}{36\,\bar{\Lambda}^3} - \frac{2\,\tilde{\tau}_{rr}^{(0,2)}\,\tilde{\tau}_{zz}^{(1,1)}}{18\,\bar{\Lambda}^3} - \frac{2\,\tilde{\tau}_{zz}^{(0,2)}\,\tilde{\tau}_{rr}^{(1,1)}}{18\,\bar{\Lambda}^3}$$
$$-\ \frac{2\,\tilde{\tau}_{rr}^{(0,2)}\,\tilde{\tau}_{rr}^{(1,1)}}{9\,\bar{\Lambda}^3} + \frac{\tilde{\tau}_{zz}^{(1,1)}\,t\tilde{z}z^{(1,1)}\,\tilde{\tau}_{zz}^{\dagger(1,1)}}{144\,\bar{\Lambda}^5} + \frac{\tilde{\tau}_{zz}^{(1,1)}\,\tilde{\tau}_{rr}^{(1,1)}\,\tilde{\tau}_{zz}^{\dagger(1,1)}}{36\,\bar{\Lambda}^5}$$
$$+\ \frac{\tilde{\tau}_{zz}^{(1,1)}\,t\tilde{z}z^{(1,1)}\,\tilde{\tau}_{rr}^{\dagger(1,1)}}{72\,\bar{\Lambda}^5} + \frac{\tilde{\tau}_{zz}^{(1,1)}\,\tilde{\tau}_{rr}^{(1,1)}\,\tilde{\tau}_{rr}^{\dagger(1,1)}}{18\,\bar{\Lambda}^5} + \frac{\tilde{\tau}_{rr}^{(1,1)}\,t\tilde{r}r^{(1,1)}\,\tilde{\tau}_{rr}^{\dagger(1,1)}}{18\,\bar{\Lambda}^5} \tag{19}$$
$$+\ \frac{\tilde{\tau}_{rr}^{(1,1)}\,t\tilde{r}r^{(1,1)}\,\tilde{\tau}_{zz}^{\dagger(1,1)}}{36\,\bar{\Lambda}^5} - \frac{\tilde{\tau}_{zz}^{\dagger(1,1)}\,\tilde{\tau}_{zz}^{(2,2)}}{36\,\bar{\Lambda}^3} - \frac{\tilde{\tau}_{rr}^{\dagger(1,1)}\,\tilde{\tau}_{zz}^{(2,2)}}{18\,\bar{\Lambda}^3} - \frac{\tilde{\tau}_{zz}^{\dagger(1,1)}\,\tilde{\tau}_{rr}^{(2,2)}}{18\,\bar{\Lambda}^3}$$
$$-\ \frac{\tilde{\tau}_{rr}^{\dagger(1,1)}\,\tilde{\tau}_{rr}^{(2,2)}}{9\,\bar{\Lambda}^3}.$$

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