Appendix A. Supplementary Material

A.1. *Calculation of $f^{(0)}$*

We seek the solution to

$$\frac{\partial}{\partial\theta}\left(yPe(1-\beta\cos 2\theta)f^{(0)}-\frac{\partial f^{(0)}}{\partial\theta}\right)+\sigma(f^{(0)}-\frac{1}{2\pi})=0, \tag{A 1}$$

with normalisation condition

$$\int_0^{2\pi} f^{(0)}(\theta;y)d\theta=1. \tag{A 2}$$

As we require periodicity $f^{(0)}(\theta;y)=f^{(0)}((2\pi+\theta);y)$, we can seek a solution as a Fourier Series:

$$f=\frac{1}{2\pi}\left(\frac{a_0}{2}+\sum_1^\infty a_k\cos k\theta+b_k\sin k\theta\right),$$

where the a_k and b_k for $k>0$ wil be functions of y, and $a_0=2$ to ensure the normalisation condition is satisfied.

On inserting this expression into equation (A 1), we obtain

$$\begin{aligned}
&yPe\sum_1^\infty -ka_k\sin k\theta+kb_k\cos k\theta\\
&\quad+yPe\frac{\beta}{2}\left(a_1\sin\theta+b_1\cos\theta+\sum_1^\infty k(a_{k+2}+a_{k-2})\sin k\theta-\sum_1^\infty k(b_{k+2}+b_{k-2})\cos k\theta\right)\\
&\quad+\sum_1^\infty(k^2+\sigma)\left(a_k\cos k\theta+b_k\sin k\theta\right)=0,
\end{aligned} \tag{A 3}$$

where $a_k=0$ for $k<0$ and $b_k=0$ for $k<1$. By the orthogonality of the Fourier modes, on truncating the expansion to a finite sum, we can obtain solutions for the coefficients for specified values of the parameters yPe, β and σ. In particular we notice that the odd and even in k modes are uncoupled, and thus find that all odd modes are equal to zero.

A.2. *$Pe \lll 1$ asymptotics*

For $Pe\ll 1$ we can obtain an approximate analytic expression for $f^{(0)}$. Specifically we expand in powers of Pe to obtain a solution correct to $O(Pe^2)$:

$$f^{(0)}=\frac{1}{2\pi}\left(f^{(00)}+Pef^{(01)}+Pe^2f^{(02)}\right). \tag{A 4}$$

with integral constraint $\int_0^{2\pi}f^{(00)}d\theta=2\pi, \int_0^{2\pi}f^{(01)}d\theta=0, \int_0^{2\pi}f^{(02)}d\theta=0$. Substituting into equation (A 1) we obtain at $O(1)$:

$$-\frac{\partial^2}{\partial\theta^2}f^{(00)}+\sigma(f^{(00)}-\frac{1}{2\pi})=0, \tag{A 5}$$

which has solution $f^{(00)}=1$. At $O(Pe)$ equation (A 1) gives

$$\frac{\partial^2}{\partial\theta^2}f^{(01)}-\sigma f^{(01)}=\frac{\partial}{\partial\theta}\left(y(1-\beta\cos 2\theta)\right)=2y\beta\sin\theta, \tag{A 6}$$

which has solution

$$f^{(01)}(\theta;y)=-\frac{\beta y}{2(1+\sigma/4)}\sin 2\theta \tag{A 7}$$

At $O(Pe^2)$ equation (A 1) gives

$$\frac{\partial^2}{\partial\theta^2}f^{(02)} - \sigma f^{(02)} = \frac{\partial}{\partial\theta}\left(y(1-\beta\cos 2\theta)f^{(01)}\right) = \frac{\beta y^2}{1+\sigma/4}(\beta\cos 4\theta - \cos 2\theta) \quad \text{(A 8)}$$

which has solution

$$f^{(02)}(\theta;y) = \frac{\beta y^2 Pe^2}{4(1+\sigma/4)}\left(\frac{\cos 2\theta}{1+\sigma/4} - \frac{\beta\cos 4\theta}{4+\sigma/4}\right). \quad \text{(A 9)}$$

The mean squared vertical swimming speed can then be computed to $O(Pe^2)$:

$$V_{MS}(y) = \int_0^{2\pi}\sin^2\theta f^{(0)}d\theta = \frac{1}{2\pi}\int\sin^2\theta\left(f^{(00)} + Pe f^{(01)} + Pe^2 f^{(02)}\right)d\theta$$
$$= \frac{1}{2}\left(1 - \frac{\beta y^2 Pe^2}{8(1+\sigma/4)^2}\right). \quad \text{(A 10)}$$

A.3. *Presence of Eigenfunctions*

Consider the governing equation for ψ when $Pe = 0$ and $d = 0$:

$$\epsilon\sin\theta\frac{\partial\psi}{\partial y} - \frac{\partial^2\psi}{\partial\theta^2} + (\sigma - \epsilon\chi\sin\theta)\psi - \frac{1}{2\pi}\int_0^{2\pi}(\sigma - \epsilon\chi\sin\theta')\psi(y,\theta')d\theta' = 0. \quad \text{(A 11)}$$

On applying the conditions $\int_0^{2\pi}\psi(y,\theta)d\theta = n(y)$, $\int_0^{2\pi}\psi(y,\theta)\sin\theta d\theta = 0$, the governing equation simplifies to

$$\epsilon\sin\theta\frac{\partial\psi}{\partial y} - \frac{\partial^2\psi}{\partial\theta^2} + (\sigma - \epsilon\chi\sin\theta)\psi - \frac{1}{2\pi}\sigma n(y) = 0. \quad \text{(A 12)}$$

As we require a solution periodic in θ, we seek a Fourier Series expansion:

$$\psi = \frac{e^{\chi y}}{2\pi}\left(\frac{a_0(y)}{2} + \sum_1^\infty a_k(y)\cos k\theta + b_k(y)\sin k\theta\right).$$

The condition $\int_0^{2\pi}\psi(y,\theta)\sin\theta d\theta = 0$ gives $b_1 = 0$. The symmetry condition $\psi(y,\theta;Pe) = \psi(y,\pi-\theta;-Pe)$ gives $a_{k+1} = b_k = 0$ for k even. The symmetry condition $\psi(y,\theta;\chi) = \psi(-y,-\theta;-\chi)$ gives $a_k(y)$ is an even function of y and $b_k(y)$ is an odd function of y. We thus rewrite the solution as

$$\psi = \frac{e^{\chi y}}{2\pi}\left(\frac{P_0(y)}{2} + \sum_1^\infty P_k(y)\cos 2k\theta + Q_k(y)\sin(2k+1)\theta\right),$$

where $P_k(y)$ is an even function of y and $Q_k(y)$ is an odd function of y. By orthogonality of the eigenmodes, we obtain the following relations between the Fourier coefficients:

$$P_k' - P_{k+1}' + \frac{2((2k+1)^2+\sigma)}{\epsilon}Q_k = 0, \quad \text{(A 13)}$$

$$Q_k' - Q_{k+1}' + \frac{2((2k)^2+\sigma)}{\epsilon}P_k = 0. \quad \text{(A 14)}$$

Setting $k = 0$ in equation (A 16) gives

$$P_0 = 1 - \int_{-1}^1 P_1(y)dy + P_1(y). \quad \text{(A 15)}$$

If P_1 contains eigenfunction solutions, P_0 and hence the cell concentration will not be

uniquely determined. If we truncate the system at $N = 1$ we have

$$P_1' + \frac{2(9+\sigma)}{\epsilon} Q_1 = 0, \tag{A 16}$$

$$Q_1' + \frac{2(4+\sigma)}{\epsilon} P_1 = 0, \tag{A 17}$$

which has solution

$$\begin{pmatrix} P_1 \\ Q_1 \end{pmatrix} = C_1 \boldsymbol{e}_+ e^{\frac{2}{\epsilon}\lambda y} + C_2 \boldsymbol{e}_- e^{-\frac{2}{\epsilon}\lambda y} \tag{A 18}$$

where

$$\boldsymbol{e}_\pm = \left(1, \pm\sqrt{\frac{\sigma+4}{\sigma+9}}\right)^T, \quad \lambda = \sqrt{(\sigma+4)(\sigma+9)}. \tag{A 19}$$

The symmetry condition that $P_k(y)$ is an even function of y and $Q_k(y)$ is an odd function of y requires $C_1 = C_2$.

In general, there are many eigenmode solutions of the form

$$P_k = \alpha_k \cosh\left(\frac{2}{\epsilon}\lambda y\right), \quad Q_k = \beta_k \sinh\left(\frac{2}{\epsilon}\lambda y\right), \tag{A 20}$$

where λ is an eigenvalue, α_k and β_k are components of the corresponding eigenvector, and both eigenvalues and eigenvectors only depend on σ and the truncation value. However, if we enforce the condition that $\sin\theta\psi = 0$ on the boundaries, then $\psi = 0$ on the boundaries and this ensures that all coefficients in the eigenfunctions are zero.

A.4. *Effect of shape on equilibrium distributionwith cross channel chemotactic gradient.* The data corresponding to figure 2 for chemotactic cells is plotted in figure A.1. Similar effects are seen with changes in β and Pe as for the non-chemotactic case, although the distribution is somewhat dominated by the exponential distribution. We see that non-monotonic centreline depletion with increasing Pe still occurs at smaller values of β, for example $\beta = 0.2, 0.4$, as shown in figure A.1f.

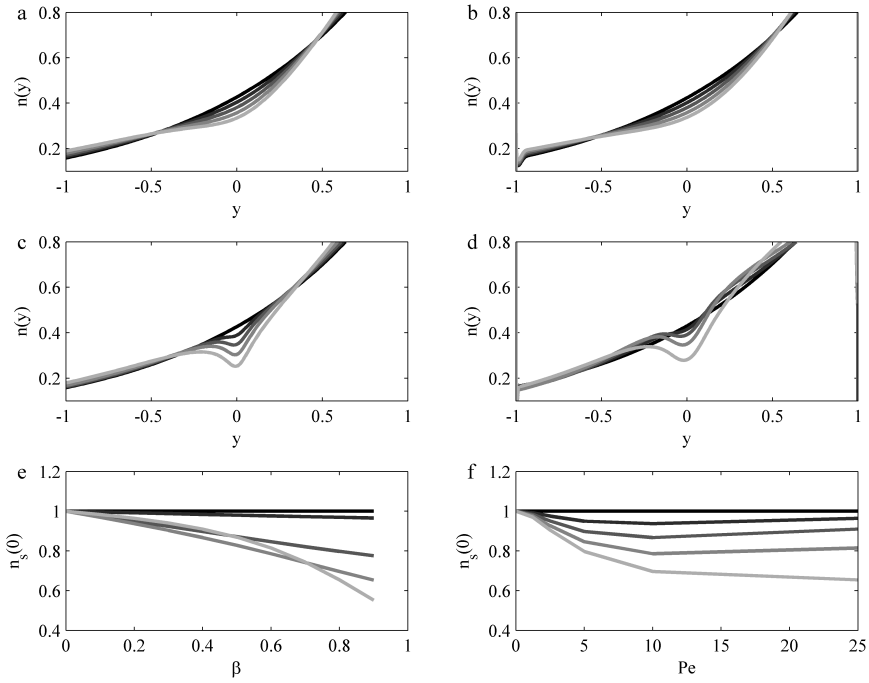

FIGURE A.1. The influence of cell shape, β, on equilibrium cell concentration with $\chi = 0.99, \sigma = 2$. (a,b) correspond to $Pe = 5$ and (c,d) to $Pe = 25$. (a,c) Leading order in ϵ asymptotic approximation, other plots correspond to numerical solution of ψ equation. In (e,f) $n_s(0)$ is the cell concentration at $y = 0$ normalised by the $\beta = 0$ solution at the corresponding Pe. In (a-d,f) greyscale indicate $\beta = [0, 0.2, 0.4, 0.6, 0.8]$ with black corresponding to 0; in (e) greyscale indicate $Pe = [0, 1.25, 5, 10, 25]$ with black corresponding to 0.