Appendix to "Statistical accuracy of scattered points filters and application to the dynamics of bubbles in gas-fluidised beds"

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Analytical derivation of the statistical relative error

Details of the analytical deduction of the statistical relative errors shown in (3.4) and (3.13) are provided in this appendix.

Interpolative filters

Expansion of the numerator of (3.1) leads to:

$$\left\langle \left(\varphi(\mathbf{x}) - \tilde{\phi}(\mathbf{x})\right)^2 \right\rangle_{\psi,k,\varepsilon}^{1/2} = \left[\left\langle \varphi(\mathbf{x})^2 \right\rangle_{\psi} - 2\left\langle \varphi(\mathbf{x})\tilde{\phi}(\mathbf{x}) \right\rangle_{\psi,k,\varepsilon} + \left\langle \tilde{\phi}(\mathbf{x})^2 \right\rangle_{\psi,k,\varepsilon} \right]^{1/2} \tag{1}$$

The statistical average of the first term on the right hand side of (1) does not sample the contribution of filter points location or the measurement error or perturbing stochastic data, since $\varphi(\mathbf{x})$ is the exact data field to be measured. Thus only the contribution of all possible data phases of the signal is taken into account. As the value of the phase ψ is considered homogeneously distributed over all its possible realisations, (1) is independent of $\mathbf{x}$ and φ can be arbitrarily referred to the origin of coordinates. Therefore:

$$\left\langle \varphi(\mathbf{x})^2 \right\rangle_{\psi} = \left\langle \varphi(0)^2 \right\rangle_{\psi} = \sum_{\mathbf{L}^{(1)},\mathbf{L}^{(2)}=0}^{\infty} \hat{\varphi}_{\mathbf{L}^{(1)}} \hat{\varphi}_{\mathbf{L}^{(2)}} \prod_{i,j=1}^{D} \left\langle \sin\left(\psi_{i,l_i^{(1)}}\right) \sin\left(\psi_{j,l_j^{(2)}}\right) \right\rangle_{\psi} \tag{2}$$

Here the compact notations $\sum_{\mathbf{L}^{(1)},\mathbf{L}^{(2)}=0}^{\infty} = \sum_{\mathbf{L}^{(1)}=0}^{\infty} \sum_{\mathbf{L}^{(2)}=0}^{\infty} = \sum_{l_1^{(1)}=0}^{\infty} \cdots \sum_{l_D^{(1)}=0}^{\infty} \sum_{l_1^{(2)}=0}^{\infty} \cdots \sum_{l_D^{(2)}=0}^{\infty}$ and $\prod_{i,j=1}^{D} = \prod_{j=1}^{D} \prod_{j=1}^{D}$ have been adopted. In a general signal, the phases between different dimensions as well as between different Fourier index components are uncorrelated, so:

$$\left\langle \sin\left(\psi_{i,l_i^{(1)}}\right) \sin\left(\psi_{j,l_j^{(1)}}\right) \right\rangle_{\psi} = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sin^2(\psi) d\psi = \dfrac{1}{2} & \text{if} \quad i = j \quad \text{and} \quad l_i^{(1)} = l_j^{(2)} \\ \left[\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \sin(\psi) d\psi\right]^2 = 0 & \text{otherwise.} \end{cases} \tag{3}$$

which greatly simplifies (2) to give:

$$\langle \varphi(\mathbf{x})^2 \rangle_{\psi} = \frac{1}{2^D} \sum_{\mathbf{L}=0}^{\infty} \hat{\varphi}_{\mathbf{L}}^2 \tag{4}$$

A similar procedure can be followed to evaluate the second term on the right hand side of (1). However, the filtered signal $\tilde{\phi}$ is now divided into the deterministic and the noise, or stochastic, components:

$$\langle \varphi(\mathbf{x})\tilde{\phi}(\mathbf{x})\rangle_{\psi,k,\varepsilon} = \langle \varphi(\mathbf{x})\tilde{\varphi}(\mathbf{x})\rangle_{\psi,k} + \langle \varphi(\mathbf{x})\tilde{\varepsilon}(\mathbf{x})\rangle_{\psi,k,\varepsilon} \tag{5}$$

Here, the second term on the right hand side of (5) is null since, by definition, $\tilde{\varepsilon}$ is uncorrelated with φ. The statistical average of the deterministic component can be expanded as a series representing the Fourier expansion over all the data points affected by the filter. The average is performed first for the phases:

$$\begin{aligned} \langle \varphi(\mathbf{x})\tilde{\varphi}(\mathbf{x})\rangle_{\psi} &= \langle \varphi(0)\tilde{\varphi}(0)\rangle_{\psi} \\ &= \frac{\sum_{k=1}^{N_p}\sum_{\mathbf{L}^{(1)},\mathbf{L}^{(2)}}^{\infty} \hat{\varphi}_{\mathbf{L}^{(1)}}\hat{\varphi}_{\mathbf{L}^{(2)}} g_I(\mathbf{x}_k)\prod_{i,j=1}^{D}\left\langle \sin\left(\psi_{i,l_i^{(1)}}\right)\sin\left(x_{k,j}F_{j,l_j^{(2)}}+\psi_{j,l_j^{(2)}}\right)\right\rangle_{\psi}}{\sum_{k=1}^{N_p} g_I(\mathbf{x}_k)} \end{aligned} \tag{6}$$

Upon applying (3) and recognizing that $\int_0^{2\pi}\sin(\psi)\cos(\psi)d\psi = 0$

$$\langle \varphi(\mathbf{x})\tilde{\varphi}(\mathbf{x})\rangle_{\psi} = \frac{\frac{1}{2^D}\sum_{k=1}^{N_p}\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}^2_{\mathbf{L}=0} g_I(\mathbf{x}_k)\prod_{i,j=1}^{D}\cos\left(x_{k,j}F_{j,l_j}\right)}{\sum_{k=1}^{N_p} g_I(\mathbf{x}_k)} \tag{7}$$

To make affordable the sampling of all the possible point locations $\mathbf{x}_k$ in the statistical average, the following approximation is made:

$$\langle \varphi(\mathbf{x})\tilde{\varphi}(\mathbf{x})\rangle_{\psi,k} \approx \frac{\frac{1}{2^D}\sum_{k=1}^{N_p}\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}^2_{\mathbf{L}}\langle g_I(\mathbf{x}_k)\prod_{i=1}^{D}\cos\left(x_{k,i}F_{i,l_i}\right)\rangle_k}{\langle\sum_{k=1}^{N_p} g_I(\mathbf{x}_k)\rangle_k} \tag{8}$$

Thus, using (2.11) for a window of rectangular section,

$$\langle \varphi(\mathbf{x})\tilde{\varphi}(\mathbf{x})\rangle_{\psi,k} \approx \frac{\frac{1}{V_F 2^D}\sum_{k=1}^{N_p}\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}^2_{\mathbf{L}}\prod_{j=1}^{D}\int_{-L_{F,j}}^{L_{F,j}} h(x_{k,j})\cos(x_j F_{j,l_j})dx_{k,j}}{\frac{1}{V_F}\sum_{k=1}^{N_p}\prod_{j=1}^{D}\int_{-L_{F,j}}^{L_{F,j}} h(x_{k,j})dx_{k,j}} \tag{9}$$

Also, from (2.7), the following result is obtained:

$$\left\langle \frac{1}{V_F}\sum_{k=1}^{N_p}\prod_{j=1}^{D}\int_{-L_{F,j}}^{L_{F,j}} h(x_{k,j})dx_{k,j}\right\rangle_k = \frac{N_p}{V_F} \tag{10}$$

The approximation of (8) is valid provided N_p is sufficiently high, since a large number of points within the filter window makes the denominator of the right hand side of (7) insensitive to the particular location of each point. Given (10), the approximation shown in (8) is equivalent of simplifying the filter denominator with (2.8),

$$\langle \varphi(\mathbf{x})\tilde{\varphi}(\mathbf{x})\rangle_{\psi,k} \approx \frac{\frac{1}{V_F 2^D}\sum_{k=1}^{N_p}\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}^2_{\mathbf{L}=0}\prod_{j=1}^{D}\int_{-L_{F,j}}^{L_{F,j}} h(x_{k,j})\cos(x_j F_{j,l_j})dx_{k,j}}{\frac{N_p}{V_F}} \tag{11}$$

Defining the pseudo-spectral transformation along the i direction as

$$S_{l_j} = \int_{-L_f}^{L_f} h(s)\cos(sF_{j,l_j})ds \tag{12}$$

so that $\delta_{\mathbf{L}} = \prod_{j=1}^{D} S_{l_j}$, the structure of (8) further simplifies

$$\langle \varphi(\mathbf{x})\tilde{\varphi}(\mathbf{x})\rangle_{\psi,k} \approx \frac{1}{2^D}\sum_{\mathbf{L}=0}^{\infty} \hat{\varphi}^2_{\mathbf{L}}\delta_{\mathbf{L}} \tag{13}$$

The third term on the right-hand side of (1) is treated similarly, but it involves the product of two filtered sets of data. This fact has implications for the way in wich the denominator of the filter in (2.6) can be handled. As above, the contribution of the deterministic and stochastic components of the filtered signal can be separated:

$$\langle \tilde{\phi}(\mathbf{x})\tilde{\phi}(\mathbf{x})\rangle_{\psi,k,\varepsilon} = \langle \tilde{\varphi}(\mathbf{x})^2\rangle_{\psi,k} + \langle \tilde{\varepsilon}(\mathbf{x})^2\rangle_{k,\varepsilon}, \tag{14}$$

with $\langle \tilde{\varphi}(\mathbf{x})\tilde{\varepsilon}(\mathbf{x})\rangle_{\psi,k,\varepsilon} = 0$ because φ and ε (and hence their filtered signals) are uncorrelated, by definition. Performing first the statistical average over phases of the deterministic filtered signal component:

$$\begin{aligned}\langle \tilde{\varphi}(\mathbf{x})^2\rangle_{\psi} &= \langle \tilde{\varphi}(0)^2\rangle_{\psi} \\ &= \frac{\sum_{k^{(1)},k^{(2)}=1}^{N_p}\sum_{\mathbf{L}^{(1)},\mathbf{L}^{(2)}=0}^{\infty} \hat{\varphi}_{\mathbf{L}^{(1)}}\hat{\varphi}_{\mathbf{L}^{(2)}} g_I(\mathbf{x}_{k^{(1)}})g_I(\mathbf{x}_{k^{(2)}})P(\mathbf{x}_{k^{(1)}},\mathbf{x}_{k^{(2)}},\mathbf{L}^{(1)},\mathbf{L}^{(2)})}{\sum_{k^{(1)},k^{(2)}=1}^{N_p} g_I(\mathbf{x}_{k^{(1)}})g_I(\mathbf{x}_{k^{(2)}})}\end{aligned} \tag{15}$$

where,

$$P(\mathbf{x}_{k^{(1)}},\mathbf{x}_{k^{(2)}},\mathbf{L}^{(1)},\mathbf{L}^{(2)}) = \prod_{i,j=1}^{D}\left\langle \sin\left(x_{k^{(1)},i}F_{i,l_i^{(1)}} + \psi_{i,l_i^{(1)}}\right)\sin\left(x_{k^{(2)},j}F_{j,l_j^{(2)}} + \psi_{j,l_j^{(2)}}\right)\right\rangle_{\psi} \tag{16}$$

In a similar fashion to the way in which (3) was treated:

$$\begin{aligned}&\left\langle \sin\left(x_{k^{(1)},i}F_{i,l_i^{(1)}} + \psi_{i,l_i^{(1)}}\right)\sin\left(x_{k^{(2)},j}F_{j,l_j^{(2)}} + \psi_{j,l_j^{(2)}}\right)\right\rangle_{\psi} \\ &= \begin{cases} \frac{1}{2}Q(x_{k^{(1)},i}, x_{k^{(2)},j}, F_{i,l_i^{(1)}}) & \text{if} \quad i=j \quad \text{and } l_i^{(1)} = l_j^{(2)} \\ 0 \quad \text{otherwise.}\end{cases}\end{aligned} \tag{17}$$

where,

$$\begin{aligned}&Q(x_{k^{(1)},i}, x_{k^{(2)},i}, F^{(1)}_{i,l_i}) \\ &= \sin(x_{k^{(1)},i}, F^{(1)}_{i,l_i})\sin(x_{k^{(2)},i}, F^{(1)}_{i,l_i}) + \cos(x_{k^{(1)},i}, F^{(1)}_{i,l_i})\cos(x_{k^{(2)},i}, F^{(1)}_{i,l_i})\end{aligned} \tag{18}$$

Following (8), the statistical average of (15) over the locations of all the sampling points is approximated by:

$$\begin{aligned}&\left\langle \tilde{\varphi}(\mathbf{x})^2\right\rangle_{\psi,k} \approx \\ &\frac{\frac{1}{2^D}\sum_{k=1}^{N_p}\sum_{\mathbf{L}=\mathbf{0}}^{\infty}\hat{\varphi}^2_{\mathbf{L}}\left\langle \prod_{i=1}^{D} h(x_{k^{(1)},i})h(x_{k^{(2)},i})Q(x_{k^{(1)},i},x_{k^{(2)},i},F_{i,l_i})\right\rangle_k}{\langle\left[\sum_{k=1}^{N_p} g_I(\mathbf{x}_k)\right]^2\rangle_k}\end{aligned} \tag{19}$$

In the summation in (19), there are $N_p^2 - N_p$ combinations of the pair $\left(k^{(1)}, k^{(2)}\right)$ in

which $\mathbf{x}_{k^{(1)}}$ and $\mathbf{x}_{k^{(2)}}$ are uncorrelated, and N_p combinations corresponding to the case $\mathbf{x}_{k^{(1)}} = \mathbf{x}_{k^{(2)}}$. Besides, due to the symmetry of h with respect to the centre of the filter, sine functions in (19) lead to zero contributions in the statistical average if $\mathbf{x}_{k^{(1)}}$ and $\mathbf{x}_{k^{(2)}}$ are uncorrelated. Therefore:

$$\begin{aligned}&\left\langle \prod_{i=1}^{D} h(x_{k^{(1)},i})h(x_{k^{(2)},i})Q(x_{k^{(1)},i}, x_{k^{(2)},i}, F_{i,l_i})\right\rangle_k \\ &= \frac{(N_p^2-N_p)}{V_F^2}\left[\prod_{i=1}^{D}\int_{-L_{F,i}}^{L_{F,i}} h(x_{k,i})cos(x_{k,i}F_{i,l_i})dx_{k,i}\right]^2 \\ &\quad + \frac{N_p}{V_F}\prod_{i=1}^{D}\int_{-L_{F,i}}^{L_{F,i}} h^2(x_{k,i})dx_{k,i}\end{aligned} \tag{20}$$

and

$$\left\langle \left[\sum_{k=1}^{N_p} g_I(\mathbf{x}_k)\right]^2\right\rangle_k = \frac{(N_p^2-N_p)}{V_F^2}\left[\prod_{i=1}^{D}\int_{-L_{F,i}}^{L_{F,i}} h(x_{k,i})dx_{k,i}\right]^2 + \frac{N_p}{V_F}\prod_{i=1}^{D}\int_{-L_{F,i}}^{L_{F,i}} h^2(x_{k,i})dx_{k,i} \tag{21}$$

Indroducing (20) and (21) into (19) the following is obtained:

$$\langle\tilde{\varphi}(\mathbf{x})^2\rangle_\psi \approx \frac{1}{2^D}\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}_{\mathbf{L}}^2\left[\frac{(N_p^2-N_p)\delta_{\mathbf{L}}^2 + N_pG_I}{N_p^2-N_p+N_pG_I}\right] \tag{22}$$

where $G_I = V_F\prod_{i=1}^{D}\int_{-L_{F,i}}^{L_{F,i}} h^2(s)ds$

The noise or non-deterministic component of the filtered signal has a contribution in (14) wich can be expressed as:

$$\begin{aligned}\left\langle\tilde{\varepsilon}(\mathbf{x})^2\right\rangle_{k,\varepsilon} &= \left\langle\frac{\sum_{k^{(1)},k^{(2)}=0}^{N_p} g_I(\mathbf{x}_{k^{(1)}})g_I(\mathbf{x}_{k^{(2)}})\varepsilon_{k^{(1)}}\varepsilon_{k^{(2)}}}{\sum_{k^{(1)},k^{(2)}=0}^{N_p} g_I(\mathbf{x}_{k^{(1)}})g_I(\mathbf{x}_{k^{(2)}})}\right\rangle_{k,\varepsilon} \\ &\approx \frac{\left\langle\sum_{k=0}^{N_p} g_I(\mathbf{x}_k)^2\right\rangle_k\sigma_\varepsilon^2}{\left\langle\left[\sum_{k=1}^{N_p} g_I(\mathbf{x}_k)\right]^2\right\rangle_k} = \frac{G_I}{(N_p-1)+G_I}\sigma_\varepsilon^2\end{aligned} \tag{23}$$

After combining all the simplified exppresions, (4), (13) and (22) in (3.1), the next closed form solution surfaces up for the estimation of the relative error of interpolation:

$$\frac{\left\langle\left(\varphi(\mathbf{x})-\tilde{\phi}(\mathbf{x})\right)^2\right\rangle_{\psi,k,\varepsilon}^{1/2}}{\langle\varphi(\mathbf{x})^2\rangle_{\psi,k}^{1/2}} \approx \frac{\left[\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}_{\mathbf{L}}^2\frac{(N_p-1)(1-\delta_L)^2+2G_I(1-\delta_L)}{N_p-1+G_I} + \frac{2^DG_I}{N_p-1+G_I}\sigma_\varepsilon^2\right]^{1/2}}{\left[\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}_{\mathbf{L}}^2\right]^{1/2}} \tag{24}$$

Differentiation filters

The statistical error of differentiation filters, (3.13), can be deduced using a similar methodology to that utilised for interpolative filters. As in (1), expansion of the numerator of (3.12) leads to three terms

$$\left\langle\left(\varphi^d(\mathbf{x})-\tilde{\tilde{\phi}}(\mathbf{x})\right)^2\right\rangle_{\psi,k,\varepsilon}^{1/2} = \left[\langle\varphi^d(\mathbf{x})^2\rangle_{\psi,\varepsilon} - 2\langle\varphi^d(\mathbf{x})\tilde{\tilde{\phi}}(\mathbf{x})\rangle_{\psi,k,\varepsilon} + \langle\tilde{\tilde{\phi}}(\mathbf{x})^2\rangle_{\psi,k,\varepsilon}\right]^{1/2} \tag{25}$$

In (25), the statistical average of the differentiated data over all phase values for each component of the Fourier transform was calculated:

$$\langle\varphi^d(\mathbf{x})^2\rangle_\psi = \langle\varphi^d(0)^2\rangle_\psi$$
$$= \sum_{\mathbf{L}^{(1)},\mathbf{L}^{(2)}=0}^{\infty} \hat{\varphi}_{\mathbf{L}^{(1)}}\hat{\varphi}_{\mathbf{L}^{(2)}} F_{1,l_1^{(1)}} F_{1,l_1^{(2)}} \left\langle \cos\left(\psi_{1,l_1^{(1)}}\right)\cos\left(\psi_{1,l_1^{(2)}}\right)\prod_{i,j=2}^{D}\sin\left(\psi_{i,l_i^{(1)}}\right)\sin\left(\psi_{j,l_j^{(2)}}\right)\right\rangle_\psi$$
$$= \tfrac{1}{2^D}\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}_{\mathbf{L}}^2 F_{1,l_1}^2 \tag{26}$$

Then, similarly to (5), the second term of ther right hand side of (25) can be descomposed into the contributions of the deterministic and stochastic signals filtered by the differentiation filter, $\tilde{\tilde{\varphi}}$ and $\tilde{\tilde{\varepsilon}}$ respectively:

$$\langle\varphi^d(\mathbf{x})\tilde{\tilde{\phi}}(\mathbf{x})\rangle_{\psi,k,\varepsilon} = \langle\varphi^d(\mathbf{x})\tilde{\tilde{\varphi}}(\mathbf{x})\rangle_{\psi,k} + \langle\varphi^d(\mathbf{x})\tilde{\tilde{\varepsilon}}(\mathbf{x})\rangle_{\psi,k,\varepsilon} \tag{27}$$

The differentition filter, (2.15), can be rearranged in a more convenient way for averaging operations:

$$\frac{\partial\tilde{\phi}(\mathbf{x})}{\partial x_1} \approx \frac{\frac{1}{N_p}\sum_{k,s=1}^{N_p}\left(g_D(\mathbf{x}_k-\mathbf{x}) - g_D(\mathbf{x}_s-\mathbf{x})\right)\phi_k}{\sum_{k=1}^{N_p} g_I(\mathbf{x}_k-\mathbf{x})} \tag{28}$$

Using the above expression, and performing a procedure analogous to that employed in interpolating filters, it can be seen that the deterministic contribution in (27) transforms to:

$$\langle\varphi^d(\mathbf{x})\tilde{\tilde{\varphi}}(\mathbf{x})\rangle_{\psi,k}$$
$$\approx \frac{\frac{1}{2^D N_p}\sum_{k,s=1}^{N_p}\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}_{\mathbf{L}}^2 F_{1,l_1}\left\langle\left(g_D(\mathbf{x}_k)-g_D(\mathbf{x}_s)\right)\sin(x_{k,1}F_{1,l_1})\prod_{i=2}^{D}\cos\left(x_{k,i}F_{i,l_i}\right)\right\rangle_{k,s}}{\left\langle\sum_{k=1}^{N_p} g_I(\mathbf{x}_k)\right\rangle_k} \tag{29}$$

Therefore:

$$\langle\varphi^d(\mathbf{x})\tilde{\tilde{\varphi}}(\mathbf{x})\rangle_{\psi,k} \approx \frac{1}{2^D}\sum_{\mathbf{L}=0}^{\infty}\hat{\varphi}_{\mathbf{L}}^2 F_{1,l_1}\gamma_{\mathbf{L}}\left(1-\frac{1}{N_p}\right) \tag{30}$$

where $\gamma_{\mathbf{L}} = R_{l_1}\prod_{j=2}^{D} S_{l_j}$ with $R_{l_i} = \int_{-L_F}^{L_F} h'(s)\sin(sF_{1,l_1})ds$. Provided g_I is null at the boundaries, it can be demostrated that $\gamma_{\mathbf{l}} = F_{l_1}\delta_{\mathbf{L}}$.

The third term on the right hand side of (25) leads to the deterministic and stochastic contributions as in interpolating filters:

$$\left\langle\tilde{\tilde{\phi}}(\mathbf{x})\tilde{\tilde{\phi}}(\mathbf{x})\right\rangle_{\psi,k,\varepsilon} = \left\langle\tilde{\tilde{\varphi}}(\mathbf{x})^2)\right\rangle_{\psi,k} + \left\langle\tilde{\tilde{\varepsilon}}(\mathbf{x})^2)\right\rangle_{k,\varepsilon} \tag{31}$$

However, the presence of $\sum_{s=1}^{N_p}\phi_s$ in the filter (2.15), makes the formulation of $\left\langle\tilde{\tilde{\varphi}}(\mathbf{x})^2)\right\rangle_{\psi,k}$ more intricate than that of $\left\langle\tilde{\varphi}(\mathbf{x})^2)\right\rangle_{\psi,k}$:

$$\left\langle\tilde{\tilde{\varphi}}(\mathbf{x})^2)\right\rangle_{\psi,k} = \left\langle\tilde{\tilde{\varphi}}(0)^2)\right\rangle_{\psi,k}$$

$$\approx \frac{\sum\limits_{k^{(1)},k^{(2)},s^{(1)},s^{(2)}=1}^{N_p} \sum\limits_{\mathbf{L}=0}^{\infty} \hat{\varphi}_{\mathbf{L}}^2 \left\langle A\left(\mathbf{x}_{k^{(1)}},\mathbf{x}_{k^{(2)}},\mathbf{x}_{s^{(1)}},\mathbf{x}_{s^{(2)}}\right) \prod\limits_{i=1}^{D} Q(x_{k^{(1)},i},x_{k^{(2)},i},F_{i,l_i}) \right\rangle_{k,s}}{2^D N_p^2 \sum\limits_{k^{(1)},k^{(2)}=1}^{N_p} \left\langle g_I(\mathbf{x}_{k^{(1)}}) g_I(\mathbf{x}_{k^{(2)}}) \right\rangle_k} \tag{32}$$

where:

$$A\left(\mathbf{x}_{k^{(1)}},\mathbf{x}_{k^{(2)}},\mathbf{x}_{s^{(1)}},\mathbf{x}_{s^{(2)}}\right) = \left(g_D(\mathbf{x}_{k^{(1)}}) - g_D(\mathbf{x}_{s^{(1)}})\right)\left(g_D(\mathbf{x}_{k^{(2)}}) - g_D(\mathbf{x}_{s^{(2)}})\right)$$

When performing the statistical average aver all possible locations of data points within the filter window, as was done in (19), several combinations of the quadruplet $\left(k^{(1)},k^{(2)},s^{(1)},s^{(2)}\right)$ appear leading to non-null terms:

$$\left\langle \tilde{\tilde{\varphi}}(\mathbf{x})^2) \right\rangle_{\psi,k}$$

$$\approx \frac{\frac{1}{2^D}\sum_{\mathbf{L}=0}^{\infty} \hat{\varphi}_{\mathbf{L}}^2 F_{1,l_1}^2 (N_p-1)\left[\left[(N_p-1)^2+1\right]\delta_L^2 - 2(N_p-1)\xi_L\alpha_L + \left[(N_p-2)\alpha_L^2+N_p\right]\frac{G_D}{F_{1,l_1}^2}\right]}{N_p^3-N_p^2+N_p^2 G_I} \tag{33}$$

Here, α_L is given by (3.22),

$$\xi_{\mathbf{L}} = \frac{V_F}{F_{1,l_1}^2}\int_{-L_{F,i}}^{L_{F,i}} h'^2(s)\cos(sF_{1,l_1})ds \prod_{i=2}^{D}\int_{-L_{F,i}}^{L_{F,i}} h^2(s)\cos(sF_{1,l_1})ds$$

by (3.23) and $G_D = V_F \int_{-L_{F,i}}^{L_{F,i}} h'^2 ds \left(\int_{-L_{F,i}}^{L_{F,i}} h^2(s)ds\right)^{D-1}$ by (3.24). Since $\tilde{\tilde{\varepsilon}}$ is uncorrelated with either φ or φ^d, the term $\langle \varphi^d(\mathbf{x})\tilde{\tilde{\varepsilon}}(\mathbf{x})\rangle_{\psi,k,\varepsilon}$ in equation (27) vanishes and the contribution of the noise or stochastic component of the signal in the statistical error of the differentiation filtering in restricted to $\langle \tilde{\tilde{\varepsilon}}(\mathbf{x}^2)\rangle_{k,\varepsilon}$ as in interpolative filters:

$$\langle \tilde{\varepsilon}(\mathbf{x}^2)\rangle_{k,\varepsilon} \approx \frac{\frac{1}{N_p^2}\left\langle \sum_{k,s=0}^{N_p}\left(g_D(\mathbf{x}_k) - g_D(\mathbf{x}_s)\right)^2\right\rangle_k \sigma_\varepsilon^2}{\left\langle \left[\sum_{k=1}^{N_p} g_I(\mathbf{x}_k)\right]^2\right\rangle_k} = \frac{(N_p-1)G_D\sigma_\varepsilon^2}{N_p^2 - N_p + N_p G_I} \tag{34}$$

Finally, incorporating (26),(30),(33) and (34) into (25) and (3.12), a closed form equation for the staticial relative error of differentiation is obtained:

$$E_D \approx \frac{\left[\sum_{\mathbf{L}=0}^{\infty} \hat{\varphi}_{\mathbf{L}}^2 F_{1,l_1}^2 \hat{e}_{D,\mathbf{L}}^2 + \frac{2^D\left(1-N_p^{-1}\right)G_D}{N_p-1+G_I}\sigma_\varepsilon^2\right]^{1/2}}{\left[\sum_{\mathbf{L}=0}^{\infty} \hat{\varphi}_{\mathbf{L}}^2 F_{1,l_1}^2\right]^{1/2}} \tag{35}$$

Where $\hat{e}_{D,\mathbf{L}}$ includes all terms of the numerator of (35) except those multiplying σ_ε^2:

$$\hat{e}_{D,\mathbf{L}} \approx \left[(1-\delta_{\mathbf{L}}^2) + \left[\frac{(N_p-1)^3+N_p-1}{N_p^3-N_p^2+N_p^2 G_I} - 1\right]\delta_{\mathbf{L}}^2 - \frac{2(N_p-1)}{N_p}\delta_{\mathbf{L}} - \frac{2(N_p-1)^2}{N_p^3-N_p^2+N_p^2 G_I}\xi_{\mathbf{L}}\alpha_{\mathbf{L}} + \frac{(N_p-2)(N_p-1)}{N_p^3-N_p^2+N_p^2 G_I}\frac{G_D\alpha_{\mathbf{L}}^2}{F_{1,l_1}^2} + \frac{N_p-1}{N_p^2-N_p+N_p G_I}\frac{G_D}{F_{1,l_1}^2}\right]^{1/2} \tag{36}$$

It should be noted that $(1-\delta_{\mathbf{L}})^2$ is usually the predominant element in the above equation when the contribution of $\hat{e}_{D,\mathbf{L}}$ to the relative error is important (i.e. for small values of F_{1,L_1}). Thus, in order to shorten (36), the following simplification, based on

the approximation $N_p + 1 - G_1 \approx N_p$, can be made without incurring any appreciable perturbation of the results given by (35)

$$\hat{e}_{D,\mathbf{L}} \approx \left[(1-\delta_{\mathbf{L}}^2) + \left[\frac{-3N_p^2+4N_p-2}{N_p^3} - 1\right]\delta_{\mathbf{L}}^2 - \frac{2(N_p-1)}{N_p}\delta_{\mathbf{L}} - \frac{2(N_p-1)^2}{N_p^3}\xi_{\mathbf{L}}\alpha_{\mathbf{L}} + \frac{(N_p-2)(N_p-1)}{N_p^3}\frac{G_D\alpha_{\mathbf{L}}^2}{F_{1,l_1}^2} + \frac{N_p-1}{N_p^2}\frac{G_D}{F_{1,l_1}^2}\right]^{1/2} \tag{37}$$