Appendix A. Weight functions and residual equation

Here we give various details related to algebraic calculation of the weight functions and the terms in the residual equation. Many of the expressions derived require the expressions in appendix B to evaluate them. Apart from the general expressions, we give below the solution for 2 Newtonian fluids, which can be used as a test case to check implementation of the more general formulae.

A.1. *Newtonian fluids*

For 2 Newtonian fluids, both effective and tangent viscosities are constant and equal, $\kappa_H = 1$, $\kappa_L = m$. After some algebra, f is given in terms of (h, b) by:

$$f = \frac{[(1-h)^4 + mh(1-h)^2(4-h)]b + 12m(1-h+mh)}{(1-h)^4 + 2mh(1-h)(2-h+h^2) + m^2h^4}. \tag{A 1}$$

The velocity in the two layers is given in terms of (h, b) by:

$$U_0(y) = -\frac{y(1-h)^2}{2(1-h+hm)}b + \frac{1-y+h(m-1)(h-y)}{2(1-h+hm)}fy, \quad y \in [0, h], \tag{A 2}$$

$$U_0(y) = -\frac{(1-y)[(1-h)(y-h) + mh(1+y-2h)]}{2m(1-h+hm)}b \tag{A 3}$$

$$+\frac{(1-y)[(1-h)(y-h) + mh(1+y-h)]}{2m(1-h+hm)}f, \qquad y \in [h, 1]. \tag{A 4}$$

The flux in the lower layer can be calculated directly and is given in terms of (h, b) by:

$$q = \frac{1}{3}\left[\frac{b[h^3(1-h)^3(1-h+mh)] + 3mh^2[(1-h)(3+h) + mh^2]}{(1-h)^4 + 2mh(1-h)(2-h+h^2) + m^2h^4}\right]. \tag{A 5}$$

We can see that there is a 1-to-1 correspondence between using b as the second variable, or using q, as is done in e.g. Amaouche *et al.* (2007).

We can see that the weight functions are quadratic in y. After some algebra the parameter a and the functions $g_H(y)$ and $g_L(y)$ are given by:

$$a = \frac{(1-h)^4 + mh(1-h)^2(4-h)}{(1-h)^4 + 2mh(1-h)(2-h+h^2) + m^2h^4}, \tag{A 6}$$

$$g_H(y) = -\frac{y^2}{2}\left[\frac{(1-h)^4 + mh(1-h)^2(4-h)}{(1-h)^4 + 2mh(1-h)(2-h+h^2) + m^2h^4}\right] \tag{A 7}$$

$$+y\left[\frac{h(1-h)^4 + mh^2(1-h)^2(2-h)}{(1-h)^4 + 2mh(1-h)(2-h+h^2) + m^2h^4}\right], \tag{A 8}$$

$$g_L(y) = \frac{(1-y)^2}{2}\left[\frac{h^2[mh^2 + (1-h)(3+h)]}{(1-h)^4 + 2mh(1-h)(2-h+h^2) + m^2h^4}\right] \tag{A 9}$$

$$-(1-y)\left[\frac{h^2(1-h)[1+h^2(m-1)]}{(1-h)^4 + 2mh(1-h)(2-h+h^2) + m^2h^4}\right]. \tag{A 10}$$

The terms in the residual equation (3.16) can then be evaluated straightforwardly using a symbolic algebra code, but are too long to list.

A.2. *General specification of the weight functions*

For more general fluids, we have seen that the weight functions are given by:

$$g_H(y) = ah[I_{H,m_H-1}(y) - L_{H,m_H}(y)] + \tau_{w,i}I_{H,m_H-1}(y), \tag{A 11}$$

$$g_L(y) = (1-a)(1-h)[L_{L,m_L}(y) - I_{L,m_L-1}(y)] - \tau_{w,i}I_{L,m_L-1}(y) \tag{A 12}$$

where $\tau_{w,i}$ and a are unknown constants. Imposing (3.7), gives one linear equation relating $\tau_{w,i}$ and a:

$$\tau_{w,i} = \frac{ah[L_{H,m_H-1}(h) - I_{H,m_H}(h)] + (1-a)(1-h)[L_{L,m_L}(h) - I_{L,m_L-1}(h)]}{[I_{H,m_H-1}(h) + I_{L,m_L-1}(h)]}. \tag{A 13}$$

We determine a by satisfying (3.12). This is a linear equation for a, with solution:

$$a = \frac{A_w}{B_w} \tag{A 14}$$

$$\begin{aligned} A_w = (1-h)\Bigg(&\left[\int_h^1 I_{L,m_L-1}\,\mathrm{d}y - \int_h^1 L_{L,m_L}\,\mathrm{d}y\right][I_{L,m_L-1}(h) + I_{H,m_H-1}(h)] \\ &+ \left[\int_h^1 I_{L,m_L-1}\,\mathrm{d}y - \int_0^h I_{H,m_H-1}\,\mathrm{d}y\right][L_{L,m_L}(h) - I_{L,m_L-1}(h)]\Bigg) \end{aligned}$$

$$\begin{aligned} B_w = &\left[h\left(\int_0^h I_{H,m_H-1}\,\mathrm{d}y - \int_0^h L_{H,m_H}\,\mathrm{d}y\right) + (1-h)\left(\int_h^1 I_{L,m_L-1}\,\mathrm{d}y - \int_h^1 L_{L,m_L}\,\mathrm{d}y\right)\right] \\ &\times[I_{L,m_L-1}(h) + I_{H,m_H-1}(h)] + \left[\int_0^h I_{H,m_H-1}\,\mathrm{d}y - \int_h^1 I_{L,m_L-1}\,\mathrm{d}y\right] \\ &\times[h(L_{H,m_H}(h) - I_{H,m_H-1}(h)) - (1-h)(L_{L,m_L}(h) - I_{L,m_L-1}(h))] \end{aligned}$$

We use the expressions given in appendix B to evaluate the various integrals of the functions $I_{k,p}$, $J_{k,p}$ and $L_{k,p}$.

A.3. *Calculating the terms in the residual equation (3.16)*

To calculate the terms in (3.16) we first evaluate the partial derivatives of (f, τ_i) with respect to (h, b). As discussed in §2.2 the velocity closure, including (f, τ_i), can be computed robustly to any given tolerance. Therefore, it is straightforward to compute these derivatives numerically. This enables us to compute partial derivatives of the wall shear stresses $\tau_H = \tau_i + fh$ and $\tau_L = \tau_i + (1-h)(b-f)$. Within the heavy fluid layer, we construct:

$$\frac{\partial \tau_H}{\partial h} = \frac{\partial \tau_i}{\partial h} + h\frac{\partial f}{\partial h} + f, \tag{A 15}$$

$$\frac{\partial \tau_H}{\partial b} = \frac{\partial \tau_i}{\partial b} + h\frac{\partial f}{\partial b}, \tag{A 16}$$

and within the light fluid layer:

$$\frac{\partial \tau_L}{\partial h} = \frac{\partial \tau_i}{\partial h} - (1-h)\frac{\partial f}{\partial h} - b + f, \tag{A 17}$$

$$\frac{\partial \tau_L}{\partial b} = \frac{\partial \tau_i}{\partial b} + (1-h)\left(1 - \frac{\partial f}{\partial b}\right). \tag{A 18}$$

Let us consider the first terms in (3.15), within the heavy fluid layer:

$$\int_0^h \frac{\partial U_0}{\partial T} g_H\,\mathrm{d}y = \int_0^h \left[\frac{\partial U_0}{\partial h}h_T + \frac{\partial U_0}{\partial b}b_T\right] g_H\,\mathrm{d}y = A_{H,h}h_T + A_{H,b}b_T, \tag{A 19}$$

$$\int_h^1 \frac{\partial U_0}{\partial T} g_L\,\mathrm{d}y = \int_h^1 \left[\frac{\partial U_0}{\partial h}h_T + \frac{\partial U_0}{\partial b}b_T\right] g_L\,\mathrm{d}y = A_{L,h}h_T + A_{L,b}b_T, \tag{A 20}$$

where we have denoted the different contributions to the acceleration terms as follows:

$$A_{H,h} = \int_0^h \frac{\partial U_0}{\partial h}(y) g_H(y)\ \mathrm{d}y, \qquad A_{H,b} = \int_0^h \frac{\partial U_0}{\partial b}(y) g_H(y)\ \mathrm{d}y,$$
$$A_{L,h} = \int_h^1 \frac{\partial U_0}{\partial h}(y) g_L(y)\ \mathrm{d}y, \qquad A_{L,b} = \int_h^1 \frac{\partial U_0}{\partial b}(y) g_L(y)\ \mathrm{d}y.$$

The weight functions are expressed as linear functions of the functions $I_{k,p}$ and $J_{k,p}$; see (3.26) & (3.28). Now we develop similar expressions for the partial derivatives of U_0 in each layer. Within the heavy fluid layer, using (2.28)

$$\frac{\partial U_0}{\partial h} = \frac{1}{m_H}\frac{\partial}{\partial h} J_{H,m_H}(y) = m_H \int_0^y \frac{(|\tau_{H,xy}| - B_H)_+^{m_H-1}}{\kappa_H^{m_H}} \frac{\partial}{\partial h}\tau_{H,xy}\ \mathrm{d}\tilde{y}$$
$$= m_H \int_0^y \frac{(|\tau_{H,xy}| - B_H)_+^{m_H-1}}{\kappa_H^{m_H}} \left[\frac{\partial \tau_H}{\partial h} + (\tau_{H,xy} - \tau_H)\left(\frac{\frac{\partial \tau_i}{\partial h} - \frac{\partial \tau_H}{\partial h}}{\tau_i - \tau_H} - \frac{1}{h} \right) \right]\ \mathrm{d}\tilde{y}$$

This last expression can be evaluated in terms of the functions $I_{H,m_H-1}(y)$ and $L_{H,m_H}(y)$. We then treat the partial derivative with respect to b analogously and those in the light fluid layer. This leads to the following expressions:

$$\frac{\partial U_0}{\partial h} = \frac{\partial \tau_H}{\partial h} I_{H,m_H-1}(y) - h\frac{\partial f}{\partial h} L_{H,m_H}(y), \tag{A 21}$$
$$\frac{\partial U_0}{\partial b} = \frac{\partial \tau_H}{\partial b} I_{H,m_H-1}(y) - h\frac{\partial f}{\partial b} L_{H,m_H}(y), \tag{A 22}$$
$$\frac{\partial U_0}{\partial h} = -\frac{\partial \tau_L}{\partial h} I_{L,m_L-1}(y) - (1-h)\frac{\partial f}{\partial h} L_{L,m_L}(y), \tag{A 23}$$
$$\frac{\partial U_0}{\partial b} = -\frac{\partial \tau_L}{\partial b} I_{L,m_L-1}(y) - (1-h)\left[\frac{\partial f}{\partial b} - 1\right] L_{L,m_L}(y), \tag{A 24}$$

We recall that the weight functions are also linear combinations of the functions I_{k,m_k-1} and L_{k,m_k}; see (A 11)-(A 12). It follows that each of the terms contributing to the linear acceleration in (3.15) can be expressed as integrals of quadratic products of the functions I_{k,m_k-1} and L_{k,m_k}. Having evaluated these terms (explained below) we proceed to construct the terms $A^h(h,b)$ and $A^b(h,b)$ in (3.15):

$$A^h(h,b) = \delta Re\,[1 + At]\, A_{H,h} + \delta Re\,[1 - At]\, A_{L,h} \tag{A 25}$$
$$A^b(h,b) = \delta Re\,[1 + At]\, A_{H,b} + \delta Re\,[1 - At]\, A_{L,b} \tag{A 26}$$

For the terms leading to $C^h(h,b)$ and $C^b(h,b)$ we follow a similar path, but the algebraic complexity increases. Basically we have an expression that involves linear, quadratic and cubic products of the functions $I_{k,p}$, $J_{k,p}$ and $L_{k,p}$, each of which can be evaluated algebraically. These expressions are then integrated across the respective fluid layers.

$$C^h(h,b) = \delta Re\,[1 + At]\, C_{H,h} + \delta Re\,[1 - At]\, C_{L,h} \tag{A 27}$$
$$C^b(h,b) = \delta Re\,[1 + At]\, C_{H,b} + \delta Re\,[1 - At]\, C_{L,b} \tag{A 28}$$

We now deal with the specific terms in heavy and light fluid layers.

A.4. *Terms in the residual equation: acceleration terms*

We have seen that the weight functions and each partial derivative of U_0 can be expressed as a linear combination of I_{k,m_k-1} and L_{k,m_k}, i.e.

$$g_k(y) = g_{c1} I_{k,m_k-1}(y) + g_{c2} L_{k,m_k}(y), \qquad \frac{\partial U_0}{\partial h}(y) = u_{c1} I_{k,m_k-1}(y) + u_{c2} L_{k,m_k}(y),$$

where the constants are specified above (and we have a similar expression for the b-derivatives). In general therefore, we have:

$$A_{k,h} = \int [u_{c1} I_{k,m_k-1}(y) + u_{c2} L_{k,m_k}(y)][g_{c1} I_{k,m_k-1}(y) + g_{c2} L_{k,m_k}(y)] \; \mathrm{d}y, \tag{A 29}$$

where the integral is over $[0, h]$ or $[h, 1]$, for heavy and light fluid layers respectively. Expressions for the integrals of the quadratic products above are given in appendix B.

A.5. *Terms in the residual equation: convective terms*

We start with the heavy layer and the terms contributing to $C^{(h)}(h, b)$, which are:

$$\int_0^h \frac{\partial U_0}{\partial h} \left[U_0 g_H - \int_0^h \frac{\partial U_0}{\partial y} g_H \; \mathrm{d}\tilde{y} + \int_0^y \frac{\partial U_0}{\partial y} g_H \; \mathrm{d}\tilde{y} \right] \; \mathrm{d}y = C_{H,h,1} + C_{H,h,2} + C_{H,h,3}.$$

For the first term: $C_{H,h,1}$ we recall that $U_0(y)$ is given in terms of $J_{H,m_H}(y)$ and therefore,

$$C_{H,h,1} = n_H \int_0^h J_{H,m_H} [u_{c1} I_{H,m_H-1}(y) + u_{c2} L_{H,m_H}(y)][g_{c1} I_{H,m_H-1}(y) + g_{c2} L_{H,m_H}(y)] \; \mathrm{d}y. \tag{A 30}$$

For the next two terms, we first to evaluate the integral expression:

$$\int_0^y \frac{\partial U_0}{\partial y} g_H \; \mathrm{d}\tilde{y} = h g_{c1} \frac{I_{H,2m_H}(y) - \mathrm{sgn}(\tau_H)(|\tau_H| - B_H)_+^{m_H} J_{H,m_H}(y)}{m_H \kappa_H^{m_H} (\tau_i - \tau_H)}$$
$$+ h g_{c2} \frac{L_{H,2m_H+1}(y) + \frac{(|\tau_H| - B_H)_+^{m_H+1} J_{H,m_H}(y) - J_{H,2m_H+1}(y)}{(m_H+1)(\tau_i - \tau_H)}}{m_H \kappa_H^{m_H} (\tau_i - \tau_H)}. \tag{A 31}$$

When arranged as above, the singularities as $\tau_i \to \tau_H$ are removed on expanding the individual terms about τ_H. Since we have

$$\frac{\partial U_0}{\partial h} = u_{c1} I_{H,m_H-1}(y) + u_{c2} L_{H,m_H}(y),$$

we see that $C_{H,h,2}$ is evaluated as integrals of quadratic products of the special functions. Equally, $C_{H,h,3}$ is the quadratic product of integrals of the special functions. To evaluate $C^{(b)}(h, b)$. the approach and expressions are identical, except that the constants u_{c1} and u_{c2} are different.

Turning to the light layer, we have that:

$$\int_h^1 \frac{\partial U_0}{\partial h} \left[U_0 g_L + \int_h^1 \frac{\partial U_0}{\partial y} g_L \; \mathrm{d}\tilde{y} - \int_y^1 \frac{\partial U_0}{\partial y} g_L \; \mathrm{d}\tilde{y} \right] \; \mathrm{d}y = C_{L,h,1} + C_{L,h,2} + C_{L,h,3}.$$

with an analogous expression for the contributions to $C^{(b)}(h, b)$. In the light fluid layer the velocity is $-n_L J_{L,m_L}(y)$ and therefore:

$$C_{L,h,1} = -n_L \int_h^1 J_{L,m_L} [u_{c1} I_{L,m_L-1}(y) + u_{c2} L_{L,m_L}(y)][g_{c1} I_{L,m_L-1}(y) + g_{c2} L_{L,m_L}(y)] \; \mathrm{d}y. \tag{A 32}$$

For the next two terms, we evaluate the integral expression:

$$\int_y^1 \frac{\partial U_0}{\partial y} g_L \, \mathrm{d}\tilde{y} = (1-h) g_{c1} \frac{I_{L,2m_L}(y) - \mathrm{sgn}(\tau_L)(|\tau_L| - B_L)_+^{m_L} J_{L,m_L}(y)}{m_L \kappa_L^{m_L}(\tau_i - \tau_L)}$$
$$+(1-h) g_{c2} \frac{L_{L,2m_L+1}(y) + \frac{(|\tau_L| - B_L)_+^{m_L+1} J_{L,m_L}(y) - J_{L,2m_L+1}(y)}{(m_L+1)(\tau_i - \tau_L)}}{m_L \kappa_L^{m_L}(\tau_i - \tau_L)}. \tag{A 33}$$

and then combine with

$$\frac{\partial U_0}{\partial h} = u_{c1} I_{L,m_L-1}(y) + u_{c2} L_{L,m_L}(y).$$

Therefore, $C_{L,h,2}$ is evaluated as integrals of quadratic products of the special functions and $C_{L,h,3}$ is the quadratic product of integrals of the special functions. To evaluate $C^{(b)}(h,b)$. the approach and expressions are identical, except that the constants u_{c1} and u_{c2} are different.

A.5.1. *Buoyancy term and flux function*

The buoyancy term $S(h,b)$ is simply:

$$\int_h^1 g_L(y) \, \mathrm{d}y = g_{c1} \int_h^1 I_{L,m_L-1}(y) \, \mathrm{d}y + g_{c2} \int_h^1 L_{L,m_L}(y) \, \mathrm{d}y. \tag{A 34}$$

The flux function is simply:

$$q = \int_0^h \frac{J_{H,m_H}(y)}{m_H} \, \mathrm{d}y. \tag{A 35}$$

Expressions for the integrals of the special functions are given in the following appendix.

Appendix B. The functions $I_{k,p}$ and $J_{k,p}$

Here we give those expressions that are useful in evaluating the functions $I_{k,p}$ and $J_{k,p}$, as well as the various integrals that contribute to the weight functions and residual terms. In each case we give expressions that are valid for $\tau_i \neq \tau_H$ (or $\tau_i \neq \tau_L$). To resolve the limiting behaviour as $\tau_i \to \tau_H$ (or $\tau_i \to \tau_L$) is not particularly difficult, but involves a series expansion about the wall shear stress, which removes any singular behaviour. This detail is omitted for brevity. We start with the heavy fluid layer.

B.1. *Heavy fluid layer*

B.1.1. $I_{H,p}$ *and* $J_{H,p}$

$$I_{H,p}(y) = \frac{hm_H[\mathrm{sgn}(\tau_{H,\xi y}(y))(|\tau_{H,\xi y}|(y) - B_H)_+^{p+1} - \mathrm{sgn}(\tau_H)(|\tau_H| - B_H)_+^{p+1}]}{(p+1)\kappa_H^{m_H}(\tau_i - \tau_H)} \tag{B 1}$$

$$J_{H,p}(y) = \frac{hm_H[(|\tau_{H,\xi y}|(y) - B_H)_+^{p+1} - (|\tau_H| - B_H)_+^{p+1}]}{(p+1)\kappa_H^{m_H}(\tau_i - \tau_H)} \tag{B 2}$$

The function $L_{H,p}$ is evaluated as a linear combination of the above.

B.1.2. *Integrals of* $I_{H,p}$ *and* $J_{H,p}$

$$\int_0^h I_{H,p}(y)\ \mathrm{d}y = \frac{hJ_{H,p+1}(h)}{(p+1)(\tau_i - \tau_H)} - \frac{h^2 m_H \mathrm{sgn}(\tau_H)(|\tau_H| - B_H)_+^{p+1}}{(p+1)\kappa_H^{m_H}(\tau_i - \tau_H)} \tag{B 3}$$

$$\int_0^h J_{H,p}(y)\ \mathrm{d}y = \frac{hI_{H,p+1}(h)}{(p+1)(\tau_i - \tau_H)} - \frac{h^2 m_H (|\tau_H| - B_H)_+^{p+1}}{(p+1)\kappa_H^{m_H}(\tau_i - \tau_H)} \tag{B 4}$$

The integral of $L_{H,p}$ is evaluated as a linear combination of the above.

B.1.3. *Integrals of quadratic products of* $I_{H,p}$ *and* $J_{H,p}$

$$\begin{aligned}\int_0^h I_{H,p}(y)I_{H,q}(y)\ \mathrm{d}y = {} & \frac{h^2 m_H I_{H,p+q+2}(h)}{(p+1)(q+1)\kappa_H^{m_H}(\tau_i - \tau_H)^2} + \frac{h^3 m_H^2 (|\tau_H| - B_H)_+^{p+q+2}}{(p+1)(q+1)\kappa_H^{2m_H}(\tau_i - \tau_H)^2} \\ & - \frac{h^2 m_H \mathrm{sgn}(\tau_H)(|\tau_H| - B_H)_+^{p+1} J_{H,q+1}(h)}{(p+1)(q+1)\kappa_H^{m_H}(\tau_i - \tau_H)^2}, \\ & - \frac{h^2 m_H \mathrm{sgn}(\tau_H)(|\tau_H| - B_H)_+^{q+1} J_{H,p+1}(h)}{(p+1)(q+1)\kappa_H^{m_H}(\tau_i - \tau_H)^2},\end{aligned} \tag{B 5}$$

$$\begin{aligned}\int_0^h I_{H,p}(y)J_{H,q}(y)\ \mathrm{d}y = {} & \frac{h^2 m_H [J_{H,p+q+2}(h) - (|\tau_H| - B_H)_+^{p+1} J_{H,q+1}(h)]}{(p+1)(q+1)\kappa_H^{m_H}(\tau_i - \tau_H)^2} \\ & - \frac{h^2 m_H \mathrm{sgn}(\tau_H)(|\tau_H| - B_H)_+^{q+1} I_{H,p+1}(h)}{(p+1)(q+1)\kappa_H^{m_H}(\tau_i - \tau_H)^2} \\ & + \frac{h^3 m_H^2 \mathrm{sgn}(\tau_H)(|\tau_H| - B_H)_+^{p+q+2}}{(p+1)(q+1)\kappa_H^{2m_H}(\tau_i - \tau_H)^2},\end{aligned} \tag{B 6}$$

$$\int_0^h J_{H,p}(y)J_{H,q}(y)\ \mathrm{d}y = \frac{h^2 m_H I_{H,p+q+2}(h)}{(p+1)(q+1)\kappa_H^{m_H}(\tau_i - \tau_H)^2} + \frac{h^3 m_H^2 (|\tau_H| - B_H)_+^{p+q+2}}{(p+1)(q+1)\kappa_H^{2m_H}(\tau_i - \tau_H)^2}$$

$$-\frac{h^2 m_H[(|\tau_H|-B_H)_+^{p+1} I_{H,q+1}(h)+(|\tau_H|-B_H)_+^{q+1} I_{H,p+1}(h)]}{(p+1)(q+1)\kappa_H^{m_H}(\tau_i-\tau_H)^2}. \tag{B 7}$$

B.1.4. *Integrals of selected cubic products of $I_{H,p}$ and $J_{H,p}$*

$$\begin{aligned}
\int_0^h J_{H,p}J_{H,q}J_{H,r}\ \mathrm{d}y = & \frac{h^3 m_H^2}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,p+q+r+3}(h) \\
& -\frac{h^3 m_H^2(|\tau_H|-B_H)_+^{p+1}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,q+r+2}(h) \\
& -\frac{h^3 m_H^2(|\tau_H|-B_H)_+^{q+1}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,p+r+2}(h) \\
& -\frac{h^3 m_H^2(|\tau_H|-B_H)_+^{r+1}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,p+q+2}(h) \\
& +\frac{h^3 m_H^2(|\tau_H|-B_H)_+^{p+q+2}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,r+1}(h) \\
& +\frac{h^3 m_H^2(|\tau_H|-B_H)_+^{p+r+2}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,q+1}(h) \\
& +\frac{h^3 m_H^2(|\tau_H|-B_H)_+^{q+r+2}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,p+1}(h) \\
& -\frac{h^4 m_H^3(|\tau_H|-B_H)_+^{p+q+r+3}}{(p+1)(q+1)(r+1)\kappa_H^{3m_H}(\tau_i-\tau_H)^3},,
\end{aligned}$$

$$\begin{aligned}
\int_0^h J_{H,p}J_{H,q}I_{H,r}\ \mathrm{d}y = & \frac{h^3 m_H^2}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} J_{H,p+q+r+3}(h) \\
& -\frac{h^3 m_H^2(|\tau_H|-B_H)_+^{p+1}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} J_{H,q+r+2}(h) \\
& -\frac{h^3 m_H^2(|\tau_H|-B_H)_+^{q+1}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} J_{H,p+r+2}(h) \\
& -\frac{h^3 m_H^2\mathrm{sgn}(\tau_H)(|\tau_H|-B_H)_+^{r+1}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,p+q+2}(h) \\
& +\frac{h^3 m_H^2(|\tau_H|-B_H)_+^{p+q+2}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} J_{H,r+1}(h) \\
& +\frac{h^3 m_H^2\mathrm{sgn}(\tau_H)(|\tau_H|-B_H)_+^{p+r+2}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,q+1}(h) \\
& +\frac{h^3 m_H^2\mathrm{sgn}(\tau_H)(|\tau_H|-B_H)_+^{q+r+2}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,p+1}(h) \\
& -\frac{h^4 m_H^3\mathrm{sgn}(\tau_H)(|\tau_H|-B_H)_+^{p+q+r+3}}{(p+1)(q+1)(r+1)\kappa_H^{3m_H}(\tau_i-\tau_H)^3},
\end{aligned}$$

$$
\begin{aligned}
\int_0^h J_{H,p} I_{H,q} I_{H,r}\ \mathrm{d}y = & \frac{h^3 m_H^2}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,p+q+r+3}(h) \\
& - \frac{h^3 m_H^2 (|\tau_H|-B_H)_+^{p+1}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,q+r+2}(h) \\
& - \frac{h^3 m_H^2 \mathrm{sgn}(\tau_H)(|\tau_H|-B_H)_+^{q+1}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} J_{H,p+r+2}(h) \\
& - \frac{h^3 m_H^2 \mathrm{sgn}(\tau_H)(|\tau_H|-B_H)_+^{r+1}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} J_{H,p+q+2}(h) \\
& + \frac{h^3 m_H^2 \mathrm{sgn}(\tau_H)(|\tau_H|-B_H)_+^{p+q+2}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} J_{H,r+1}(h) \\
& + \frac{h^3 m_H^2 \mathrm{sgn}(\tau_H)(|\tau_H|-B_H)_+^{p+r+2}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} J_{H,q+1}(h) \\
& + \frac{h^3 m_H^2 (|\tau_H|-B_H)_+^{q+r+2}}{(p+1)(q+1)(r+1)\kappa_H^{2m_H}(\tau_i-\tau_H)^3} I_{H,p+1}(h) \\
& - \frac{h^4 m_H^3 (|\tau_H|-B_H)_+^{p+q+r+3}}{(p+1)(q+1)(r+1)\kappa_H^{3m_H}(\tau_i-\tau_H)^3}.
\end{aligned}
$$

B.2. *Light fluid layer*

For the light fluid layer the expressions are similar.

B.2.1. $I_{L,p}$ *and* $J_{L,p}$

$$
I_{L,p}(y) = \frac{(1-h)m_L[\mathrm{sgn}(\tau_{L,\xi y}(y))(|\tau_{L,\xi y}|(y)-B_L)_+^{p+1} - \mathrm{sgn}(\tau_L)(|\tau_L|-B_L)_+^{p+1}]}{(p+1)\kappa_L^{m_L}(\tau_i-\tau_L)} \tag{B 8}
$$

$$
J_{L,p}(y) = \frac{(1-h)m_L[(|\tau_{L,\xi y}|(y)-B_L)_+^{p+1} - (|\tau_L|-B_L)_+^{p+1}]}{(p+1)\kappa_L^{m_L}(\tau_i-\tau_L)} \tag{B 9}
$$

The function $L_{L,p}$ is evaluated as a linear combination of the above.

B.2.2. *Integrals of* $I_{L,p}$ *and* $J_{L,p}$

$$
\int_h^1 I_{L,p}(y)\ \mathrm{d}y = \frac{(1-h)J_{L,p+1}(h)}{(p+1)(\tau_i-\tau_L)} - \frac{(1-h)^2 m_L \mathrm{sgn}(\tau_L)(|\tau_L|-B_L)_+^{p+1}}{(p+1)\kappa_L^{m_L}(\tau_i-\tau_L)} \tag{B 10}
$$

$$
\int_h^1 J_{L,p}(y)\ \mathrm{d}y = \frac{(1-h)I_{L,p+1}(h)}{(p+1)(\tau_i-\tau_L)} - \frac{(1-h)^2 m_L (|\tau_L|-B_L)_+^{p+1}}{(p+1)\kappa_L^{m_L}(\tau_i-\tau_L)} \tag{B 11}
$$

The integral of $L_{L,p}$ is evaluated as a linear combination of the above.

B.2.3. *Integrals of quadratic products of* $I_{L,p}$ *and* $J_{L,p}$

$$
\begin{aligned}
\int_h^1 I_{L,p}(y) I_{L,q}(y)\ \mathrm{d}y = & \frac{(1-h)^2 m_L I_{L,p+q+2}(h)}{(p+1)(q+1)\kappa_L^{m_L}(\tau_i-\tau_L)^2} + \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{p+q+2}}{(p+1)(q+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^2} \\
& - \frac{(1-h)^2 m_L \mathrm{sgn}(\tau_L)(|\tau_L|-B_L)_+^{p+1} J_{L,q+1}(h)}{(p+1)(q+1)\kappa_L^{m_L}(\tau_i-\tau_L)^2},
\end{aligned}
$$

$$-\frac{(1-h)^2 m_L \mathrm{sgn}(\tau_L)(|\tau_L|-B_L)_+^{q+1} J_{L,p+1}(h)}{(p+1)(q+1)\kappa_L^{m_L}(\tau_i-\tau_L)^2}, \tag{B 12}$$

$$\int_h^1 I_{L,p}(y) J_{L,q}(y)\ \mathrm{d}y = \frac{(1-h)^2 m_L [J_{L,p+q+2}(h) - (|\tau_L|-B_L)_+^{p+1} J_{L,q+1}(h)]}{(p+1)(q+1)\kappa_L^{m_L}(\tau_i-\tau_L)^2} - \frac{(1-h)^2 m_L \mathrm{sgn}(\tau_L)(|\tau_L|-B_L)_+^{q+1} I_{L,p+1}(h)}{(p+1)(q+1)\kappa_L^{m_L}(\tau_i-\tau_L)^2} + \frac{(1-h)^3 m_L^2 \mathrm{sgn}(\tau_L)(|\tau_L|-B_L)_+^{p+q+2}}{(p+1)(q+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^2}, \tag{B 13}$$

$$\int_h^1 J_{L,p}(y) J_{L,q}(y)\ \mathrm{d}y = \frac{(1-h)^2 m_L I_{L,p+q+2}(h)}{(p+1)(q+1)\kappa_L^{m_L}(\tau_i-\tau_L)^2} + \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{p+q+2}}{(p+1)(q+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^2} - \frac{(1-h)^2 m_L [(|\tau_L|-B_L)_+^{p+1} I_{L,q+1}(h) + (|\tau_L|-B_L)_+^{q+1} I_{L,p+1}(h)]}{(p+1)(q+1)\kappa_L^{m_L}(\tau_i-\tau_L)^2}. \tag{B 14}$$

B.2.4. *Integrals of selected cubic products of $I_{L,p}$ and $J_{L,p}$*

$$\begin{aligned}
\int_h^1 J_{L,p} J_{L,q} J_{L,r}\ \mathrm{d}y = {} & \frac{(1-h)^3 m_L^2}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,p+q+r+3}(h) \\
& - \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{p+1}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,q+r+2}(h) \\
& - \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{q+1}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,p+r+2}(h) \\
& - \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{r+1}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,p+q+2}(h) \\
& + \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{p+q+2}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,r+1}(h) \\
& + \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{p+r+2}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,q+1}(h) \\
& + \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{q+r+2}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,p+1}(h) \\
& - \frac{(1-h)^3 m_L^3 (|\tau_L|-B_L)_+^{p+q+r+3}}{(p+1)(q+1)(r+1)\kappa_L^{3m_L}(\tau_i-\tau_L)^3},
\end{aligned}$$

$$\begin{aligned}
\int_h^1 J_{L,p} J_{L,q} I_{L,r}\ \mathrm{d}y = {} & \frac{(1-h)^3 m_L^2}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} J_{L,p+q+r+3}(h) \\
& - \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{p+1}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} J_{L,q+r+2}(h) \\
& - \frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{q+1}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} J_{L,p+r+2}(h)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{(1-h)^3 m_L^2 \operatorname{sgn}(\tau_L)(|\tau_L|-B_L)_+^{r+1}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,p+q+2}(h) \\
&+\frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{p+q+2}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} J_{L,r+1}(h) \\
&+\frac{(1-h)^3 m_L^2 \operatorname{sgn}(\tau_L)(|\tau_L|-B_L)_+^{p+r+2}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,q+1}(h) \\
&+\frac{(1-h)^3 m_L^2 \operatorname{sgn}(\tau_L)(|\tau_L|-B_L)_+^{q+r+2}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,p+1}(h) \\
&-\frac{(1-h)^3 m_L^3 \operatorname{sgn}(\tau_L)(|\tau_L|-B_L)_+^{p+q+r+3}}{(p+1)(q+1)(r+1)\kappa_L^{3m_L}(\tau_i-\tau_L)^3},
\end{aligned}
$$

$$
\begin{aligned}
\int_h^1 J_{L,p} I_{L,q} I_{L,r}\,\mathrm{d}y = &\frac{(1-h)^3 m_L^2}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,p+q+r+3}(h) \\
&-\frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{p+1}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,q+r+2}(h) \\
&-\frac{(1-h)^3 m_L^2 \operatorname{sgn}(\tau_L)(|\tau_L|-B_L)_+^{q+1}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} J_{L,p+r+2}(h) \\
&-\frac{(1-h)^3 m_L^2 \operatorname{sgn}(\tau_L)(|\tau_L|-B_L)_+^{r+1}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} J_{L,p+q+2}(h) \\
&+\frac{(1-h)^3 m_L^2 \operatorname{sgn}(\tau_L)(|\tau_L|-B_L)_+^{p+q+2}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} J_{L,r+1}(h) \\
&+\frac{(1-h)^3 m_L^2 \operatorname{sgn}(\tau_L)(|\tau_L|-B_L)_+^{p+r+2}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} J_{L,q+1}(h) \\
&+\frac{(1-h)^3 m_L^2 (|\tau_L|-B_L)_+^{q+r+2}}{(p+1)(q+1)(r+1)\kappa_L^{2m_L}(\tau_i-\tau_L)^3} I_{L,p+1}(h) \\
&-\frac{(1-h)^3 m_L^3 (|\tau_L|-B_L)_+^{p+q+r+3}}{(p+1)(q+1)(r+1)\kappa_L^{3m_L}(\tau_i-\tau_L)^3}.
\end{aligned}
$$

Appendix C. Long-wave limit of the Orr-Sommerfeld problem

Here we outline the steps needed in semi-analytical solution of the long-wave limit of the Orr-Sommerfeld problem, $\alpha \longrightarrow 0$.

C.1. *Solution of the leading order problem*

We assume that $\bar{h}_0 \neq 0$ and divide through by $\bar{h}_0$, as we can see that only $(\psi_0/\bar{h}_0)$ is determined by the system (5.23)-(5.28). There is a similarity between the leading order problem and the earlier derivation of the weight functions, in §3.3. Since $\eta_{k,t} \sim \dot{\gamma}^{n_k-1}$, it seems that $D^2(\psi_0/\bar{h}_0)$ must mimic the behaviour of the strain rate, in each layer. This prompts definition of $\eta_{k,t}D^2(\psi_0/\bar{h}_0)$ as a linear function of the shear stress of the base flow in each layer. We find that (5.23) and (5.27)-(5.28) are satisfied by:

$$\eta_{H,t} D^2 \left(\frac{\psi_0}{\bar{h}_0} \right) = A_p \tau_{H,\xi y}(y) + \tau_{pi} + \frac{1}{2} \left(\chi - A_p \frac{\chi}{\chi - f} \tau_i \right), \tag{C 1}$$

$$\eta_{L,t} D^2 \left(\frac{\psi_0}{\bar{h}_0} \right) = -\frac{f}{\chi - f} A_p \tau_{L,\xi y}(y) + \tau_{pi} - \frac{1}{2} \left(\chi - A_p \frac{\chi}{\chi - f} \tau_i \right). \tag{C 2}$$

where A_p and τ_{pi} are arbitrary constants to be determined. The other parameters are defined via the base flow. These expressions may be rearranged and integrated, via the functions $I_{k,p}(y)$ and $J_{k,p}(y)$, defined earlier. The perturbed axial velocity is given by:

$$\begin{aligned} D \left(\frac{\psi_0}{\bar{h}_0} \right) &= A_p J_{H,m_H}(y) + A_p B_H J_{H,m_H-1}(y) + \\ &\left[\frac{1}{2} \left(\chi - A_p \frac{\chi}{\chi - f} \tau_i \right) + \tau_{pi} \right] I_{H,m_H-1}(y), \qquad y \in [0, h_0), \end{aligned} \tag{C 3}$$

$$\begin{aligned} D \left(\frac{\psi_0}{\bar{h}_0} \right) &= \frac{f}{\chi - f} A_p J_{L,m_L}(y) + \frac{f}{\chi - f} A_p B_L J_{L,m_L-1}(y) + \\ &\left[\frac{1}{2} \left(\chi - A_p \frac{\chi}{\chi - f} \tau_i \right) - \tau_{pi} \right] I_{L,m_L-1}(y), \qquad y \in (h_0, 1]. \end{aligned} \tag{C 4}$$

Note that we have also satisfied (5.24) in deriving the above.

We now determine the constants A_p and τ_{pi}. Using (5.26) we have:

$$\begin{aligned} DU|_{h_0^+}^{h_0^-} = &\frac{f}{\chi - f} A_p J_{L,m_L}(h_0) + \frac{f}{\chi - f} A_p B_L J_{L,m_L-1}(h_0) - A_p J_{H,m_H}(h_0) \\ &- A_p B_H J_{H,m_H-1}(h_0) + \left[\frac{1}{2} \left(\chi - A_p \frac{\chi}{\chi - f} \tau_i \right) - \tau_{pi} \right] I_{L,m_L-1}(h_0) - \\ &\left[\frac{1}{2} \left(\chi - A_p \frac{\chi}{\chi - f} \tau_i \right) + \tau_{pi} \right] I_{H,m_H-1}(h_0). \end{aligned} \tag{C 5}$$

A second equation for A_p and τ_{pi} is obtained by integrating (C 3) and (C 4) across their respective layers and applying the continuity condition (5.25):

$$\begin{aligned} 0 = &A_p \int_0^{h_0} J_{H,m_H}(y) \, \mathrm{d}y + A_p B_H \int_0^{h_0} J_{H,m_H-1}(y) \, \mathrm{d}y + \frac{A_p f}{\chi - f} \int_{h_0}^1 J_{L,m_L}(y) \, \mathrm{d}y + \\ &\frac{A_p B_L f}{\chi - f} \int_{h_0}^1 J_{L,m_L-1}(y) \, \mathrm{d}y + \left[\frac{1}{2} \left(\chi - A_p \frac{\chi}{\chi - f} \tau_i \right) + \tau_{pi} \right] \int_0^{h_0} I_{H,m_H-1}(y) \, \mathrm{d}y \\ &+ \left[\frac{1}{2} \left(\chi - A_p \frac{\chi}{\chi - f} \tau_i \right) - \tau_{pi} \right] \int_{h_0}^1 I_{L,m_L-1}(y) \, \mathrm{d}y. \end{aligned} \tag{C 6}$$

Algebraic expressions for the integrals above are given in appendix B. Having solved the two linear equations for A_p and τ_{pi}, equation (5.29) gives the leading order eigenvalue:

$$c_0 = \frac{J_{H,m_H}(h_0)}{m_H} + A_p \int_0^{h_0} J_{H,m_H}(y)\ \mathrm{d}y + A_p B_H \int_0^{h_0} J_{H,m_H-1}(y)\ \mathrm{d}y$$
$$+ \left[\frac{1}{2}\left(\chi - A_p \frac{\chi}{\chi - f}\tau_i\right) + \tau_{pi}\right] \int_0^{h_0} I_{H,m_H-1}(y)\ \mathrm{d}y \qquad \text{(C 7)}$$

We see that c_0 is always real, as is $(\psi_0/\bar{h}_0)$.

C.2. *Solution of the first order problem*

The first order problem has a similar structure to the leading order problem. Using linearity and superposition, we can see that

$$\frac{\psi_1}{\bar{h}_0} = \frac{\bar{h}_1}{\bar{h}_0}\frac{\psi_0}{\bar{h}_0} + \mathrm{i}\phi,$$

where $\phi(y)$ is real and satisfies the following problem:

$$D^2(\eta_{k,t} D^2\phi) + Re_k[(c_0 - U)D^2(\psi_0/\bar{h}_0) + (\psi_0/\bar{h}_0)D^2 U] = 0 \qquad \text{(C 8)}$$

with boundary and interface conditions:

$$\phi = D\phi = 0, \quad y = 0,\ 1 \qquad \text{(C 9)}$$
$$0 = \phi|_L^H \quad y = h_0, \qquad \text{(C 10)}$$
$$0 = D\phi|_L^H \quad y = h_0, \qquad \text{(C 11)}$$
$$\eta_{H,t} D^2\phi = \eta_{L,t} D^2\phi, \quad y = h_0, \qquad \text{(C 12)}$$
$$\chi \tan\beta = \left(Re_H[(c_0 - U)D(\psi_0/\bar{h}_0) + (\psi_0/\bar{h}_0)DU] + D[\eta_{H,t} D^2\phi]\right) \qquad \text{(C 13)}$$
$$- \left(Re_L[(c_0 - U)D(\psi_0/\bar{h}_0) + (\psi_0/\bar{h}_0)DU] + D[\eta_{L,t} D^2\phi]\right), \quad y = h_0,$$

The stability of the system is found from

$$c_1 = \mathrm{i}\phi(y = h_0).$$

Let $\mathcal{F}_k(y)$ be defined by:

$$\mathcal{F}_H(y) = -Re_H[(c_0 + U)(\psi_0/\bar{h}_0) - 2\int_0^y UD(\psi_0/\bar{h}_0)\ \mathrm{d}s],$$

$$\mathcal{F}_L(y) = -Re_L[(c_0 + U)(\psi_0/\bar{h}_0) - 2\int_1^y UD(\psi_0/\bar{h}_0)\ \mathrm{d}s],$$

so that $[\eta_{k,t}D^2\phi - \mathcal{F}_k(y)]$ is linear in each layer. We find that:

$$\eta_{H,t} D^2\phi = \mathcal{F}_H(y) + A_H \tau_{H,\xi y} + D_H, \qquad \text{(C 14)}$$
$$\eta_{L,t} D^2\phi = \mathcal{F}_L(y) - \left(\frac{f}{\chi - f}A_H + \frac{\chi}{\chi - f}\tan\beta\right)\tau_{L,\xi y} + D_H + \mathcal{F}_H(h_0) - \mathcal{F}_L(h_0)$$
$$+ \frac{\chi\tau_i}{\chi - f}(A_H + \tan\beta), \qquad \text{(C 15)}$$

for constants A_H and D_H. These expressions already satisfy (C 12) & (C 13). The constants A_H and D_H are determined by integrating twice with respect to y, away from each wall, using (C 9), and then applying (C 9) & (C 10). The relevant expressions are as follows.

$$D\phi(h_0^-) = \int_0^{h_0} \frac{\mathcal{F}_H}{\eta_{H,t}}\ \mathrm{d}y + A_H J_{H,m_H}(h_0) + A_H B_H J_{H,m_H-1}(h_0) + D_H I_{H,m_H-1}(h_0),$$

$$\tag{C 16}$$

$$D\phi(h_0^+) = -\int_{h_0}^{1} \frac{\mathcal{F}_L}{\eta_{L,t}}\ \mathrm{d}y + \left(\frac{f}{\chi - f} A_H + \frac{\chi}{\chi - f}\tan\beta\right)\left[J_{L,m_L}(h_0) + B_L J_{L,m_L-1}(h_0)\right] - \left[D_H + \mathcal{F}_H(h_0) - \mathcal{F}_L(h_0) + \frac{\chi\tau_i}{\chi - f}(A_H + \tan\beta)\right] I_{L,m_L-1}(h_0). \tag{C 17}$$

$$\phi(h_0^-) = \int_0^{h_0} (h_0 - y)\frac{\mathcal{F}_H}{\eta_{H,t}}\ \mathrm{d}y + A_H \int_0^{h_0} J_{H,m_H}(y)\ \mathrm{d}s + A_H B_H \int_0^{h_0} J_{H,m_H-1}(y)\ \mathrm{d}s + D_H \int_0^{h_0} I_{H,m_H-1}(y)\ \mathrm{d}s \tag{C 18}$$

$$\phi(h_0^+) = \int_{h_0}^{1} (y - h_0)\frac{\mathcal{F}_L}{\eta_{L,t}}\ \mathrm{d}y - \left(\frac{f}{\chi - f} A_H + \frac{\chi}{\chi - f}\tan\beta\right)\int_{h_0}^{1} J_{L,m_L}(y)\ \mathrm{d}y - \left(\frac{f}{\chi - f} A_H + \frac{\chi}{\chi - f}\tan\beta\right) B_L \int_{h_0}^{1} J_{L,m_L-1}(y)\ \mathrm{d}y + + \left[D_H + \mathcal{F}_H(h_0) - \mathcal{F}_L(h_0) + \frac{\chi\tau_i}{\chi - f}(A_H + \tan\beta)\right]\int_{h_0}^{1} I_{L,m_L-1}(y)\ \mathrm{d}y. \tag{C 19}$$