Appendix D. Numerical methods

D.1. *Numerical solution of rim equation*

Here, we describe our numerical method for solving the boundary value problem for the rim profile, defined by equations (3.9), (3.10), and (4.13). For a fixed value of the parameter q, equations (3.9) and (4.13) were recast as

$$\frac{d^3\tilde{h}^{(n+1)}(\tilde{x})}{d\tilde{x}^3} = \frac{q}{\tilde{h}^{(n)}(\tilde{x})}\frac{2}{\pi}\int_{-\infty}^{\infty}\log|\tilde{x}-\tilde{y}|\frac{d^2}{d\tilde{y}^2}\left(1/\tilde{h}^{(n)}(\tilde{x})\right)d\tilde{y}\,, \tag{D 1}$$

with boundary conditions

$$\tilde{h}^{(n+1)}(0) = 1, \quad \left.\frac{d\tilde{h}^{(n+1)}(\tilde{x})}{d\tilde{x}}\right|_{\tilde{x}=0} = 0\,, \quad \left.\frac{d^2\tilde{h}^{(n+1)}(\tilde{x})}{d\tilde{x}^2}\right|_{\tilde{x}=0} = 1. \tag{D 2}$$

Boundary condition (3.10) was enforced by rescaling $\tilde{x}$ after the converged profile $\tilde{h}(\tilde{x})$ is obtained.

Equations (D 1) and (D 2) were solved by iteration, where $\tilde{h}^{(n+1)}(\tilde{x})$ is the profile corresponding to $n+1$ iterations, and $\tilde{h}^{(0)}(\tilde{x}) = \tilde{x}^2$. The transformation (D 1) and (D 2) is a contracting map; we observed that $\tilde{h}^{(n)}(\tilde{x})$ converges exponentially to a fixed point $\tilde{h}(\tilde{x})$. The convergence exponent vanishes for $q \to q^*$, and we were unable to obtain a solution for $q > q^*$, indicating that $q = q^*$ is a turning point. Solutions corresponding to $q > q^*$ collapse at a finite negative value of $\tilde{x}$ and are thus unphysical.

The critical parameter value (4.16) was obtained by by advancing the parameter q using interval halving. The coefficient (4.18) of the square-root far-field film profile (4.17) was extracted from the profile corresponding to $q = q^*$.

The iterative solution was obtained on the finite computational domain, $-X_1 < x < X_2$, using approximations for the tails of the integral in equation (D 1), $x > X_2$ and $x < -X_1$, based on the asymptotic far-field forms (3.10) and (4.17). The cut-off parameters X_1 and X_2 were increased starting with an iterative solution for modest values of these parameters, until convergent results were obtained.

D.2. *Numerical solution of dome equation*

Integral equation (5.24) was solved using a Galerkin formulation, in which the L_2 norm of the residual was minimized as a functional of $D_2\left[\bar{\omega}_{\mathrm{d}}\right](r)$. By symmetry, the film profile has an expansion in even powers of r about $r = 0$ as does the velocity profile $\bar{\omega}_{\mathrm{d}}(r)$, but $D_2\left[\bar{\omega}_{\mathrm{d}}\right](r)$ has an odd-power expansion about $r = 0$. At the edge of the dome region, $D_2\left[\bar{\omega}_{\mathrm{d}}\right](r)$ has the singular behavior (5.14). Based on the form of $D_2\left[\bar{\omega}_{\mathrm{d}}\right](r)$ for $r \to 0$ and $r \to 1^-$, the function $D_2\left[\bar{\omega}_{\mathrm{d}}\right](r)$ was represented in terms of basis functions given by

$$D_2\left[\bar{\omega}_{\mathrm{d}}\right](r) = \sum_{k=0}^{N} a_k\, r(1-r^2)^{k-5/2}\,, \tag{D 3}$$

where a_k $(k = 0, 1, 2\cdots)$ are the expansion coefficients. Our numerical solution of equation (5.24) was insensitive to the form of the basis functions; the same results were obtained with other basis functions that have the proper symmetry at $r = 0$ and proper singular behavior at $r = 1$.

After solving equation (5.24) for $D_2\left[\bar{\omega}_{\mathrm{d}}\right](r)$, as described above, the velocity profile was obtained by integrating equation (5.9) from $r = 0$ using initial conditions compatible with equation (5.4):

$$\bar{\omega}_{\mathrm{d}}(r) = \tfrac{1}{2}\int_0^r \left(r^2 - R^2\right) D_2\left[\bar{\omega}_{\mathrm{d}}\right]\, dR + \tfrac{1}{2}r^2\,. \tag{D 4}$$

The volume profile $\bar{\Omega}(r)$ and the film thickness $\bar{h}_{\rm d}(r)$ were derived from $\bar{\omega}_{\rm d}(r)$ using equations (5.27)–(5.28).

Inserting the basis-function expansion (D 3) into equation (5.14) yields the amplitude of the square-root film profile at the rim (5.11),

$$\bar{\alpha} = 3\sqrt{2}a_0^{-1}\bar{\Omega}_1\,, \tag{D 5}$$

where a_0 is the first expansion coefficient in (D 3) and $\bar{\Omega}_1$ is the dome volume. According to our numerical solution, $a_0 \doteq 1.500$ and $\bar{\Omega}_1$ is given by (5.31), thus we obtain the numerical value of $\bar{\alpha}$ by equation (5.32).

D.3. *Discretization for thin film simulations*

In our thin film simulations, the interface was discretized on the finite interval $0 \geq r \geq R_\infty$ (where R_∞ is a cutoff parameter). To resolve the lateral length scales in the rim as well as in the dome and outer regions the spatial coordinate r was discretized using non-uniform adaptive mesh, defined by the time-dependent mesh-point density function

$$\rho(r,t) = N_0\left[(1-x_{\rm r})R_\infty^{-1} + x_{\rm r}\,g\,(\tilde{x})\right]. \tag{D 6}$$

Here, N_0 is the number of mesh points, $x_{\rm r}$ is the fraction of mesh points used to discretize the rim region, $g(\tilde{x})$ is a normalized distribution function that resolves the curvature, curvature gradient, and tangential stress in the rim region, and $\tilde{x}$ is the rim variable (3.6). Accurate resolution of the far-field rim stress was found to be important, thus a distribution function with algebraic decay compatible with equation (4.19) was used in our simulations. Given the normalization $\int_{-\infty}^{+\infty} g(t)dt = 1$, the node-point density function (D 6) satisfies $\int_0^{R_\infty} \rho(r,t)\,{\rm d}r = N_0$ at long times, $h_{\rm m}^{-1/2}(t) \gg 1$.

In our simulations, approximately two-thirds of the nodes were used to resolve the rim region, i.e., $x_{\rm r} \approx 2/3$. Our long-time numerical results were insensitive to the truncation in r for $R_\infty > 3$; however, we used $R_\infty = 20$ in order to resolve the short-time behavior also. For $x_{\rm r}$ and R_∞ fixed, we observed $O(N_0^{-2})$ numerical convergence. The results depicted in figures 6–9 were obtained using $N_0 = 300$ and are converged to the resolution of the figures.