Supplementary Material to the paper "Analytically approximate natural sloshing modes for a spherical tank shape"

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— A —

The algorithm starts with the two functions

$$w_0^{(m)}(a,b) = \left[-\frac{2(\sqrt{a^2+b^2}+b)}{a}\right]^m, \quad w_{-1}^{(m)}(a,b) = -\frac{1}{\sqrt{a^2+b^2}}w_0^{(m)}(a,b) \qquad (1)$$

and the two recursive relations

$$w_k^{(m)}(a,b) = \frac{(2k-1)bw_{k-1}^{(m)}(a,b) - (k-m-1)(a^2+b^2)w_{k-2}^{(m)}(a,b)}{m+k}, \quad k = 1,2,\ldots; \qquad (2a)$$

$$w_k^{(m)}(a,b) = \frac{-(2k+3)bw_{k+1}^{(m)}(a,b) + (k+m+2)w_{k+2}^{(m)}(a,b)}{(a^2+b^2)(m-k-1)}, \quad k = -2,-3,\ldots. \qquad (2b)$$

Eqs. (1) and (2) introduce $w_k^{(m)}(a,b)$ for arbitrary integer k and $m \geqslant 0$. The formulas

$$\begin{aligned} \frac{\partial w_k^{(m)}(a,b)}{\partial a} &= \frac{1}{a}\left[kw_k^{(m)}(a,b) - (k-m)bw_{k-1}^{(m)}(a,b)\right] \\ \frac{\partial w_k^{(m)}(a,b)}{\partial b} &= (k-m)w_{k-1}^{(m)}(a,b) \end{aligned} \qquad (3)$$

make it possible to find partial derivatives.

Based on $w_k^{(m)}(a,b)$, we can define

$$\phi_j^{(m)}(r,z) = \frac{2}{\sqrt{r^2+(z-1)^2}}\psi_j^{(m)}\left(\frac{2r}{r^2+(z-1)^2}, \frac{2(z-1)}{r^2+(z-1)^2}\right), \qquad (4)$$

where

$$\psi_j^{(m)}(\xi,\zeta) = (2j-1)w_{m+2j-2}^{(m)}(\xi,\zeta+1) + w_{m+2j-1}^{(m)}(\xi,\zeta+1) - 2(m+j)w_{m+2j}^{(m)}(\xi,\zeta+1) \qquad (5)$$

for $m \geqslant 0$ and $j \geqslant 1$.

There is an extra function $\phi_0^{(m)}(r,z)$ defined by (4) with $\psi_0^{(m)}(r,z)$. For $m = 0$, $\psi_0^{(0)}(r,z) = 1$, but, when $m \geqslant 1$, $\psi_0^{(m)}(\xi,\zeta) = \sum_{k=-1}^{m} b_k w_k^{(m)}(\xi,\eta)$, where $b_m = 1$, $b_{m-1} = -1/(2m)$ and $b_{k-1} = -2b_k(k+1)/(k+m)$, $k = (m-1),\ldots,0$. For instance, $b_1 = 1, b_0 = -1/2$ and $b_{-1} = 1$ as $m = 1$.

One should note that $\psi_j^{(m)}$ is always defined as a linear combination of $w_k^{(m)}$ and, therefore, one can use formulas (3) to compute partial derivatives of $\psi_j^{(m)}$. The Trefftz method requires $\partial\phi_j^{(m)}/\partial z$ which can be computed by the formula

$$\frac{\partial\phi_j^{(m)}}{\partial z}(r,z)=\frac{2}{\sqrt{r^2+(z-1)^2}}\left[-\frac{z-1}{r^2+(z-1)^2}\psi_j^{(m)}(\xi,\eta)\right.$$
$$\left.-\frac{4r(z-1)}{(r^2+(z-1)^2)^2}\frac{\partial\psi_j^{(m)}}{\partial\xi}(\xi,\zeta)+\frac{2(r^2-(z-1)^2)}{(r^2+(z-1)^2)^2}\frac{\partial\psi_j^{(m)}}{\partial\zeta}(\xi,\zeta)\right],$$

where

$$\xi=\frac{2r}{r^2+(z-1)^2},\quad \zeta=\frac{2(z-1)}{r^2+(z-1)^2}.$$

— B —

The Trefftz solution needs the projections

$$\bar{\phi}_k^{(m)}(r,z_0)=\frac{1}{4\pi}\int_0^{\theta_0}\left[-2I_1(m;r,\theta)+I_2(m;r,\theta)\right]\mathcal{V}_{mk}(\theta)\mathrm{d}\theta, \tag{6a}$$

and

$$\frac{\partial\bar{\phi}_k^{(m)}}{\partial z}(r,z_0)=\frac{1}{2\pi}\int_0^{\theta_0}(z_0-\cos\theta)I_3(m;r,\theta)\mathcal{V}_{mk}(\theta)\mathrm{d}\theta$$
$$+\frac{1}{4\pi}\int_0^{2\pi}I_4(m,k;r,\varphi)\cos(m\varphi)\mathrm{d}\varphi, \tag{6b}$$

where

$$I_1(m;r,\theta)=\int_0^{2\pi}\frac{\cos(m\varphi)}{R_\tau}\,\mathrm{d}\varphi,$$
$$I_2(m;r,\theta)=\int_0^{2\pi}\ln|R_\tau-F_\tau|\cos(m\varphi)\mathrm{d}\varphi,$$
$$I_3(m;r,\theta)=\int_0^{2\pi}\frac{\cos(m\varphi)}{R_\tau^3}\,\mathrm{d}\varphi,$$
$$I_4(m,k;r,\varphi)=\int_0^{\theta_0}\frac{1}{R_\tau-F_\tau}\left[\frac{z_0-\cos\theta}{R_\tau}-\cos\theta\right]\mathcal{V}_{mk}(\theta)\,\mathrm{d}\theta.$$

The kernels I_2 and I_4 should be found numerically by using Kress'† quadrature rules handling the asymptotic behavior at the integration ends, but I_1 and I_3 admit analytical expressions. In particular, $I_1(m;r,\theta)=$

$$=\frac{4K(c)}{\sqrt{a+b}}\quad\text{for}\quad m=0,$$
$$=\frac{4}{b\sqrt{a+b}}\left[aK(c)-(a+b)E(c)\right]\quad\text{for}\quad m=1,$$
$$=\frac{4}{3b^2\sqrt{a+b}}\left[(4a^2-b^2)K(c)-4a(a+b)E(c)\right]\quad\text{for}\quad m=2,$$

† Kress, R. (1990) A Nyström method for boundary integral equations in domains with corners. *Numerische Mathematik*, **58**, 145-161.

$$= \frac{4}{15b^3\sqrt{a+b}}\left[a(32a^2-17b^2)K(c)+(a+b)(9b^2-32a^2)E(c)\right] \quad \text{for} \quad m=3,$$

$$= \frac{4}{105b^4\sqrt{a+b}}\left[(384a^4+25b^4-304a^2b^2)K(c)+a(a+b)(208b^2-384a^2)E(c)\right] \quad \text{for} \quad m=4,$$

$$= \frac{4}{315b^5\sqrt{a+b}}[a(2084a^4+411b^4-2144a^2b^2)K(c) + (a+b)(-147b^4-2048a^4+1632a^2b^2)E(c)] \quad \text{for} \quad m=5,$$

$$\ldots\ldots\ldots\ldots;$$

and $I_3(m;r,\theta) =$

$$= \frac{4E(c)}{\sqrt{a+b}} \quad \text{for} \quad m=0,$$

$$= \frac{4}{b\sqrt{a+b}}[aE(c)-(a-b)K(c)] \quad \text{for} \quad m=1,$$

$$= \frac{4}{b^2\sqrt{a+b}}\left[(4a^2-3b^2)E(c)-4a(a-b)K(c)\right] \quad \text{for} \quad m=2,$$

$$= \frac{4}{3b^3\sqrt{a+b}}\left[a(32a^2-29b^2)E(c)-(a-b)(32b^2-5a^2)K(c)\right] \quad \text{for} \quad m=3,$$

$$= \frac{4}{5b^4\sqrt{a+b}}\left[(128a^4+21b^4-144a^2b^2)E(c)-16a(a-b)(8a^2-3b^2)K(c)\right] \quad \text{for} \quad m=4,$$

$$= \frac{4}{35b^5\sqrt{a+b}}[a(2084a^4+771b^4-2784a^2b^2)E(c) - (a-b)(75b^4+2048a^4-1248a^2b^2)E(c)] \quad \text{for} \quad m=5,$$

$$\ldots\ldots\ldots\ldots,$$

with

$$a = 1 + r^2 + z_0^2 - 2z_0\cos\theta; \quad b = 2r\sin\theta; \quad c = \sqrt{\frac{2b}{a+b}},$$

where $K(\cdot)$ and $E(\cdot)$ are complete elliptic integrals† of the first and second kind, respectively.

† Gradstein, I.S. & Ryschik J.M. (2007) Table of Integrals, Series, and Products. Academic Press.

Table 1. Nondimensional eigenvalues κ_{0i} computed within the six stabilized significant figures. The numerical results are consistent with McIver (1989) where his results exist. Computations become unstable with our FORTRAN code for $h > 1.99$. The numerical values $(r_0\kappa_{0i}), i = 1, ..., 4$ at $h = 1.99$ are equal to 4.098, 7.309, 10.48, 13.63 that agrees with the limiting values 4.1213, 7.34208, 10.51708, 13.6773 for $h \to 2$ McIver (1989).

h	κ_{01}	κ_{02}	κ_{03}	κ_{04}	κ_{05}	κ_{06}
0.2	3.82612	9.25613	14.7556	20.1188	25.4184	30.9443
0.3	3.75982	8.46939	13.0211	17.4824	21.8736	28.2305
0.4	3.70804	7.91895	11.9412	15.9077	19.8565	23.7968
0.5	3.67129	7.53290	11.2272	14.8840	18.5284	22.1672
0.6	3.65014	7.26596	10.7450	14.1964	17.6381	21.0755
0.7	3.64543	7.09015	10.4255	13.7389	17.0441	20.3453
0.8	3.65836	6.98858	10.2311	13.4553	16.6723	19.8858
0.9	3.69072	6.95190	10.1415	13.3152	16.4828	19.6511
1.0	3.74517	6.97636	10.1475	13.3042	16.4549	19.5998
1.1	3.82565	7.06317	10.2484	13.4203	16.5867	19.7416
1.2	3.93812	7.21881	10.4521	13.6727	16.8877	20.0936
1.3	4.09169	7.45662	10.7768	14.0843	17.3865	20.6834
1.4	4.30102	7.80055	11.2559	14.6984	18.1350	21.5764
1.5	4.59096	8.29330	11.9501	15.5934	19.2305	22.8344
1.6	5.00753	9.01565	12.9748	16.9190	20.8567	24.7903
1.7	5.64767	10.1393	14.5755	18.9946	23.4062	27.8138
1.8	6.76419	12.1139	17.3959	22.6569	27.9085	33.1552
1.9	9.36800	16.7394	24.0139	31.2579	38.4884	45.7117
1.92	10.4324	18.6330	26.7249	34.7823	42.8245	50.8585
1.93	11.1312	19.8765	28.5052	37.0971	45.6725	54.2392
1.94	12.0002	21.4233	30.7201	39.9768	49.2157	58.4451
1.96	14.6429	26.1284	37.4582	48.7385	59.9965	71.2427
1.98	20.6250	36.7859	52.7255	68.5948	84.4324	100.253
1.99	29.0480	51.8093	74.2630	96.6202	118.935	141.228

Table 2. The same as in Table 1, but for κ_{1i}. The numerical values $(r_0\kappa_{1i}), i = 1, ..., 4$ at $h = 1.99$ are equal to 2.678, 5.833, 8.978, and 12.12 that agrees with the limiting values 2.75476, 5.89215, 9.03285, and 12.1741 for $h \to 2$ McIver (1989).

h	κ_{11}	κ_{12}	κ_{13}	κ_{14}	κ_{15}	κ_{16}
0.2	1.07232	6.20081	11.8821	17.3588	22.7068	30.4793
0.3	1.11334	5.90650	10.6580	15.1880	19.6479	24.0492
0.4	1.15826	5.67422	9.85513	13.8685	17.8381	21.7926
0.5	1.20771	5.49688	9.31187	13.0047	16.6661	20.3151
0.6	1.26250	5.36832	8.94181	12.4233	15.8797	19.3274
0.7	1.32363	5.28386	8.69710	12.0371	15.3557	18.6282
0.8	1.39239	5.24058	8.55088	11.7996	15.0297	18.2602
0.9	1.47048	5.23745	8.48869	11.6855	14.8649	18.0363
1.0	1.56016	5.27555	8.50444	11.6834	14.8461	18.0019
1.1	1.66457	5.35842	8.59880	11.7923	14.9703	18.1411
1.2	1.78818	5.49298	8.77928	12.0208	15.2474	18.4670
1.3	1.93761	5.69110	9.06200	12.3898	15.7028	19.0092
1.4	2.12320	5.97283	9.47622	12.9380	16.3853	19.8257
1.5	2.36224	6.37304	10.0744	13.7355	17.3826	21.0229
1.6	2.68635	6.95710	10.9557	14.9158	18.8621	22.8018
1.7	3.16160	7.86338	12.3315	16.7631	21.1813	25.5931
1.8	3.95929	9.45348	14.7549	20.0224	25.2772	30.5258
1.9	5.76092	13.1737	20.4406	27.6778	34.9033	42.1229
1.92	6.48719	14.6958	22.7696	30.8148	38.8487	46.8769
1.94	7.55156	16.9379	26.2018	35.4385	44.6643	53.8845
1.96	9.33708	20.7176	31.9902	43.2374	54.4741	65.7056
1.98	13.3608	29.2811	45.1129	60.9219	76.7212	92.5154
1.99	18.9830	41.3481	63.6388	85.9083	108.168	130.424

TABLE 3. The same as in Table 1 but for κ_{2i}. The numerical values $(r_0\kappa_{2i}), i = 1, ..., 4$ at $h = 1.99$ are equal to 4.052, 7.282, 10.46, and 13.62 that agrees with the limiting values 4.1213, 7.34208, 10.51708, and 13.6773 for $h \to 2$ McIver (1989).

h	κ_{21}	κ_{22}	κ_{23}	κ_{24}	κ_{25}	κ_{26}
0.2	2.10792	8.39523	14.2944	19.8094	25.3629	33.6626
0.3	2.16868	7.84180	12.6767	17.2401	21.7254	26.5550
0.4	2.23491	7.42178	11.6525	15.6994	19.6938	23.7711
0.5	2.30753	7.11008	10.9705	14.6962	18.3809	22.0526
0.6	2.38767	6.88669	10.5082	14.0217	17.4991	20.9779
0.7	2.47678	6.73722	10.2015	13.5729	16.9125	20.2568
0.8	2.57671	6.65229	10.0156	13.2951	16.5493	19.8222
0.9	2.68986	6.62678	9.93152	13.1584	16.3572	19.5506
1.0	2.81969	6.65941	9.94129	13.1499	16.3312	19.5027
1.1	2.97061	6.75287	10.0450	13.2675	16.4633	19.6514
1.2	3.14917	6.91454	10.2508	13.5208	16.7655	19.9986
1.3	3.36510	7.15835	10.5773	13.9328	17.2641	20.5828
1.4	3.63358	7.50871	11.0578	14.5472	18.0125	21.4656
1.5	3.98004	8.00810	11.7546	15.4352	19.1076	22.7569
1.6	4.45122	8.73895	12.7817	16.7691	20.7336	24.6865
1.7	5.14489	9.87213	14.3856	18.8436	23.2821	27.7083
1.8	6.31546	11.8582	17.2101	22.5091	27.7850	33.0491
1.9	8.97725	16.4986	23.8340	31.1127	38.3658	45.6052
1.92	10.0546	18.3960	26.5465	34.6377	42.7021	50.7521
1.94	11.6361	21.1904	30.5435	39.8330	49.0936	58.3386
1.96	14.2941	25.9005	37.2838	48.5955	59.8746	71.1360
1.98	20.2984	36.5719	52.5640	68.4646	84.3235	100.160
1.99	28.7250	51.6187	74.1398	96.5393	118.883	141.197

TABLE 4. The same as in Table 1 but for κ_{3i}. The values $(r_0\kappa_{3i}), i = 1, ..., 4$ at $h = 1.99$ are equal to 5.332, 8.656, 11.88, and 15.07 that agrees with the limiting values 5.4, 8.71829, 11.94062, and 15.1293 for $h \to 2$ McIver (1989).

h	κ_{31}	κ_{32}	κ_{33}	κ_{34}	κ_{35}	κ_{36}
0.2	3.12948	10.4883	16.5802	22.1594	27.7926	36.0779
0.3	3.20245	9.66514	14.5968	19.2145	23.7359	28.7241
0.4	3.28209	9.05998	13.3674	17.4636	21.4917	25.6446
0.5	3.36959	8.62166	12.5554	16.3272	20.0433	23.7434
0.6	3.46642	8.31214	12.0064	15.5635	19.0700	22.5704
0.7	3.57445	8.10557	11.6416	15.0548	18.4237	21.8545
0.8	3.69610	7.98544	11.4186	14.7392	18.0296	21.4233
0.9	3.83457	7.94237	11.3148	14.5808	17.8058	21.0272
1.0	3.99416	7.97281	11.3200	14.5668	17.7736	20.9630
1.1	4.18079	8.07852	11.4330	14.6906	17.9131	21.1155
1.2	4.40306	8.26740	11.6645	14.9711	18.2404	21.4899
1.3	4.67360	8.55541	12.0330	15.4253	18.7807	22.1112
1.4	5.01220	8.97086	12.5788	16.1042	19.5941	23.0647
1.5	5.45227	9.56460	13.3693	17.0943	20.7843	24.4564
1.6	6.05461	10.4325	14.5350	18.5607	22.5501	26.5241
1.7	6.94366	11.7783	16.3573	20.8571	25.3238	29.7619
1.8	8.46298	14.1374	19.5642	24.9123	30.2209	35.5088
1.9	11.9293	19.6472	27.0868	34.4301	41.7274	48.9990
1.92	13.3358	21.9000	30.1667	38.3296	46.4430	54.5285
1.94	15.4025	25.2178	34.7049	44.0767	53.3933	62.6790
1.96	18.8789	30.8102	42.3576	53.7697	65.1169	76.4276
1.98	26.7397	43.4817	59.7065	75.7486	91.7030	107.608
1.99	37.7945	61.3581	84.2090	106.808	129.287	151.698

TABLE 5. The same as in Table 1 but for κ_{4i}.

h/R_0	κ_{41}	κ_{42}	κ_{43}	κ_{44}	κ_{45}	κ_{46}
0.2	4.14409	12.5164	18.7820	24.4363	30.0902	40.5171
0.3	4.22556	11.4190	16.4514	21.1320	25.6964	29.9179
0.4	4.31448	10.5989	14.9661	19.1324	23.1547	26.9415
0.5	4.41286	10.0731	14.0891	17.9112	21.6623	25.3750
0.6	4.52260	9.68257	13.4567	17.0625	20.6014	24.1122
0.7	4.64537	9.42359	13.0366	16.4960	19.8952	23.2535
0.8	4.78446	9.26862	12.7785	16.1381	19.4129	22.7132
0.9	4.94374	9.21300	12.6552	15.9652	19.2186	22.4647
1.0	5.12735	9.24215	12.6574	15.9458	19.1790	22.4280
1.1	5.34664	9.36082	12.7811	16.0817	19.3330	22.5654
1.2	5.60203	9.56912	12.9698	16.3490	19.6676	22.9535
1.3	5.92822	9.90664	13.4473	16.8767	20.2495	23.6063
1.4	6.33167	10.3851	14.0550	17.6193	21.1389	24.6292
1.5	6.85941	11.0689	14.9360	18.7028	22.4224	26.1089
1.6	7.58677	12.0686	16.2378	20.3071	24.3288	28.3221
1.7	8.66936	13.6182	18.2702	22.8077	27.3188	31.7886
1.8	10.5165	16.3363	21.8290	27.2311	32.5790	37.9166
1.9	14.7576	22.6836	30.2397	37.6585	45.0091	52.3204
1.92	16.4810	25.2789	33.6753	41.9222	50.0947	58.2243
1.94	19.0148	29.1013	38.7377	48.2059	57.5904	66.9266
1.96	23.2790	35.5445	47.2744	58.8039	70.2337	81.6058
1.98	32.9268	50.1446	66.6273	82.8348	98.9053	114.897
1.99	46.5093	70.7483	93.9637	116.796	139.438	161.971

TABLE 6. The same as in Table 1 but for κ_{5i}.

h/R_0	κ_{51}	κ_{52}	κ_{53}	κ_{54}	κ_{55}	κ_{56}
0.2	5.15470	14.4990	20.9238	26.6713	33.0975	45.8110
0.3	5.24252	13.1254	18.2586	23.0054	27.6237	33.0731
0.4	5.33907	12.1605	16.6436	20.8541	24.9618	29.3109
0.5	5.44608	11.4845	15.5857	19.4628	23.2540	27.0294
0.6	5.56506	11.0158	14.8714	18.5173	22.0993	25.6382
0.7	5.70068	10.7077	14.3969	17.9007	21.3388	24.7648
0.8	5.84282	10.5244	14.1024	17.5168	20.8567	24.1362
0.9	6.03181	10.4528	13.9634	17.3196	20.6055	23.8837
1.0	6.23959	10.4817	13.9590	17.2953	20.5637	23.7954
1.1	6.48626	10.6123	14.0972	17.4400	20.7176	23.9597
1.2	6.78404	10.8541	14.3776	17.7661	21.0913	24.3777
1.3	7.15127	11.2262	14.8286	18.3021	21.7150	25.1276
1.4	7.61698	11.7649	15.4972	19.1047	22.6541	26.1765
1.5	8.22762	12.5266	16.4633	20.1452	24.0096	27.6616
1.6	9.07688	13.6638	17.9002	22.0154	26.0706	30.0919
1.7	10.3447	15.4131	20.1377	24.7360	29.2735	33.7737
1.8	12.5146	18.4783	24.0782	29.5397	34.9300	40.2808
1.9	17.5097	25.6400	33.3161	40.8158	48.2251	55.5810
1.92	19.5420	28.5685	37.0987	45.4354	53.6729	61.8522
1.94	22.5306	32.8817	42.6719	52.2434	61.7026	71.0957
1.96	27.5611	40.1513	52.0694	63.7249	75.2458	86.6872
1.98	38.9115	56.5911	73.3398	89.7251	105.925	122.014
1.99	54.7266	79.6196	103.209	126.291	149.113	171.785