1. Comparison of stability analysis to previous work

The stability problem (6.4) can be understood in the context of previous work. Benjamin (1957) and Yih (1963) have studied the stability of fluid flowing down an inclined plane with a free surface in two dimensions, and Kao (1965a,b) has extended this by considering the effects of a stratified two-fluid system flowing down an inclined plane, although he did not consider arbitrary density profiles. In addition Yih (1967) has studied the stability of flow with viscous stratification in two dimensions, which although quite different physically, bears many mathematical similarities to the present study.

Finally, Sangster (1964) studied the problem of the flow of a two-fluid system down an inclined surface with a rigid lid (no free-surface) in two dimensions, with and without surface tension at the fluid-fluid interface. Therefore, the present study corresponds to the problem studied by Sangster for an inclination of 90°, zero surface-tension, and the addition that one of the boundaries is towed at a fixed rate. Sangster did not investigate the time-evolution of the thickness of one stream relative to the other, but only considered the stability analysis for a fixed ratio of the bottom fluid thickness to top fluid thickness. Through the use of a long-wave expansion (as used here) in addition to a truncated Frobenius series for the eigenfunction (not used here), Sangster obtained approximations to the real and imaginary components of the eigenvalue c. The analysis of Sangster was limited to the condition that the thickness of the lower stratum be greater than that of the upper, which is exactly the opposite of the situation of primary attention in the current investigation. In the current study, primary attention is given to the situation in which the size of the entrained layer (corresponding to the lower-stratum in Sangster's study) is small, although much of the analysis presented here does not require this condition. Finally, Sangster's calculations are rather laborious and the resulting eigenvalue expressions are not shown to simplify as do the calculations presented here, which result in the comparatively compact expressions (6.10) and (6.11). In the axisymmetric problem, the corresponding expressions do not simplify to such compact expressions.

In the investigations (Kao 1965a,b; Yih 1963, 1967), there were found "hidden neutral modes" which are analogous to the modes found here. These modes are termed hidden since they arise either from the misalignment of the density gradient with gravity (Benjamin 1957; Yih 1963; Kao 1965a,b), or from the viscous stratification (Yih 1967), and they vanish if these physical conditions are not present, for instance in homogeneous flows. Further the modes are neutral to leading order for long waves. Thus the modes found in the present study lie under this classification of hidden neutral modes.

In order to understand the connections to these previous investigations, we consider a two-dimensional stratified fluid flow bounded by two walls of infinite extent which enclose a channel (no free surface present) that is inclined with respect to gravity by an angle θ. One wall of the channel is towed at speed U_0 which may be zero, and the density profile is arbitrary. See figure (1) for a diagram.

We let $\mathbf{x} = (x_1, x_2)$ be the horizontal and vertical coordinates with respect to the channel walls and let $\mathbf{u} = (u_1, u_2)$ be the corresponding velocities. The no-ndimensional Boussinesq approximation equations are given by

$$Re\frac{D\mathbf{u}}{Dt} = -\nabla p + \Delta\mathbf{u} + \frac{Re}{Fr^2}\,\rho\,(\sin\theta, -\cos\theta)\,, \qquad \nabla\cdot\mathbf{u} = 0\,, \qquad \frac{D\rho}{Dt} = 0\,. \tag{1.1}$$

We allow for an arbitrary density profile varying only in the vertical direction $\rho_B(x_2)$, and a shear velocity profile $u_1 = U(x_2)$ as our background solution. The density and velocity are related by

$$U''(x_2) + \frac{Re}{Fr^2}\sin\theta\;\rho_B(x_2) + \beta = 0\,, \tag{1.2}$$

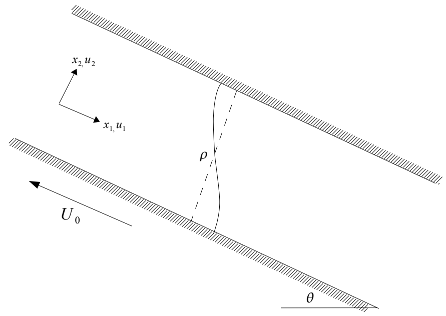

Figure 1: Diagram of stratified flow through a channel inclined with respect to gravity.

where β is a constant arising for the horizontal pressure gradient.

A stream-function ψ exists with relation to the velocities $u_1 = -\partial\psi/\partial x_2$ and $u_2 = \partial\psi/\partial x_1$. We perturb the background solution to obtain a stability equation for the perturbation stream-function ψ

$$\begin{aligned}(D^2-k^2)^2\psi - i\,k\,Re\,\left((U_B-c)(D^2-k^2)\psi - U_B''\psi\right)\\ +\,ik\cos\theta\,\frac{Re}{Fr^2}\frac{\rho_B'}{U_B-c}\psi + \sin\theta\,\frac{Re}{Fr^2}D\left(\frac{\rho_B'}{U_B-c}\psi\right) = 0\,.\end{aligned} \tag{1.3}$$

To the knowledge of the authors, this stability operator for arbitrary background density profiles has not been previously documented. The previous investigation (Benjamin 1957; Yih 1963; Kao 1965a,b; Yih 1967) have focused on the special case of a two-fluid system and thus the effects of the terms involving ρ_B in (1.3) entered analysis in the boundary conditions at the fluid-fluid interface only.

We now give an understanding of each component of the stability operator (1.3) in light of previous work and the present study. The top line of the operator is simply the classic Orr-Somerfeld equation for homogeneous fluid flow and does not give rise to neutral modes at leading order in the long-wave expansion. Indeed all modes are damped for sufficiently long waves or low Reynolds number in the Orr-Somerfeld equation (see Drazin & Reid 1981, pp. 158-164).

The term multiplying $\cos\theta$ arises from the horizontal component of density layering, and can be understood as part of the Taylor-Goldstein equation for the inviscid limit. Indeed the characteristic value of the term $-\rho_B'/Fr^2$ is the overall Richardson number for the inviscid Boussinesq approximation, and in the case of $\theta = 0$ and $Re \to \infty$ we obtain the Taylor-Goldstein equation exactly. Therefore, we will refer to this term multiplying $\cos\theta$ as the Taylor-Goldstein term. Notice that the wavenumber k multiplies the Taylor-Goldstein term and so it does not enter at the leading order of the long-wave expansion and therefore cannot give rise to leading order neutral modes. Therefore, in the case of $\theta = 0$ and with the same scaling of the eigenvalue as in the homogeneous case, the Taylor-Goldstein term drops out of the leading order problem and the leading order behavior of the eigenvalue will be identical to that as in the homogeneous case.

The term multiplying $\sin\theta$ arises from the vertical component of the density layering and we will refer to it as the vertical density layering term. Since this term is not multiplied by the wavenumber, it enters the leading order equation for the long-wave expansion, and further since the eigenvalue c is present in this term, it allows for $c = O(1)$ to leading order in this expansion. Therefore, this term is necessary in (1.3) for the presence of hidden neutral modes, and it is the presence of this term in the stability operator that is the focus of the current study.

For further simplification we can substitute the relationship between the background density and velocity profiles

$$\sin\theta\,\frac{Re}{Fr^2}\rho_B'(x_2) = -U_B'''(x_2)\,, \tag{1.4}$$

into (1.3) to obtain a stability operator in which we have eliminated ρ_B

$$\begin{aligned}(D^2-k^2)^2\psi - i\,k\,Re\,\big((U_B-c)(D^2-k^2)\psi - U_B''\psi\big)\\ - ik\cot\theta\,\frac{U_B'''}{U_B-c}\psi - D\left(\frac{U_B'''}{U_B-c}\psi\right) = 0\,.\end{aligned} \tag{1.5}$$

The previous investigations (Benjamin 1957; Yih 1963; Kao 1965a,b) have only studied the effects of the vertical density layering term in conjunction with the Taylor-Goldstein term, and once again these terms only entered in the boundary conditions at the fluid-fluid interface. In the Orr-Somerfeld equation the first power of k only enters in product with Re, while in the Taylor-Goldstein term k is not in product with Re. Therefore, in the long-wave expansion of c, the first correction will be a function of Re if the Taylor-Goldstein term is present and therefore there is the potential for a stability transition to occur as the Reynolds number is varied, as has been seen in these previous studies (Benjamin 1957; Yih 1963; Kao 1965a,b). In the current study however, the vertical density layering term has been isolated so that there is no Taylor-Goldstein term present in the stability operator. In this case, the first power of k only enters the operator in product with Re and so the first correction to c has trivial dependence on Re as it is only a scaling factor in the magnitude of the first correction, as seen in (6.8), (6.9) and (6.11). Therefore, the flow configuration under current investigation in which the density layering is completely vertical has the interesting feature that *its stability or instability to long waves is independent of the Reynolds number, and only depends on the density profile.* Of course the magnitude of the growth or decay rates of disturbances does depend on the Reynolds number. This interesting feature was also found by Yih (1967) in the stability analysis of a flow with viscous stratification.

2. Stability subtleties in the limit to the unbounded domain

For the two-dimensional stability problem, consider the limit to the semi-infinite domain with $L \to \infty$ and all other dimensional parameters fixed. This implies that $\kappa \to \infty$, and in this limit the lubrication theory gives $h_\infty \sim \kappa^{-1/2}$. Substitution of this relationship into (6.11) gives $c_1 \sim 3\,i/140\;\kappa^{-1/2}$. Since this value is positive it indicates that the entrained layer will grow to a size that is unstable. We note that, however, the magnitude of the instability tends to zero in this limit.

Now consider the same limit to an unbounded domain in the axisymmetric stability problem. This implies that $\kappa \to \infty$, $a \to 0$, and $h \to 0$, and so the leading order correction for c_1 given by (C4) becomes valid with the $\log h$ term asymptotically dominating. This gives the asymptotic expression for c_1

$$c_1 \sim -\frac{i\,\kappa\eta^2\,(\log h - \log a)\,\log h}{96(1+\log a)^3}\left(1-\frac{\kappa}{2}\eta^2\right) \qquad \text{for } \kappa \gg 1,\ a \ll h \ll 1,\ |\log h| \gg 1\,, \tag{2.1}$$

where η is given by (4.31). Therefore, the asymptotic position of the neutral stability curve satisfies $\kappa h^2 = 2$, which is the same relationship as for the limiting layer size given by the lubrication theory as can be seen in (4.35). Therefore it can be concluded that in the limit to the semi-infinite domain, the entrained layer grows to a size asymptotically

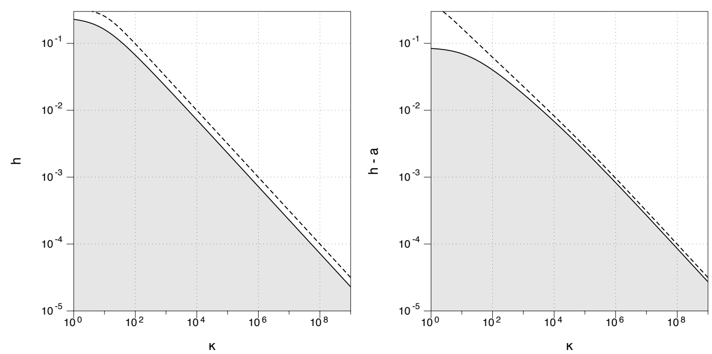

Figure 2: Log-log plot of neutral stability curves (solid lines) and zero-propagation speed, or h_∞, curves (dashed lines) for large κ in two-dimensons (left) and the axisymmetric geometric (right) with the fibre radius $a = 0.004$ subtracted.

matching the size of a neutrally stable layer - an unexpected coincidence that does not occur in the two-dimensional case.

Now consider the entire expression (C4), in the long-time lubrication limit of $\kappa\eta^2 \to 2$,

$$c_1 \sim -\frac{i\,5\,(\log h - \log a)}{512(1+\log a)^3} \qquad \text{for } \kappa \gg 1,\ a \ll h \ll 1,\ \kappa\eta^2 = 2\,. \tag{2.2}$$

As $a \to 0$ the term $(1 + \log a)^3$ becomes negative, so that the imaginary component of the eigenvalue is positive which indicates instability, and further the magnitude of the instability only decays logarithmically with a as $a \to 0$.

There is a subtle difference between the two-dimensional case and the axisymmetric case in the limit to the unbounded domain. In the two-dimensional case, the layer grows to an unstable size whereas the magnitude of instability decays algebraically with L (through the dependence on κ). On the other hand in the the axisymmetric case, the layer grows to a size that asymptotically matches the size of a neutrally stable layer, however the limiting layer size remains unstable with a magnitude that decays very slowly (logarithmically) with L (through the dependence on a). Figure 2 shows the curves of neutral stability and the curves of h_∞ for both cases, and it can be seen that in the two-dimensional case the two curves remain separated on a log-log plot as $\kappa \to \infty$, whereas in the axisymmetric case the curves approach one another in this same limit.

3. Smooth density transition velocity profiles

Here, we find a class of exact velocity profiles for the case of a vertically layered parallel flow with a smoothed density profile, with density transition located at $x = h$, and a transition length-scale of $\lambda \ll h$. We employ an arctangent functional form for the density profile so that the velocity profile may be obtained in closed form. Let the density profile be given by

$$\rho(x) = 1 - \frac{\Delta\rho}{\rho_0}\left(\frac{1}{\pi}\arctan\left(\frac{x-h}{\lambda}\right) - \frac{1}{2}\right). \tag{3.1}$$

The differential equation for the velocity profile becomes

$$w''(x) = \beta - \kappa\left(\frac{1}{\pi}\arctan\left(\frac{x-h}{\lambda}\right) - \frac{1}{2}\right). \tag{3.2}$$

The exact solution to this differential equation can be written in closed form as

$$\begin{aligned} w(x) =& A_0(h,\lambda) + A_1(h,\lambda)\,(x-1) + A_2(h,\lambda)\,x(x-1) \\ &- \frac{\kappa}{2\pi}\left((x-h)^2-\lambda^2\right)\arctan\left(\frac{x-h}{\lambda}\right) \\ &+ \frac{\kappa\lambda}{2\pi}(x-h)\log\left(1+\frac{(x-h)^2}{\lambda^2}\right). \end{aligned} \tag{3.3}$$

The coefficients A_0 and A_1 are chosen to satisfy the boundary conditions, giving

$$A_0(h,\lambda) = \frac{\kappa}{2\pi}\left((1-h)^2-\lambda^2\right)\arctan\left(\frac{1-h}{\lambda}\right) - \frac{\kappa\lambda}{2\pi}(1-h)\log\left(1+\frac{(1-h)^2}{\lambda^2}\right) \tag{3.4}$$

$$A_1(h,\lambda) = -1 + A_0(h,\lambda) + \frac{\kappa}{2\pi}(h^2-\lambda^2)\arctan\frac{h}{\lambda} - \frac{\kappa\lambda h}{2\pi}\log\left(1+\frac{h^2}{\lambda^2}\right). \tag{3.5}$$

A_2 is obtained by enforcing the vanishing flux condition. For convenience we define the quantities

$$I_1(h,\lambda) \equiv 3\int_0^1 \left((x-h)^2-\lambda^2\right)\arctan\left(\frac{x-h}{\lambda}\right)dx \tag{3.6}$$

$$I_2(h,\lambda) \equiv 2\int_0^1 (x-h)\log\left(1+\frac{(x-h)^2}{\lambda^2}\right)dx\,. \tag{3.7}$$

These quantities are given in closed form by

$$I_1(h,\lambda) = (x-h)\left((x-h)^2-3\lambda^2\right)\arctan\left(\frac{x-h}{\lambda}\right)\Bigg|_0^1 \tag{3.8}$$

$$-\frac{\lambda}{2}(1-2h) + 2\lambda^3\log\left(\frac{(1-h)^2+\lambda^2}{h^2+\lambda^2}\right), \tag{3.9}$$

$$I_2(h,\lambda) = (x-h)^2\log\left(1+\frac{(x-h)^2}{\lambda^2}\right)\Bigg|_0^1 + 2h - 1 + \lambda^2\log\left(\frac{(1-h)^2+\lambda^2}{h^2+\lambda^2}\right).$$

The coefficient A_2 can be expressed in terms of these quantities as

$$A_2(h,\lambda) = 6A_0(h,\lambda) - 3A_1(h,\lambda) - \frac{\kappa}{\pi}I_1(h,\lambda) + \frac{3\kappa\lambda}{2\pi}I_2(h,\lambda)\,. \tag{3.10}$$

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