A study of two-fluid model equations

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Appendix A. Derivation of (2.4)

Let us integrate the term $\frac{\partial}{\partial t}(\rho_c \boldsymbol{u}_c)$, at a point of interest, over a duration Λ from time $t - \frac{\Lambda}{2}$ to $t + \frac{\Lambda}{2}$. Assuming that the point of interest is covered by the continuous phase at $t - \frac{\Lambda}{2}$ and $t + \frac{\Lambda}{2}$, (A1) is obtained:

$$\int_{t-\frac{\Lambda}{2}}^{t+\frac{\Lambda}{2}} [H_c \frac{\partial}{\partial t}(\rho_c \boldsymbol{u}_c)]dt = (\rho_c \boldsymbol{u}_c)|_{t+\frac{\Lambda}{2}} - (\rho_c \boldsymbol{u}_c)|_{t-\frac{\Lambda}{2}} + \sum_{i=1}^{N} \{(\rho_c \boldsymbol{u}_c)|_{T_i^a} - (\rho_c \boldsymbol{u}_c)|_{T_i^l}\} \quad (A\,1)$$

Here, T_i^a and T_i^l respectively denote the arriving time and leaving time of the particle i at the point of interest.

Equation (A2) now holds:

$$(1-\epsilon)\overline{\rho_c \boldsymbol{u}_c}^c = \frac{\Lambda_c}{\Lambda}\frac{1}{\Lambda_c}\int_{t-\frac{\Lambda}{2}}^{t+\frac{\Lambda}{2}} H_c \rho_c \boldsymbol{u}_c dt = \frac{1}{\Lambda}\int_{t-\frac{\Lambda}{2}}^{t+\frac{\Lambda}{2}} H_c \rho_c \boldsymbol{u}_c dt \quad (A\,2)$$

from which it follows that

$$\frac{\partial}{\partial t}\{(1-\epsilon)\overline{\rho_c \boldsymbol{u}_c}^c\} = \frac{1}{\Lambda}\{(\rho_c \boldsymbol{u}_c)|_{t+\frac{\Lambda}{2}} - (\rho_c \boldsymbol{u}_c)|_{t-\frac{\Lambda}{2}}\} \quad (A\,3)$$

Upon substituting (A3) into (A1), (A4) is obtained:

$$\frac{1}{\Lambda}\int_{t-\frac{\Lambda}{2}}^{t+\frac{\Lambda}{2}} [H_c \frac{\partial}{\partial t}(\rho_c \boldsymbol{u}_c)]dt = \frac{\partial}{\partial t}\{(1-\epsilon)\overline{\rho_c \boldsymbol{u}_c}^c\} + \frac{1}{\Lambda}\sum_{i=1}^{N} \{(\rho_c \boldsymbol{u}_c)|_{T_i^a} - (\rho_c \boldsymbol{u}_c)|_{T_i^l}\} \quad (A\,4)$$

In fact, (A4) holds regardless of whether the point of interest is covered by the continuous phase or the dispersed phase at the beginning and end of the interval of time-integration.

Since T_i^a and T_i^l are functions of position in space, they can be expressed as:

$$t = T_i^a(x, y, z) \tag{A 5}$$

$$t = T_i^l(x, y, z) \tag{A 6}$$

By differentiating both sides of (A5) and (A6) with respect to time, we have

$$1 = \frac{dx}{dt}\frac{\partial T_i^a}{\partial x} + \frac{dy}{dt}\frac{\partial T_i^a}{\partial y} + \frac{dz}{dt}\frac{\partial T_i^a}{\partial z} = \boldsymbol{u}^s \cdot \boldsymbol{\xi}_i^a \tag{A 7}$$

$$1 = \frac{dx}{dt}\frac{\partial T_i^l}{\partial x} + \frac{dy}{dt}\frac{\partial T_i^l}{\partial y} + \frac{dz}{dt}\frac{\partial T_i^l}{\partial z} = \boldsymbol{u}^s \cdot \boldsymbol{\xi}_i^l \tag{A 8}$$

Here, $\boldsymbol{u}^s$ is a moving velocity vector of surface and $\boldsymbol{\xi}_i^a$ and $\boldsymbol{\xi}_i^l$ are gradient vectors of the surfaces $t = T_i^a$ and $t = T_i^l$, respectively. Since surface gradient vectors are perpendicular to the surface, (A 9)and (A 10) hold for the case with no mass transfer across the surface.

$$1 = \boldsymbol{u} \cdot \boldsymbol{\xi}_i^a \tag{A 9}$$

$$1 = \boldsymbol{u} \cdot \boldsymbol{\xi}_i^l \tag{A 10}$$

Here, $\boldsymbol{u}$ is a velocity vector for either the continuous or dispersed phase at the interface.

Time-integration of $\nabla \cdot (\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)$ can be transformed to (A11) by applying Leibniz' rule:

$$\begin{aligned}
&\int_{t-\frac{\Lambda}{2}}^{t+\frac{\Lambda}{2}} [H_c \nabla(\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)]dt \\
&= \int_{t-\frac{\Lambda}{2}}^{T_1^a} \{\nabla \cdot (\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)\}dt + \sum_{i=1}^{N-1} \int_{T_i^l}^{T_{i+1}^a} \{\nabla \cdot (\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)\}dt + \int_{T_N^l}^{t+\frac{\Lambda}{2}} \{\nabla \cdot (\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)\}dt \\
&= \nabla \cdot \{\int_{t-\frac{\Lambda}{2}}^{T_1^a} (\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)dt + \sum_{i=1}^{N-1} \int_{T_i^l}^{T_{i+1}^a} (\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)dt + \int_{T_N^l}^{t+\frac{\Lambda}{2}} (\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)dt\} \\
&\quad + \sum_{i=1}^{N} \{(\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)|_{T_i^l} \cdot \boldsymbol{\xi}_i^l - (\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)|_{T_i^a} \cdot \boldsymbol{\xi}_i^a\} \\
&= \nabla \cdot [\int_{t-\frac{\Lambda}{2}}^{t-\frac{\Lambda}{2}} \{H_c(\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)\}dt] + \sum_{i=1}^{N} \{(\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)|_{T_i^l} \cdot \boldsymbol{\xi}_i^l - (\rho_c \boldsymbol{u}_c \boldsymbol{u}_c)|_{T_i^a} \cdot \boldsymbol{\xi}_i^a\} \\
&= \Lambda \nabla \cdot \{(1-\epsilon)\overline{\rho_c \boldsymbol{u}_c \boldsymbol{u}_c}^c\} + \sum_{i=1}^{N} \{(\rho_c \boldsymbol{u}_c|_{T_i^l} - (\rho_c \boldsymbol{u}_c)|_{T_i^a}\}
\end{aligned} \tag{A 11}$$

Here, (2.9), (A9) and (A10) have been used.

Equation (2.4) in the main text is obtained from (A4) and (A11),

$$\frac{1}{\Lambda}\int_{t-\frac{\Lambda}{2}}^{t+\frac{\Lambda}{2}}[H_c\{\frac{\partial}{\partial t}(\rho_c\boldsymbol{u}_c)+\nabla\cdot(\rho_c\boldsymbol{u}_c\boldsymbol{u}_c)]dt=\frac{\partial}{\partial t}\{(1-\epsilon)\overline{\rho_c\boldsymbol{u}_c}^c\}+\nabla\cdot\{(1-\epsilon)\overline{\rho_c\boldsymbol{u}_c\boldsymbol{u}}_c\} \quad (2.4)$$

Appendix B. Derivation of (3.8)

Let us determine the time-averaged force acting on a portion of particle surface in a spherical control volume V, having radius γ which is smaller than the particle radius R, and with center at $\boldsymbol{X}_0$. Higher order terms with respect to γ in the equations that follow will ultimately be neglected, because we are focusing on the force per unit volume applicable to infinitesimal volume, which is obtained for the limit $\gamma \to 0$.

We can count number of particles, surfaces of which appear in the volume V during the total time-averaging duration Λ. Figures 4 and 5 show the spatial relation between the control volume V and the particle k. Two spherical coordinate systems, $\{\boldsymbol{X}_0 : r, \theta, \phi\}$ and $\{\boldsymbol{X}'_k : r', \theta', \phi'\}$, are introduced to denote coordinates relative to the center of the volume V and the center of the particle k, respectively. Here, $\boldsymbol{X}_0$ and $\boldsymbol{X}'_k$ are the centers of the control volume and the particle k, and the origins of each spherical coordinate system. In the figures, $\boldsymbol{X}_0$ and $\boldsymbol{X}'_k$ are denoted as O and O'.

When a particle center $\boldsymbol{X}'_k$ is in a spherical shell with inner and outer radii $R-\gamma$ and $R+\gamma$, in the coordinate system $\{\boldsymbol{X}_0 : r, \theta, \phi\}$, a portion of the surface of the particle is within the volume V. Upon denoting the partial surface of the particle k in V as S_k^V, the force acting on S_k^V due to static pressure and shear stress of the continuous phase is given by:

$$\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k) = -\int_{S_k^V} d\theta' d\phi' \left[R^2 sin\theta' \boldsymbol{n}'(\theta', \phi') \cdot \boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta', \phi')\right] \quad \text{(B 1)}$$

Here, $\int_{S_k^V} d\theta' d\phi'$ denotes solid-angular integration over the surface S_k^V with respect to θ'

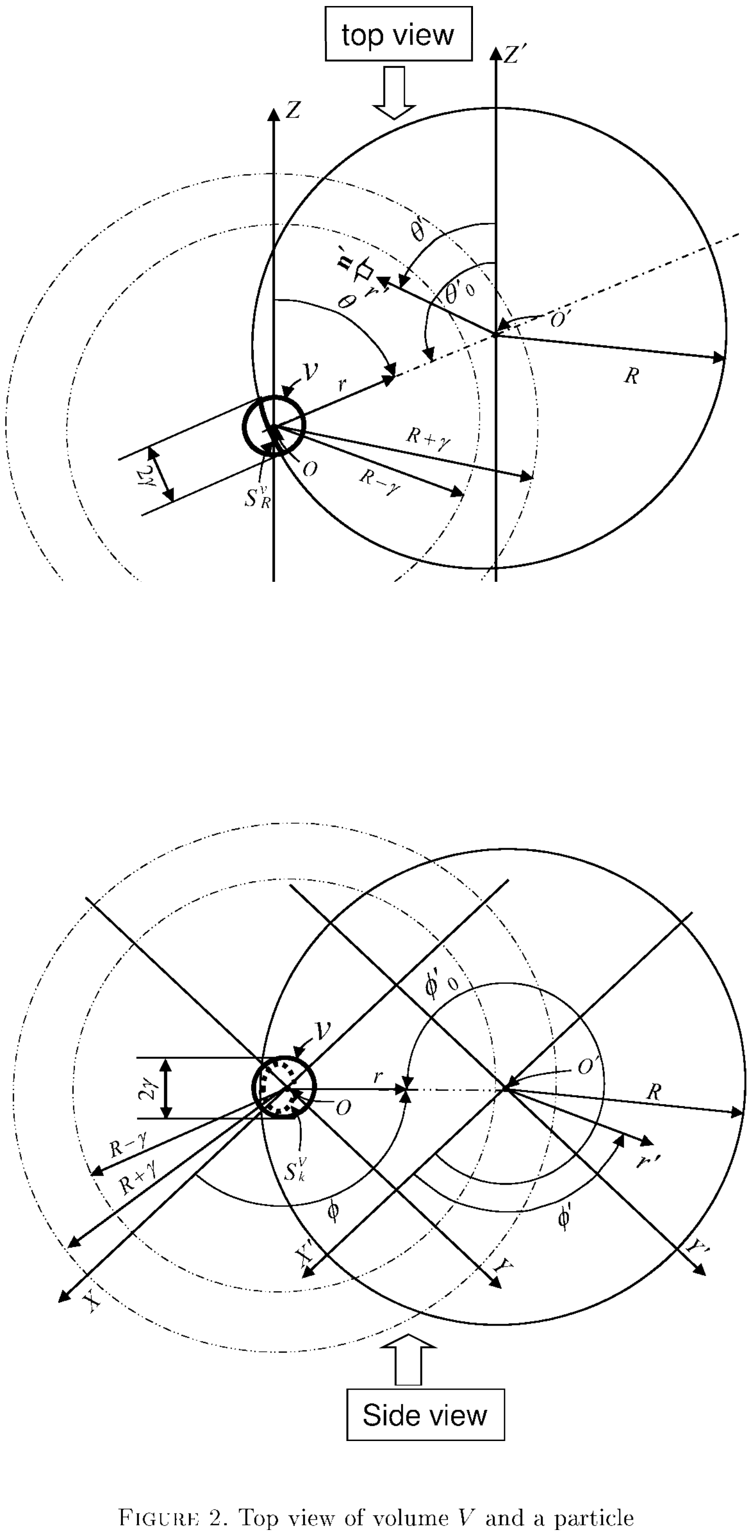

FIGURE 2. Top view of volume V and a particle

and ϕ' in the spherical coordinate system $\{\boldsymbol{X}'_k : r', \theta', \phi'\}$, $\boldsymbol{f}^{\boldsymbol{T}}_k(\boldsymbol{X}'_k)$ is the force due to static pressure and shear stress of the continuous phase on a particle surface S^V_k with center at $\boldsymbol{X}'_k$, and $\boldsymbol{n}'(\theta', \phi')$ is a unit vector along the r' axis. The quantity $\boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta', \phi')$ in the integrand is a tensor defined by (B 2) below, where P_c, $\boldsymbol{\tau}_c$ and $\boldsymbol{I}$ respectively denote the static pressure of continuous phase, the shear stress tensor of the continuous phase, and the unit tensor:

$$\boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta', \phi') = P_c(\boldsymbol{X}'_k : R, \theta', \phi')\boldsymbol{I} + \boldsymbol{\tau}_c(\boldsymbol{X}'_k : R, \theta', \phi') \tag{B 2}$$

Although time t is not explicit in (B1) and (B2), it should be noted that both particle center $\boldsymbol{X}'_k$ and particle surface S^V_k move with time in accordance with the movement of the particle k.

We may choose angular coordinates (θ'_0, ϕ'_0) such that the vector $\boldsymbol{n}'(\theta'_0, \phi'_0)$ is oriented along the line $\boldsymbol{X}'_k - \boldsymbol{X}$. Then, the point $\boldsymbol{X}'_k$ is given by;

$$\boldsymbol{X}'_k = \boldsymbol{X}_0 + r\boldsymbol{n}(\pi - \theta'_0, \pi + \phi'_0), \qquad (R - \gamma) < r < (R + \gamma) \tag{B 3}$$

Here, r and $\boldsymbol{n}(\pi - \theta'_0, \pi + \phi'_0)$ are radial coordinate and unit vector along r axis, respectively, of the spherical coordinatesystem $\{\boldsymbol{X}_0 : r, \theta, \phi\}$.

On the surface S^V_k, the following expressions are obtained as Taylor series around the center of the surface S^V_k, that is, $(\boldsymbol{X}'_k : R, \theta'_0, \phi'_0)$:

$$\begin{aligned}\boldsymbol{n}'(\theta', \phi') &= \boldsymbol{n}'(\theta'_0, \phi'_0) + \left(\frac{\partial \boldsymbol{n}'}{R\partial\theta'}\right)|_{\theta'_0, \phi'_0} R\Delta\theta' \\ &\quad + \left(\frac{\partial \boldsymbol{n}'}{Rsin\theta'\partial\phi}\right)|_{\theta'_0, \phi'_0} Rsin\theta'_0\Delta\theta' + \cdots\cdots \\ &= \boldsymbol{n}'(\theta'_0, \phi'_0) + \boldsymbol{o}(\gamma)\end{aligned} \tag{B 4}$$

$$\begin{aligned}\boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta', \phi') &= \boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta'_0, \phi'_0) + \left(\frac{\partial \boldsymbol{T}_c}{R\partial\theta'}\right)|_{R, \theta'_0, \phi'_0} R\Delta\theta \\ &\quad + \left(\frac{\partial \boldsymbol{T}_c}{Rsin\theta'\partial\phi'}\right)|_{R, \theta'_0, \phi'_0} Rsin\theta'_0\Delta\theta' + \cdots\cdots \\ &= \boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta'_0, \phi'_0) + \boldsymbol{o}'(\gamma)\end{aligned} \tag{B 5}$$

Notations $\boldsymbol{o}(\gamma)$ and $\boldsymbol{o}'(\gamma)$ are vector and tensor quantities, respectively, whose values are zero in the limit $\gamma \to 0$, because the values of $R\Delta\theta'$ and $Rsin\theta'_0\Delta\phi'$ are not larger than γ on S_k^V.

Upon substituting (B4) and (B5) into (B1), we have

$$
\begin{aligned}
&\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k) \\
&= -\int_{S_k^V} d\theta' d\phi' \left[R^2 sin\theta' \{\boldsymbol{n}'(\theta'_0, \phi'_0) + \boldsymbol{o}(\gamma)\} \cdot \{\boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta'_0, \phi'_0) + \boldsymbol{o}(\gamma)\}\right] \\
&= -\{\boldsymbol{n}'(\theta'_0, \phi'_0) + \boldsymbol{o}(\gamma)\} \cdot \{\boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta'_0, \phi'_0) + \boldsymbol{o}(\gamma)\} \int_{S_k^V} d\theta' d\phi' \left[R^2 sin\theta'\right] \\
&= -\{\boldsymbol{n}'(\theta'_0, \phi'_0) + \boldsymbol{o}(\gamma)\} \cdot \{\boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta'_0, \phi'_0) + \boldsymbol{o}(\gamma)\}\pi\{\gamma^2 - (R-r)^2\}(1 + o(\gamma)) \\
&= -\boldsymbol{n}'(\theta'_0, \phi'_0) \cdot \boldsymbol{T}_c(\boldsymbol{X}'_k : R, \theta'_0, \phi'_0)\pi\{\gamma^2 - (R-r)^2\}(1 + o(\gamma))
\end{aligned}
\qquad \text{(B 6)}
$$

In (B6), $\gamma^2 - (R-r)^2$ is the cross sectional area of the volume V perpendicular to the r' axis at (R, θ'_0, ϕ'_0), and the value of r varies with time corresponding to the position vector $\boldsymbol{X}'_k$ defined by (B3).

Denote the contribution of those particles having centers in the volume element $d\theta d\phi\left[\int_{R-\gamma}^{R+\gamma}(r^2 sin\theta)dr\right]$ to the time-averaged force acting on particle surfaces in the control volume V in the time duration Λ, as $\overline{\boldsymbol{f}^{\boldsymbol{T}}}$. Then,

$$
\overline{\boldsymbol{f}^{\boldsymbol{T}}} = \frac{\sum_k \int_{\tau_k} \boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)dt}{\Lambda} \qquad \text{(B 7)}
$$

Here τ_k is the residence time of $\boldsymbol{X}'_k$ in the volume element $d\theta d\phi\left[\int_{R-\gamma}^{R+\gamma}(r^2 sin\theta)dr\right]$, and $\sum_k$ denotes summation over particles whose centers are in the volume element for the time duration Λ. Since the particle k moves in a volume element $d\theta d\phi\left[\int_{R-\gamma}^{R+\gamma}(r^2 sin\theta)dr\right]$, the time-integration in $\int_{\tau_k} \boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)dt$ can be converted to a radial integration by using the r component of the moving velocity of S_k^V, which we denote by v_r^k.

$$\left.\begin{aligned}\int_{\tau_k} \boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)dt &= \int_{r_k}^{r_k+l_k} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{v_r^k}dr \quad \text{for} \quad v_r^k > 0, \\ \int_{\tau_k} \boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)dt &= \int_{r_k+l_k}^{r_k} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|}dr \quad \text{for} \quad v_r^k < 0,,\end{aligned}\right\} \tag{B 8}$$

Here, l_k is the radial displacement of $\boldsymbol{X}'_k$ in the volume element $d\theta d\phi\big[\int_{R-\gamma}^{R+\gamma}(r^2 sin\theta)dr\big]$ during the time duration Λ, the value of which is positive for $v_r^k > 0$ and negative for $v_r^k < 0$, and r_k is the initial radial position of S_k^V when the center of the particle k, that is $\boldsymbol{X}'_k$, appears in the volume element $d\theta d\phi\big[\int_{R-\gamma}^{R+\gamma}(r^2 sin\theta)dr\big]$.

It follows from (B7) and (B8) that

$$\overline{\boldsymbol{f}^{\boldsymbol{T}}} = \frac{1}{\Lambda}\sum_k \int_{Min(r_k,r_k+l_k)}^{Max(r_k,r_k+l_k)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|}dr \tag{B 9}$$

In the summation on the right-hand side the radial integration involves overlap, and the number of overlaps is countable and varies with radial position. Let us introduce the overlap number $N(r)$, which depends on radial position, and is such that

$$\sum_k \int_{Min(r_k,r_k+l_k)}^{Max(r_k,r_k+l_k)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|}dr = \int_{R-\gamma}^{R+\gamma}\{\sum_{N(r)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|}\}dr \tag{B 10}$$

Here, $\sum_{N(r)}$ refers to summation over all overlaps.

The right-hand side of (B10) can be manipulated as:

$$\begin{aligned}\int_{R-\gamma}^{R+\gamma}\{\sum_{N(r)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|}\}dr &= \lim_{N\to\infty}\sum_{i=l}^{N}[\{\sum_{N(r_i)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|}\}|_{r=r_i}](\frac{2\gamma}{N}) \\ &= \lim_{N\to\infty}\sum_{i=l}^{N}[\sum_{N(r_i)}\{(\Delta t)_i^k \boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'|_{r=r_i})\}] \\ &= \lim_{N\to\infty}\sum_{i=l}^{N}[\{\sum_{N(r_i)}(\Delta t)_i^k\}\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i})]\end{aligned} \tag{B 11}$$

The total radial section $[R-\gamma, R+\gamma]$ is divided into N subsections with equal length $\frac{2\gamma}{N}$, and $r_i = R-\gamma+i\frac{2\gamma}{N}$ is the radial position of the subsection i. In the second and the last expressions on the right-hand side of (B11), $(\Delta t)_i^k$ is the residence time of the particle center $\boldsymbol{X}'_k$ in the subsection $[r_i, r_{i+1}]$, and $\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i})$ is the time-averaged value of $\boldsymbol{f}^{\boldsymbol{T}}_k(\boldsymbol{X}'|_{r=r_i})$ for all drops overlapping in the subsection. These are given by the following equations:

$$(\Delta t)^k_i = \frac{1}{|v^k_r|_{r=r_i}|}\frac{2\gamma}{N} \tag{B 12}$$

$$\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i}) = \frac{\sum\limits_{N(r_i)} \{(\Delta t)^k_i \boldsymbol{f}^{\boldsymbol{t}}_k(\boldsymbol{X}'|_{r=r_i})\}}{\sum\limits_{N(r_i)} (\Delta t)^k_i} \tag{B 13}$$

In the last expression on the right-hand side of (B11), $\sum\limits_{N(r_i)} (\Delta t)^k_i$ can be transformed, using the definition of λ_R, as follows:

$$\begin{aligned}\sum_{N(r_i)} (\Delta t)^k_i &= \frac{2\gamma}{N} d\theta d\phi\big[{r_i}^2 sin\theta\big]\frac{\sum\limits_{N(r_i)} (\Delta t)^k_i}{\frac{2\gamma}{N} d\theta d\phi\big[{r_i}^2 sin\theta\big]} \\ &= \frac{2\gamma}{N} d\theta d\phi\big[{r_i}^2 sin\theta\big]\lambda_R(r_i)\end{aligned} \tag{B 14}$$

By introducing (B14) into (B11), we obtain,

$$\begin{aligned}\int_{R-\gamma}^{R+\gamma} \{\sum_{N(r)} \frac{\boldsymbol{f}^{\boldsymbol{T}}_k(\boldsymbol{X}'_k)}{|v^k_r|}\} dr &= \lim_{N\to\infty} \sum_{i=l}^{N} [\{\sum_{N(r_i)} (\Delta t)^k_i\} \overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i})] \\ &= \lim_{N\to\infty} \sum_{i=l}^{N} [\frac{2\gamma}{N} d\theta d\phi\{{r_i}^2 sin\theta\}\lambda_R(r_i)\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i})] \\ &= d\theta d\phi\big[sin\theta \int_{R-\gamma}^{R+\gamma} \{r^2 \lambda_R(r)\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}')\} dr\big]\end{aligned} \tag{B 15}$$

From (B9), (B10) and (B15), it follows that,

$$\overline{\boldsymbol{f}^{\boldsymbol{T}}} = \frac{1}{\Lambda} d\theta d\phi\big[sin\theta \int_{R-\gamma}^{R+\gamma} r^2 \lambda_R(r)\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}') dr\big] \tag{B 16}$$

By summing the contributions of particles having centers in a spherical shell with inner and outer radii $R-\gamma$ and $R+/\gamma$, in the time duration Λ, we determine the time-averaged force acting on particle surfaces in the control volume V during the total time duration Λ is given by

$$\overline{\boldsymbol{F}^V_{R,S}}^{\Lambda} = \frac{1}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta \int_{R-\gamma}^{R+\gamma} \{r^2 \lambda_R(r)\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}')\} dr\big] \tag{B 17}$$

Since the vector $\boldsymbol{n}'(\theta'_0, \phi'_0)$ is common to $\boldsymbol{f}^{\boldsymbol{T}}_k(\boldsymbol{X}'_k)$ for all particles having centers in the volume element $d\theta d\phi\big[\int_{R-\gamma}^{R+\gamma}(r^2 sin\theta)dr\big]$, as shown in (B 6), we can define the time-averaged value of $\boldsymbol{T}_c(\boldsymbol{X}'_k|_{r=r_i} : R, \theta'_0, \phi'_0)$ for all particles overlapping in the subsection $[r_i, r_{i+1}]$ as

$$\overline{\boldsymbol{T}^{\Lambda}_c}(\boldsymbol{X}'|_{r=r_i} : R, \theta'_0, \phi'_0) = \frac{\sum_{N(r_i)} \{(\Delta t)^k_i \boldsymbol{T}_c(\boldsymbol{X}'|_{r=r_i} : R, \theta'_0, \phi'_0)\}}{\sum_{N(r_i)} (\Delta t)^k_i} \qquad \text{(B 18)}$$

Here, $\overline{\boldsymbol{T}^{\Lambda}_c}(\boldsymbol{X}'|_{r=r_i} : R, \theta'_0, \phi'_0)$ is the time-averaged value of $\boldsymbol{T}_c(\boldsymbol{X}'|_{r=r_i} : R, \theta'_0, \phi'_0)$ for all particles overlapping in the subsection. Then,

$$\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'_k|_{r=r_i}) = \boldsymbol{n}'(\theta'_0, \phi'_0) \cdot \overline{\boldsymbol{T}^{\Lambda}_c}(\boldsymbol{X}'_k|_{r=r_i} : R, \theta'_0, \phi'_0)\pi\{\gamma^2 - (R-r)^2\}(1 + o(\gamma)) \qquad \text{(B 19)}$$

From (B17) and (B19) we have,

$$\begin{aligned} &\overline{\boldsymbol{F}^{V}_{R,S}}^{\Lambda} \\ &= \frac{1}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \\ &\big[sin\theta \int_{R-\gamma}^{R+\gamma}[r^2\lambda_R(r)\boldsymbol{n}'(\theta'_0, \phi'_0) \cdot \overline{\boldsymbol{T}^{\Lambda}_c}(\boldsymbol{X}' : R, \theta'_0, \phi'_0)\pi\{\gamma^2 - (R-r)^2\}(1 + o(\gamma)]dr\big] \end{aligned} \qquad \text{(B 20)}$$

In the entire radial section $[R-\gamma, R+\gamma]$ it follows that

$$\begin{aligned} &\| \lambda_R(\boldsymbol{X}')\{\boldsymbol{n}'(\theta'_0, \phi'_0) \cdot \overline{\boldsymbol{T}^{\Lambda}_c}(\boldsymbol{X}' : R, \theta'_0, \phi'_0)\} \| \\ &\approx \| \lambda_R(\boldsymbol{X}'|_{r=R})\{\boldsymbol{n}'(\theta'_0, \phi'_0) \cdot \overline{\boldsymbol{T}^{\Lambda}_c}(\boldsymbol{X}'|_{r=R} : R, \theta'_0, \phi'_0)\} \| (1 + o(\gamma)) \end{aligned} \qquad \text{(B 21)}$$

Here, $\boldsymbol{X}'|_{r=R} = \boldsymbol{X}_0 + R\boldsymbol{n}(\theta, \phi)$. We now find from (B20) and (B21) that

$$\begin{aligned} &\overline{\boldsymbol{F}^{V}_{R,S}}^{\Lambda} \\ &= \frac{1}{\Lambda}\frac{4\pi}{3}\gamma^3 \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[R^2 sin\theta\lambda_R(\boldsymbol{X}_0 + R\boldsymbol{n}(\theta, \pi)) \\ &\qquad \times\boldsymbol{n}'(\theta'_0, \phi'_0) \cdot \overline{\boldsymbol{T}^{\Lambda}_c}(\boldsymbol{X}_0 + R\boldsymbol{n}(\theta, \phi) : R, \theta'_0, \phi'_0)\big](1 + o(\gamma)) \end{aligned} \qquad \text{(B 22)}$$

Finally, using the relation $V = \frac{4\pi}{3}\gamma^3$, we derive

$$\lim_{V\to \boldsymbol{X}_0} \frac{1}{V}\overline{\boldsymbol{F}_{R,S}^{V}}^{\Lambda} = -\frac{1}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi \big[R^2 sin\theta \lambda_R(\boldsymbol{X}_0 - R\boldsymbol{n}(\theta,\phi)) \times \boldsymbol{n}(\theta,\phi)\cdot \overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0 - R\boldsymbol{n}(\theta,\phi) : R,\theta,\phi)\big] \quad \text{(B 23)}$$

The following relations have been used in obtaining the final expression on the right-hand side of (B23).

$$\theta_0' = \pi - \theta \quad \text{(B 24)}$$

$$\phi_0' = \pi + \phi \quad \text{(B 25)}$$

$$\boldsymbol{n}'(\theta_0', \phi_0') = -\boldsymbol{n}(\theta,\phi) \quad \text{(B 26)}$$

Appendix C. Comparison of (3.8) with the impact term

In the Navier-Stokes equations time-averaged for multiphase flow(Ueyama and Miyauchi, 1976), the interaction term is given by

$$\boldsymbol{D}_c = \frac{1}{\Lambda}\sum_{i=1}^{N}\{(P_c|_{T_i^a}\boldsymbol{I} + \boldsymbol{\tau}_c|_{T_i^a})\cdot \boldsymbol{\xi}_i^a - (P_c|_{T_i^l}\boldsymbol{I} + \boldsymbol{\tau}_c|_{T_i^l})\cdot \boldsymbol{\xi}_i^l\} \quad \text{(C 1)}$$

Here, $\sum_{i=1}^{N}$ denotes summation for all particle surfaces appearing at the point of time-averaging during the total time duration Λ, and T_i^a and T_i^l are the arrival and leaving time of the particlep i at the point of time-averaging, which are functions of spatial coordinates; $\boldsymbol{\xi}_i^a$ and $\boldsymbol{\xi}_i^l$ are gradient vectors of the respective surfaces T_i^a and T_i^l, as defined by the following two equations:

$$\boldsymbol{\xi}_i^a = \nabla \boldsymbol{T}_i^a \quad \text{(C 2)}$$

$$\boldsymbol{\xi}_i^l = \nabla \boldsymbol{T}_i^l \quad \text{(C 3)}$$

We know that

$$\boldsymbol{\xi}_i^a \cdot \boldsymbol{u}_s = 1 \quad \text{(C 4)}$$

$$\boldsymbol{\xi}_i^l \cdot \boldsymbol{u}_s = 1 \tag{C 5}$$

Here, $\boldsymbol{u}_s$ is a velocity vector describing the motion of the surface.

The Navier-Stokes equation for the continuous phase, time-averaged for the dispersed multiphase flow(Ueyama & Miyauchi, 1976), is given by:

$$\begin{aligned} &\frac{\partial}{\partial t}\{(1-\epsilon)\overline{\rho_c \boldsymbol{u}_c}^c\} + \nabla \cdot \{(1-\epsilon)\overline{\rho_c \boldsymbol{u}_c \boldsymbol{u}_c}^c\} \\ &= -\nabla\{(1-\epsilon)\overline{P_c}^c\} - \nabla \cdot \{(1-\epsilon)\overline{\boldsymbol{\tau}_c}^c\} + (1-\epsilon)\overline{\rho_c}^c \boldsymbol{g} + \boldsymbol{D}_c \end{aligned} \tag{C 6}$$

Upon comparing (C6) with (2.13), we expect $\boldsymbol{D}_c$ to be identical to $-\lim_{V \to \boldsymbol{X}_0} \frac{1}{V}\overline{\boldsymbol{F}_{R,S}^V}^\Lambda$.

From (B6), (B9) and (B10), we have

$$\overline{\boldsymbol{f}^{\boldsymbol{T}}} = -\frac{1}{\Lambda}\int_{R-\gamma}^{R+\gamma}\{\sum_{N(r)} \frac{\boldsymbol{n}'(\theta_0', \phi_0') \cdot \boldsymbol{T}_c(\boldsymbol{X}_k' : R, \theta_0', \phi_0')\pi\{\gamma^2 + (R-r)^2\}(1+o(\gamma))}{|v_r^k|}\}dr \tag{C 7}$$

The radial integration on the right-hand side for a total section $[R-\gamma, R+\gamma]$ can be given as the product of the value of the integrand at $r = R$ and the representative radial length which gives the volume under consideration, that is $\int_{R-\gamma}^{R+\gamma}\{\gamma^2 + (R-r)^2\}dr$, since we shall take the limit for $\gamma \to 0$. The representative radial length is obtained as

$$\frac{\int_{R-\gamma}^{R+\gamma}\{\gamma^2 + (R-r)^2\}dr}{\{\gamma^2 + (R-r)^2\}_{r=R}} = \frac{4}{3}\gamma \tag{C 8}$$

From (C7) and (C8), it follows that

$$\overline{\boldsymbol{f}^{\boldsymbol{T}}} = -\frac{1}{\Lambda}\{\sum_{N(R)} \frac{\boldsymbol{n}'(\theta_0', \phi_0') \cdot \boldsymbol{T}_c(\boldsymbol{X}_k'|_{r=R} : R, \theta_0', \phi_0')\pi(1+o(\gamma))}{|v_r^k|_{r=R}|}\}\frac{4}{3}\gamma^3 \tag{C 9}$$

The value of $\frac{1}{V}\overline{\boldsymbol{F}_{R,S}^V}^\Lambda$ is given, from (C9), as

$$\begin{aligned} &\frac{1}{V}\overline{\boldsymbol{F}_{R,S}^V}^\Lambda = \frac{1}{V}\int_0^\pi d\theta \int_0^{2\phi} d\phi[R^2 sin\theta \overline{\boldsymbol{f}^{\boldsymbol{T}}}] \\ &= -\frac{1}{\Lambda}\int_0^\pi d\theta \int_0^{2\phi} d\phi[R^2 sin\theta\{\sum_{N(R)} \frac{\boldsymbol{n}'(\theta_0', \phi_0') \cdot \boldsymbol{T}_c(\boldsymbol{X}_k'|_{r=R} : R, \theta_0', \phi_0')}{|v_r^k|_{r=R}|}\}](1+o(\gamma)) \end{aligned} \tag{C 10}$$

Upon taking the limit $\gamma \to 0$, we have

$$-\lim_{\gamma\to 0}\frac{1}{V}\overline{\boldsymbol{F}^{V}_{R,S}}^{\Lambda} = -\frac{1}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\phi} d\phi[R^2 sin\theta\{\sum_{N(r)} \frac{\boldsymbol{n}'(\theta_0', \phi_0') \cdot \boldsymbol{T}_c(\boldsymbol{X}'_k|_{r=R} : R, \theta_0', \phi_0')}{|v_r^k|_{r=R}|}\}] \tag{C 11}$$

The right-hand side of (C11) is the sum for duration Λ of all the terms at point $\boldsymbol{X}_0$, given as the product of a tensor $\boldsymbol{T}_c$ on a particle surface and an outwardly directed unit normal vector, divided by the absolute value of the normal component of the surface moving velocity, which appears at every instant when a particle center arrives at the surface of a sphere with radius R centered at $\boldsymbol{X}_0$, in the total time duration Λ.

The right-hand side of (C1) can be transformed, using (C4) and (C5), to

$$\boldsymbol{D}_c = \frac{1}{\Lambda}\sum_{i=1}^{N}\{\boldsymbol{T}_c|_{T_i^a} \cdot \frac{\boldsymbol{n}_i^a}{\boldsymbol{u}_s \cdot \boldsymbol{n}_i^a} - \boldsymbol{T}_c|_{T_i^l} \cdot \frac{\boldsymbol{n}_i^l}{\boldsymbol{u}_s \cdot \boldsymbol{n}_i^l}\} \tag{C 12}$$

Here, $\boldsymbol{n}_i^a$ and $\boldsymbol{n}_i^l$ respectively denote unit normal vectors at the arriving and leaving surfaces, with a positive component along the movement of the surfaces. The directions of $\boldsymbol{n}_i^a$ and $\boldsymbol{n}_i^l$ are respectively outward and inward at the particle surface, and the expressions $\boldsymbol{u}_s \cdot \boldsymbol{n}_i^a$ and $\boldsymbol{u}_s \cdot \boldsymbol{n}_i^l$ are both positive and correspond to $|v_r^k|_{r=R}|$. The first term in the brackets on the right-hand side of (C12) is a product of a tensor $\boldsymbol{T}_c$ on a surface and an outwardly directed unit normal vector, divided by the absolute value of the normal component of the velocity of motion of the surface. The second term is a product of a tensor $\boldsymbol{T}_c$ on a surface and an inwardly directed unit normal vector, divided by the absolute value of the normal component of the velocity of motion of the surface.

The physical meaning of the right hand sides of (C11) and (C12) are therefore identical.

Appendix D. Derivation of (3.9):Non-spherical particle

By using a spherical coordinate system with its origin at $\boldsymbol{X}'$, an arbitrary point on the particle surface can be specified as $(\boldsymbol{X}' : R'(\theta', \phi'), \theta', \phi')$. The point $\boldsymbol{X}'$ will henceforth be referred to a reference point of the particle.

Let us determine the time-averaged force acting on a portion of the particle surface in a spherical control volume V, having radius γ less than the particle size, and with center at $\boldsymbol{X}_0$. Higher order terms with respect to γ in the equations that follow will ultimately be neglected, because we are focusing on the force per unit volume applicable to infinitesimal volume, which corresponds to the limit $\gamma \to 0$.

Analogously to a spherical particle (Appendix B), the force acting on S_k^V due to static pressure and shear stress of the continuous phase is given by:

$$\begin{aligned}
&\boldsymbol{f}_k^T(\boldsymbol{X}'_k) \\
&= -\int_{S_k^V} d\theta' d\phi' \big[sin\theta' \{R'(\theta', \phi')\}^2 C(\theta', \phi') \\
&\qquad \times \boldsymbol{n}'(R'(\theta', \phi'), \theta', \phi') \cdot \boldsymbol{T}_c(\boldsymbol{X}'_k : R'(\theta', \phi'), \theta', \phi') \big] \\
&= -\boldsymbol{n}'(R'(\theta'_0, \phi'_0), \theta'_0, \phi'_0) \cdot \boldsymbol{T}_c(\boldsymbol{X}'_k : R'(\theta'_0, \phi'_0), \theta'_0, \phi'_0) \\
&\qquad \times \pi\{\gamma^2 - (R'(\theta'_0, \phi'_0) - r)^2\} C(\theta'_0, \phi'_0)(1 + 0(\gamma^2)) \qquad \text{(D 1)}
\end{aligned}$$

for $R'(\theta'_0, \phi'_0) - \gamma < r(\theta, \phi) < R'(\theta'_0, \phi'_0) + \gamma$. Here, $\boldsymbol{n}'(R'(\theta'_0, \phi'_0), \theta'_0, \phi'_0)$ in (D-1) is a unit normal vector on the surface S_k^V directed outward; it is distinct from a unit vector $\boldsymbol{i}_{r'}(\theta'_0, \phi'_0)$ in a spherical coordinate system $(\boldsymbol{X}' : r', \theta', \phi')$. The function $C(\theta', \phi)$ is a correction factor corresponding to the shape of the particle surface suitably defined so that the surface area element at $(R'(\theta', \phi'), \theta', \phi')$ is precisely given as $sin\theta' C'(\theta', \phi')\{R'(\theta', \phi')\}^2 d\theta' d\phi'$, and $\pi\{\gamma^2 - (R'(\theta'_0, \phi'_0) - r)^2\}$ is a cross sectional area of the volume V perpendicular to the $r'(\theta'_0, \phi'_0)$ axis at $(R'(\theta'_0, \phi'_0), \theta'_0, \phi'_0)$. The value of $r(\theta, \phi)$ varies with time according to the position of the reference point $\boldsymbol{X}'_k = \boldsymbol{X}_0 + r\boldsymbol{i}_r(\theta, \phi)$ for $R'(\theta'_0, \phi'_0) - \gamma <$

$r(\theta,\phi) < R'(\theta_0',\phi_0')+\gamma$, where $\boldsymbol{i}_r(\theta,\phi)$ is a unit vector along the radial direction specified by spherical polar angles (θ,ϕ).

Denote by $\overline{\boldsymbol{f}^T}$ the contribution of those particles having reference points in the volume element $d\theta d\phi[\int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma}(r^2 sin\theta)C(\theta_0',\phi_0')dr$ to the time-averaged force acting on particle surfaces in the control volume V in the time duration Λ. Then,

$$\overline{\boldsymbol{f}^T} = \frac{\sum_k \int_{\tau_k} \boldsymbol{f}^T(\boldsymbol{X}_k')dt}{\Lambda} \tag{D 2}$$

where τ_k is the residence time of the surface S_k^V in the control volume V, and $\sum_k$ denotes summation over particles whose reference points are in the volume element $d\theta d\phi \int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma}(r^2 sin\theta)C(\theta_0',\phi_0')dr$ for the time duration Λ. Since the particle k moves in a volume element $d\theta d\phi \int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma}(r^2 sin\theta)C(\theta_0',\phi_0')dr$, the time-integration in the expression $\int_{\tau_k}\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}_k')dt$ can be converted to a radial integration using the r component of the moving velocity of S_k^V , which we denote by v_r^k:

$$\left.\begin{aligned} \int_{\tau_k}\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}_k')dt &= \int_{r_k}^{r_k+l_k}\frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}_k')}{v_r^k}dr \quad \text{for} \quad v_r^k > 0, \\ \int_{\tau_k}\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}_k')dt &= \int_{r_k+l_k}^{r_k}\frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}_k')}{|v_r^k|}dr \quad \text{for} \quad v_r^k < 0,, \end{aligned}\right\} \tag{D 3}$$

Here, l_k is the radial displacement of particle k in the time duration Λ, and r_k is the initial radial position of S_k^V when the center of the particle k, that is $\boldsymbol{X}_k'$, appears in the volume element $d\theta d\phi[\int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma}(r^2 sin\theta)C(\theta_0',\phi_0')dr$.

It follows from (D2) and (D3) that

$$\overline{\boldsymbol{f}^T} = \frac{1}{\Lambda}\sum_k \int_{Min(r_k,r_k+l_k)}^{Max(r_k,r_k+l_k)}\frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}_k')}{|v_r^k|}dr \tag{D 4}$$

In the summation on the right-hand side, the radial integration involves overlap, and the number of overlaps is countable and varies with radial position. Introduce the overlap number $N(r)$, which depends on radial position and is such that

$$\sum_k \int_{Min(r_k, r_k+l_k)}^{Max(r_k, r_k+l_k)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|} dr = \int_{R'(\theta'_0,\phi'_0)-\gamma}^{R'(\theta'_0,\phi'_0)+\gamma} \sum_{N(r)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|} dr \tag{D 5}$$

Here, $\sum_{N(r)}$ refers to summation over all overlaps.

The right-hand side of Equation (D5) can be manipulated as:

$$\begin{aligned}
\int_{R'(\theta'_0,\phi'_0)-\gamma}^{R'(\theta'_0,\phi'_0)+\gamma} \{\sum_{N(r)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|}\} dr &= \lim_{N\to\infty} \sum_{i=1}^{N} \big[\{\sum_{N(r)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|}\}|_{r=r_i} \big] (\frac{2\gamma}{N}) \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} \big[\sum_{N(r)} \{(\Delta t)_i^k \boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k|_{r=r_i})\} \big] \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} \big[\{\sum_{N(r)} \{(\Delta t)_i^k\} \overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'_k|_{r=r_i})\} \big]
\end{aligned} \tag{D 6}$$

The total radial section $\left[R'(\theta'_0,\phi'_0)-\gamma, R'(\theta'_0,\phi'_0)+\gamma\right]$ is divided into N subsections with equal length $\dfrac{2\gamma}{N}$, and $r_i = R'(\theta'_0,\phi'_0) - r + \frac{2\gamma}{N}$ is the radial position of the i-th subsection. In the second and in the last expression on the right-hand side of (D6), $(\Delta t)_i^k$ is the residence time of the surface S_k^V in the subsection $[r_i, r_{i+1}]$, and $\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i})$ is the time-averaged value of $\boldsymbol{f}^{\Lambda}(\boldsymbol{X}'|_{r=r_i})$ for all particles overlapping in the subsection. These are given by the following equations:

$$(\Delta t)_i^k = \frac{1}{|v_r^k|_{r=r_i}|} \frac{2\gamma}{N} \tag{D 7}$$

$$\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i}) = \frac{\sum_{N(r_i)} \{(\Delta t)_i^k \boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'|_{r=r_i})\}}{\sum_{N(r_i)} (\Delta t)_i^k} \tag{D 8}$$

In the last expression on the right-hand side of (D6), $\sum_{N(r_i)} (\Delta t)_i^k$ can be transformed, using the definition of λ_R, as follows:

$$\begin{aligned}
\sum_{N(r_i)} (\Delta t)_i^k &= \frac{2\gamma}{N} d\theta d\phi (r_i^2 sin\theta) C(\theta'_0,\phi'_0) \frac{\sum_{N(r_i)} (\Delta t)_i^k}{\frac{2\gamma}{N} d\theta d\phi (r_i^2 sin\theta) C(\theta'_0,\phi'_0)} \\
&= \frac{2\gamma}{N} d\theta d\phi (r_i^2 sin\theta) C(\theta'_0,\phi'_0) \lambda(r_i)
\end{aligned} \tag{D 9}$$

By introducing (D9) into Equation (D6), we obtain,

$$\int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma} \sum_{N(r)} \frac{\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)}{|v_r^k|} dr = \lim_{N\to\infty} \sum_{i=1}^{N} [\{\sum_{N(r)} (\Delta t)_i^k\} \overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i})]$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N} [\frac{2\gamma}{N} d\theta d\phi (r_i^2 sin\theta) C(\theta_0', \phi_0') \lambda_R(r_i) \overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i})]$$

$$= d\theta d\phi [sin\theta C(\theta_0', \phi_0') \int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma} r^2 \overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}')dr] \qquad \text{(D 10)}$$

From (D4), (D5) and (D10), it follows that,

$$\overline{\boldsymbol{f}^{\boldsymbol{T}}} = \frac{1}{\Lambda} d\theta d\phi [sin\theta C(\theta_0', \phi_0') \int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma} r^2 \overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}')dr] \qquad \text{(D 11)}$$

By summing the contributions of particles having centers in a spherical shell with inner and outer radii $R'(\theta_0', \phi_0') - \gamma$ and $R'(\theta_0', \phi_0') + \gamma$, in the time duration Λ, we determine the time-averaged force acting on particle surfaces in the control volume V during the total time duration Λ to be:

$$\overline{\boldsymbol{F}_{R,S}^{V}}^{\Lambda} = \frac{1}{\Lambda} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi [sin\theta C(\theta_0', \phi_0') \int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma} r^2 \overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}')dr] \qquad \text{(D 12)}$$

Since the vector $\boldsymbol{n}'(\theta_0', \phi_0')$ is common to $\boldsymbol{f}_k^{\boldsymbol{T}}(\boldsymbol{X}'_k)$ for all particles having reference points in the volume element $d\theta d\phi [sin\theta C(\theta_0', \phi_0') \int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma} r^2 dr]$, as shown in Equation (D-1), we can define the time-averaged value of $\boldsymbol{T}_c(\boldsymbol{X}'_k|_{r=r_i} : R'(\theta_0', \phi_0'), \theta_0', \phi_0')$ for all particles overlapping in the subsection $[r_i, r_{i+1}]$ as

$$\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}'_k|_{r=r_i} : R'(\theta_0', \phi_0'), \theta_0', \phi_0') = \frac{\sum_{N(r_i)} \{(\Delta t)_i^k \boldsymbol{T}_c(\boldsymbol{X}'_k|_{r=r_i} : R'(\theta_0', \phi_0'))\}}{\sum_{N(r_i)} (\Delta t)_i^k} \qquad \text{(D 13)}$$

Here, $\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}'_k|_{r=r_i} : R'(\theta_0', \phi_0'), \theta_0', \phi_0')$ is the time-averaged value of $\boldsymbol{T}_c(\boldsymbol{X}'_k|_{r=r_i} : R'(\theta_0', \phi_0'), \theta_0', \phi_0')$ for all particles overlapping in the subsection. Then,

$$\overline{\boldsymbol{f}^{\Lambda}}(\boldsymbol{X}'|_{r=r_i}) = \pi\{\gamma^2 - (R'(\theta_0', \phi_0') - r)^2\}(1 + o(\gamma))$$

$$\times \boldsymbol{n}'(\theta_0', \phi_0') \cdot \overline{\boldsymbol{T}_c^{k}}(\boldsymbol{X}'_k|_{r=r_i} : R'(\theta_0', \phi_0'), \theta_0', \phi_0') \qquad \text{(D 14)}$$

From (D12) and (D14) we have,

$$\overline{\boldsymbol{F}_{R,S}^{V}}^{\Lambda}$$
$$= -\frac{\pi}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta C(\theta_0', \phi_0')\boldsymbol{n}'(\theta_0', \phi_0') \cdot \int_{R'(\theta_0',\phi_0')-\gamma}^{R'(\theta_0',\phi_0')+\gamma} \{r^2\{\gamma^2 - (R'(\theta_0', \phi_0') - r)^2\}$$
$$\times\lambda_R(r)\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}' : R'(\theta_0', \phi_0'), \theta_0', \phi_0')\}dr\big](1 + o(\gamma)) \quad \text{(D 15)}$$

In the entire radial section $[R'(\theta_0', \phi_0') - \gamma, R'(\theta_0', \phi_0') + \gamma]$, it follows that

$$\lambda_R(\boldsymbol{X}')\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}' : R'(\theta_0', \phi_0'), \theta_0', \phi_0')$$
$$\approx \lambda_R(\boldsymbol{X}'|_{r=R'(\theta_0',\phi_0')})\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}'|_{r=R'(\theta_0',\phi_0')} : R'(\theta_0', \phi_0'), \theta_0', \phi_0')(1 + o(\gamma)) \quad \text{(D 16)}$$

Here, $\boldsymbol{X}'|_{r=R'(\theta_0',\phi_0')} = \boldsymbol{X}_0 + R'(\theta_0', \phi_0')\boldsymbol{i}_r(\theta, \phi)$. W find now from (D15) and (D16) that;

$$\overline{\boldsymbol{F}_{R,S}^{\Lambda}}$$
$$= -\frac{1}{\Lambda}\frac{4\pi}{3}\gamma^3 \int_0^{\pi} d\theta \int_0^{2\pi} d\theta\big[\{R'(\theta_0', \phi_0')\}^2 C(\theta_0', \phi_0')sin\theta\lambda_R(\boldsymbol{X}_0 + R'(\theta_0', \phi_0')\boldsymbol{i}_r(\theta, \phi))$$
$$\times\boldsymbol{n}'(\theta_0', \phi_0') \cdot \overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0 + R'(\theta_0', \phi_0')\boldsymbol{i}_r(\theta, \phi) : R'(\theta_0', \phi_0'), \theta_0', \phi_0')\big](1 + \boldsymbol{o}(\gamma)) \quad \text{(D 17)}$$

where, $\boldsymbol{n}'(\theta_0', \phi_0')$ is a unit normal vector on a surface of the particle at the angular coordinates (θ_0', ϕ_0').

Finally, using the relation $V = \frac{4\pi}{3}\gamma^3$, we derive

$$\lim_{V\to\boldsymbol{X}_0} \frac{1}{V}\overline{\boldsymbol{F}_{R,S}^{V}}^{\Lambda} = -\frac{1}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[\{R(\theta, \phi)\}^2 C(\theta, \phi)sin\theta\lambda_R(\boldsymbol{X}_0 - R(\theta, \phi)\boldsymbol{i}_r(\theta, \phi))$$
$$\times\boldsymbol{n}(\theta, \phi) \cdot \overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0 - R(\theta, \phi)\boldsymbol{i}_r(\theta, \phi) : R(\theta, \phi), \theta, \phi)\big] \quad \text{(D 18)}$$

Here, $\boldsymbol{n}(\theta.\phi)$ is a unit normal vector on a surface of the particle at the angular coordinates (θ, ϕ). The following relations have been used in obtaining the final expression on the right-hand side of (D18).

$$\theta_0' = \pi - \theta \quad \text{(D 19)}$$

$$\phi_0' = \pi + \phi \quad \text{(D 20)}$$

$$\boldsymbol{i}_r(\theta, \phi) = -\boldsymbol{i}_r'(\theta_0', \phi_0') \quad \text{(D 21)}$$

Appendix E. Derivation of (3.15)

A following equation holds because a tensor $\overline{\boldsymbol{T}_c}^c(\boldsymbol{X}_0)$ is constant in space.

$$-\frac{1}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta\boldsymbol{n}(\theta,\phi)\cdot\overline{\boldsymbol{T}_c}^c(\boldsymbol{X}_0)\lambda_R(\boldsymbol{X}_0 - R\boldsymbol{n}(\theta,\phi))R^2\big]$$
$$= \frac{1}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta\boldsymbol{n}(\theta,\phi)\lambda_R(\boldsymbol{X}_0 + R\boldsymbol{n}(\theta,\phi))R^2\big]\cdot\overline{\boldsymbol{T}_c}^c(\boldsymbol{X}_0) \qquad \text{(E 1)}$$

By neglecting higher order terms of Taylor expansion, a surface integral on the right hand side of (E1) is calculated to

$$\frac{1}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta\boldsymbol{n}(\theta,\phi)\lambda_R(\boldsymbol{X}_0 + R\boldsymbol{n}(\theta,\phi))R^2\big]$$
$$\approx \frac{\lambda_R(\boldsymbol{X}_0)}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta\boldsymbol{n}(\theta,\phi)R^2\big]$$
$$+\frac{R^3}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta\boldsymbol{n}(\theta,\phi)\{\boldsymbol{n}(\theta,\phi)\cdot(\nabla\lambda_R)|_{\boldsymbol{X}_0}\}\big]$$
$$= \frac{R^3}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta\boldsymbol{n}(\theta,\phi)\{\boldsymbol{n}(\theta,\phi)\cdot(\nabla\lambda_R)|_{\boldsymbol{X}_0}\}\big] \qquad \text{(E 2)}$$

Let us take the Z axis of the Cartesian coordinate system along the vector $(\nabla\lambda_R)|_{\boldsymbol{X}_0}$.

$$(\nabla\lambda_R)|_{\boldsymbol{X}_0} = a\boldsymbol{i}_Z \qquad (a > 0) \qquad \text{(E 3)}$$

By using the Cartesian coordinate system, the unit vector on the r axis, $\boldsymbol{i}_r$, is given by

$$\boldsymbol{n}(\boldsymbol{X}_0 : \theta,\phi) = \boldsymbol{i}_r = sin\theta cos\phi\boldsymbol{i}_X + sin\theta sin\phi\boldsymbol{i}_Y + cos\theta\boldsymbol{i}_Z \qquad \text{(E 4)}$$

Here, $\boldsymbol{i}_X$, $\boldsymbol{i}_Y$ and $\boldsymbol{i}_Z$ are unit vectors along the X, Y and Z axes, respectively.

The angular integration in the final expression on the right hand side of (E2) is now calculated to

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta\boldsymbol{n}(\theta,\phi)\{\boldsymbol{n}(\theta,\phi)\cdot(\nabla\lambda_R)|_{\boldsymbol{X}_0}\}\big]$$
$$= a\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta^2 cos\theta cos\phi\boldsymbol{i}_X + sin\theta^2 cos\theta sin\phi\boldsymbol{i}_Y + sin\theta cos\theta^2\boldsymbol{i}_Z\big]$$
$$= \frac{4}{3}\pi a\boldsymbol{i}_Z = \frac{4}{3}\pi(\nabla\lambda_R)|_{\boldsymbol{X}_0} \qquad \text{(E 5)}$$

Upon substituting (E2) and (E5) into (E1), we have

$$-\frac{1}{\Lambda}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\big[sin\theta\boldsymbol{n}(\theta,\phi)\cdot\overline{\boldsymbol{T}_c}^c(\boldsymbol{X}_0)\lambda_R(\boldsymbol{X}_0 - R\boldsymbol{n}(\theta,\phi))R^2\big]$$
$$\approx \frac{4}{3\Lambda}\pi R^3(\nabla\lambda_R)|_{\boldsymbol{X}_0}\cdot\overline{\boldsymbol{T}_c}^c(\boldsymbol{X}_0) \qquad \text{(E 6)}$$

From (E6) and (F5) derived in Appendix F, (3.15) is obtained.

Appendix F. Further calculations on the first and second terms on the right-hand side of (3.19)

F.1. *First term*

The first term on the right-hand side of (3.19) can be expressed, using (3.11), as

$$-\frac{1}{\Lambda}\lambda_R(\boldsymbol{X}_0)\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta\boldsymbol{n}(\theta,\phi)\cdot\overline{\Delta\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0:R,\theta,\phi)R^2]$$
$$=\frac{\lambda_R(\boldsymbol{X}_0)}{\Lambda}\overline{\boldsymbol{f}_p^{\Delta\boldsymbol{T}}}(\boldsymbol{X}_0) = \frac{V_p\lambda_R(\boldsymbol{X}_0)}{\Lambda}\frac{\overline{\boldsymbol{f}_p^{\Delta\boldsymbol{T}}}(\boldsymbol{X}_0)}{V_p} \qquad \text{(F 1)}$$

A following equation is obtained from (3.3) by neglecting higher order terms in a Taylor expansion.

$$\Lambda_d(\boldsymbol{X}_0)$$
$$=\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta\int_0^R\{r^2\lambda_R(\boldsymbol{X}_0:r,\theta,\phi)\}dr]$$
$$\approx\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta\int_0^R\{r^2(\lambda_R(\boldsymbol{X}_0)+r\boldsymbol{n}(\theta,\phi)\cdot(\nabla\lambda_R)|_{\boldsymbol{X}_0})\}dr]$$
$$=\lambda_R(\boldsymbol{X}_0)\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta\int_0^R r^2dr]$$
$$+\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta\int_0^R\{r^3\boldsymbol{n}(\theta,\phi)\cdot(\nabla\lambda_R)|_{\boldsymbol{X}_0}\}dr] \qquad \text{(F 2)}$$

The second term on the right hand side of (F2) can be calculated using (E3) and (E4),

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta\int_0^R\{r^3\boldsymbol{n}(\theta,\phi)\cdot(\nabla\lambda_R)|_{\boldsymbol{X}_0}\}dr]$$
$$=a\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta cos\theta\int_0^R r^3dr]=0 \qquad \text{(F 3)}$$

The integral of the first term on the right hand side of (F2) gives the volume of a sphere with radius R.

$$\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta \int_0^{R} r^2 dr] = \frac{4\pi R^3}{3} = V_p \tag{F 4}$$

By substituting (F3) and (F4) into (F2), we have

$$V_p \lambda_R(\boldsymbol{X}_0) \approx \Lambda_d(\boldsymbol{X}_0) \tag{F 5}$$

From (3.1), (F1) and (F5), it follows that

$$-\frac{1}{\Lambda}\lambda_R(\boldsymbol{X}_0)\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta \boldsymbol{n}(\boldsymbol{X}_0:\theta,\phi)\cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0:R,\theta,\phi)R^2]$$
$$\approx \epsilon(\boldsymbol{X}_0)\frac{\overline{\boldsymbol{f}_p^{\Delta \boldsymbol{T}}}(\boldsymbol{X}_0)}{V_p} \tag{F 6}$$

The right hand side of (F6) is the same as the conventional interaction term.

F.2. *Second term*

The second term on the right-hand side of (3.19) can be calculated from (E3) and (E4) as

$$\frac{1}{V}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta \boldsymbol{n}(\theta,\phi)\cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0:R,\theta,\phi)\{\boldsymbol{n}(\theta,\phi)\cdot(\nabla\lambda_R)|_{\boldsymbol{X}_0}\}R^3]$$
$$= a\frac{R}{\Lambda}\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi[sin\theta cos\theta \boldsymbol{n}(\theta,\phi)\cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0:R,\theta,\phi)R^2]$$
$$+a\frac{R}{\Lambda}\int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta cos\theta \boldsymbol{n}(\theta,\phi)\cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0:R,\theta,\phi)R^2] \tag{F 7}$$

The surface integration in the first term on the right-hand side of (F7) is a weighted surface integration of the force due to $\Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0:R,\theta,\phi)$ on a hemispherical surface with positive Z axis, with weight function $cos\theta$. Since the value of $cos\theta$ is between 0 and 1 for $0<\theta<\frac{\pi}{2}$, the following equation is proposed to see the main property of the term:

$$a\frac{R}{\Lambda}\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi[sin\theta cos\theta \boldsymbol{n}(\theta,\phi)\cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0:R,\theta,\phi)R^2]$$
$$\approx c^+ a\frac{R}{\Lambda}\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi[sin\theta \boldsymbol{n}(\theta,\phi)\cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0:R,\theta,\phi)R^2]$$
$$= -c^+ a\frac{R}{\Lambda}\overline{\boldsymbol{f}_p^{\Delta \boldsymbol{T}+}}(\boldsymbol{X}_0) = -c^+ R(\frac{\partial\epsilon}{\partial Z})|_{\boldsymbol{X}_0}\frac{\overline{\boldsymbol{f}_p^{\Delta \boldsymbol{T}+}}(\boldsymbol{X}_0)}{V_p} \tag{F 8}$$

Here, $\overline{\boldsymbol{f}_p^{\Delta \boldsymbol{T}+}}(\boldsymbol{X}_0)$ is the contribution of $\Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0 : R, \theta, \phi)$ on a hemispherical surface with positive Z axis, defined as follows (c^+ is a fitting parameter between 0 and 1, and represents the effect of the weight function, $cos\theta$):

$$\overline{\boldsymbol{f}_p^{\Delta \boldsymbol{T}+}}(\boldsymbol{X}_0) = -\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi[sin\theta \boldsymbol{n}(\theta,\phi) \cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0 : R, \theta, \phi)R^2] \tag{F 9}$$

Similarly, the second term on the right-hand side of (F7) is given by

$$\begin{aligned} & a\frac{R}{\Lambda}\int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta cos\theta \boldsymbol{n}(\theta,\phi) \cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0 : R, \theta, \phi)R^2] \\ & \approx -c^- a\frac{R}{\Lambda}\int_0^{\frac{\pi}{2}} d\theta \int_0^{2\pi} d\phi[sin\theta \boldsymbol{n}(\theta,\phi) \cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0 : R, \theta, \phi)R^2] \\ & = c^- a\frac{R}{\Lambda}\overline{\boldsymbol{f}_d^{\Delta \boldsymbol{T}+}}(\boldsymbol{X}_0) = c^- R(\frac{\partial \epsilon}{\partial Z})|_{\boldsymbol{X}_0}\frac{\overline{\boldsymbol{f}_p^{\Delta \boldsymbol{T}-}}(\boldsymbol{X}_0)}{V_p} \end{aligned} \tag{F 10}$$

Here, $\overline{\boldsymbol{f}_p^{\Delta \boldsymbol{T}-}}(\boldsymbol{X}_0)$ is the contribution of $\Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0 : R, \theta, \phi)$ on a hemispherical surface with negative Z axis, defined as follows (c^- is a fitting parameter between 0 and 1, representing the effect of the weight function $cos\theta$):

$$\overline{\boldsymbol{f}_p^{\Delta \boldsymbol{T}-}}(\boldsymbol{X}_0) = -\int_{\frac{\pi}{2}}^{\pi} d\theta \int_0^{2\pi} d\phi[sin\theta \boldsymbol{n}(\theta,\phi) \cdot \Delta\overline{\boldsymbol{T}_c^{\Lambda}}(\boldsymbol{X}_0 : R, \theta, \phi)R^2] \tag{F 11}$$

Appendix G. Derivation of (4.5) and (4.6)

Let us take a time-averaged value of the velocities of the dispersed phase observed at point $\boldsymbol{X}_0$. For simplicity, no rotation is assumed for particles. When the center of the particle is in a sphere with radius R centered at $\boldsymbol{X}_0$, the dispersed phase velocity measured at $\boldsymbol{X}_0$ is the same as the moving velocity of the particle center at the volume element in the sphere. Upon denoting the residence time of a center of particle i in a volume element $r^2 sin\theta d\theta r\phi dr$ at $(\boldsymbol{X}_0 : r, \theta, \phi)$ as $(\Delta t)^i_{r,\theta,\phi}$, the contribution of those particles, for which the center has been in the volume element during the total time length Λ, to the time integral of $\boldsymbol{u}_d$ at $\boldsymbol{X}_0$ is given as $\sum_{N(r,\theta,\phi)} \{(\Delta t)^i_{r,\theta,\phi}\boldsymbol{u}^i_p(\boldsymbol{X}_0 : r, \theta, \phi)\}$. Here, $N(r,\theta,\phi)$ and $\boldsymbol{u}^i_p(\boldsymbol{X}_0 : r, \theta, \phi)$ respectively denote the total number of particles that appeared in the volume element $r^2 sin\theta d\theta r\phi dr$ at $(\boldsymbol{X}_0 : r,\theta,\phi)$ during the total time duration Λ, and the moving velocity of particle i centered at $(\boldsymbol{X}_0 : r,\theta,\phi)$. Let us introduce the time-averaged moving velocity of a particle with center at $(\boldsymbol{X}_0 : r,\theta,\phi)$ as

$$\overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}_0 : r,\theta,\phi) = \frac{\sum_{N(r,\theta,\phi)} \{(\Delta t)^i_{r,\theta,\phi} \boldsymbol{u}^i_p(\boldsymbol{X}_0 : r,\theta,\phi)\}}{\sum_{N(r,\theta,\phi)} (\Delta t)^i_{r,\theta,\phi}} \tag{G 1}$$

The contribution can now be transformed to:

$$\begin{aligned} &\sum_{N(r,\theta,\phi)} \{(\Delta t)^i_{r,\theta,\phi} \boldsymbol{u}^i_p(\boldsymbol{X}_0 : r,\theta,\phi)\} \\ &= \{\sum_{N(r,\theta,\phi} (\Delta t)^i_{r,\theta,\phi}\} \overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}_0 : r,\theta,\phi) \\ &= r^2 sin\theta d\theta d\phi dr \lambda_R(\boldsymbol{X}_0 : r,\theta,\phi) \overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}_0 : r,\theta,\phi) \end{aligned} \tag{G 2}$$

The final expression is obtained from the definition of λ_R.

The total time integral of the dispersed phase velocity at $\boldsymbol{X}_0$ during Λ is obtained by summing the contribution of volume elements in the sphere:

$$\begin{aligned} &\epsilon(\boldsymbol{X}_0)\overline{\boldsymbol{u}_d}^d(\boldsymbol{X}_0 \\ &= \int_0^\pi d\theta \int_0^{2\pi} d\phi [sin\theta \int_0^R r^2 \lambda_R(\boldsymbol{X}_0 : r,\theta,\phi) \overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}_0 : r,\theta,\phi) dr] \\ &= \int_{V_R} \lambda_R(\boldsymbol{X}) \overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}) dV \end{aligned} \tag{G 3}$$

Here, V_R is a spherical volume with radius R centered at $\boldsymbol{X}_0$. (G3) is a rigorous relation between the time-averaged moving velocity of particles $\overline{\boldsymbol{u}_p^\Lambda}$ and the time-averaged velocity of the dispersed phase $\overline{\boldsymbol{u}_d}^d$, and it shows that these two physical quantities are not generally the same.

In the case where the time-averaged moving velocity of particles $\overline{\boldsymbol{u}_p^\Lambda}$ is constant in space, we know from (G3) that the time-averaged moving velocity of particles $\overline{\boldsymbol{u}_p^\Lambda}$ is equal to the time-averaged velocity of the dispersed phase $\overline{\boldsymbol{u}_d}^d$, since

$$\epsilon\Lambda\overline{\boldsymbol{u}_d}^d(\boldsymbol{X}_0) = \int_{V_R} \lambda_R(\boldsymbol{X}) \overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}) dV = \overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}_0) \int_{V_R} \lambda_R(\boldsymbol{X}) dV = \epsilon\Lambda\overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}_0) \tag{G 4}$$

When $\overline{\boldsymbol{u}_p^\Lambda}$ exhibits spatial variation, the right hand side of (G3) can be calculated, neglecting higher order terms of Taylor expansion, as follows:

$$\int_{V_R} \lambda_R(\boldsymbol{X}_0)\overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}_0)dV$$
$$\approx \int_0^\pi d\theta \int_0^{2\pi} d\phi[sin\theta \int_0^R r^2\{\lambda_R|_{\boldsymbol{X}_0} + r\boldsymbol{i}_r \cdot (\nabla\lambda_R)|_{\boldsymbol{X}_0}\}\{\overline{\boldsymbol{u}_p^\Lambda}|_{\boldsymbol{X}_0} + r\boldsymbol{i}_r \cdot (\nabla\overline{\boldsymbol{u}_p^\Lambda})|_{\boldsymbol{X}_0}\}dr] \tag{G 5}$$

By using (E4), the right hand side of (G5) can be calculated to

$$\int_0^\pi d\theta \int_0^{2\pi} d\phi[sin\theta \int_0^R r^2\{\lambda_R|_{\boldsymbol{X}_0} + r\boldsymbol{i}_r \cdot (\nabla\lambda_R)|_{\boldsymbol{X}_0}\}\{\overline{\boldsymbol{u}_p^\Lambda}|_{\boldsymbol{X}_0} + r\boldsymbol{i}_r \cdot (\nabla\overline{\boldsymbol{u}_p^\Lambda})|_{\boldsymbol{X}_0}\}dr]$$
$$\approx \epsilon(\boldsymbol{X}_0)\Lambda\overline{\boldsymbol{u}_p^\Lambda}(\boldsymbol{X}_0) + \frac{R^2}{5}\Lambda(\nabla\epsilon)|_{\boldsymbol{X}_0} \cdot (\nabla\overline{\boldsymbol{u}_p^\Lambda})|_{\boldsymbol{X}_0} \tag{G 6}$$

From (G3), (G5) and (G6), we have:

$$\overline{\boldsymbol{u}_d}^d \approx \overline{\boldsymbol{u}_p^\Lambda} + \frac{R^2}{5}\frac{\nabla\epsilon}{\epsilon} \cdot (\nabla\overline{\boldsymbol{u}_p^\Lambda}) \tag{G 7}$$

The terms in (G7) are all given at the same point in space. It is interesting to study the physical background of the second term on the right hand side of (G7), however, it is not investigated here.

Appendix H. Derivation of (4.5)

The Navier-Stokes equations, time-averaged, for the continuous and dispersed phases are given by (Ueyama and Miyauchi, 1976):

$$\frac{\partial}{\partial t}\{(1-\epsilon)\overline{\rho_c\boldsymbol{u}_c}^c\} + \nabla \cdot \{(1-\epsilon)\overline{\rho_c\boldsymbol{u}_c\boldsymbol{u}_c}^c\}$$
$$= -\nabla\{(1-\epsilon)\overline{P_c}^c\} - \nabla \cdot \{(1-\epsilon)\overline{\boldsymbol{\tau}_c}^c\} + (1-\epsilon)\overline{\rho_c}^c\boldsymbol{g} + \boldsymbol{D}_c \tag{H 1}$$

$$\frac{\partial}{\partial t}(\epsilon\overline{\rho_d\boldsymbol{u}_d}^d) + \nabla \cdot (\epsilon\overline{\rho_d\boldsymbol{u}_d\boldsymbol{u}_d}^d) = -\nabla(\epsilon\overline{P_d}^d) - \nabla \cdot (\epsilon\overline{\boldsymbol{\tau}_d}^d) + \epsilon\overline{\rho_d}^d\boldsymbol{g} - \boldsymbol{D}_c \tag{H 2}$$

Here, $\boldsymbol{D}_c$ is an interaction term given by (C1) in Appendix C. By summing Equations (H1) and (H2), we have:

$$\rho_m \frac{\partial}{\partial t}\overline{\boldsymbol{u}_c}^c + \rho_m \overline{\boldsymbol{u}_c}^c \cdot (\nabla \overline{\boldsymbol{u}_c}^c)$$
$$= -\nabla \cdot (\overline{P_c}^c \boldsymbol{I} + \overline{\boldsymbol{\tau}_c}^c) - \nabla \cdot \overline{\boldsymbol{\tau}_R} + \rho_m \boldsymbol{g} - \epsilon \rho_d \overline{\boldsymbol{u}_R} \cdot (\nabla \overline{\boldsymbol{u}_c}^c) \qquad \text{(H 3)}$$

Here, $\rho_m = (1-\epsilon)\overline{\rho_c}^c + \epsilon\overline{\rho_d}^d$ is the density of the two-phase mixture, $\overline{\boldsymbol{\tau}_R} = (1-\epsilon)\overline{\rho_c \boldsymbol{u}'_c \boldsymbol{u}'_c}^c + \epsilon\overline{\rho_d \boldsymbol{u}'_d \boldsymbol{u}'_d}^d$ is the Reynolds stress tensor of the two-phase flow, and $\overline{\boldsymbol{u}_R} = \overline{\boldsymbol{u}_d}^d - \overline{\boldsymbol{u}_c}^c$ is a relative velocity vector between the time-averaged velocity vectors of the dispersed and continuous phases. In deriving (H3), the equations of continuity, time-averaged for both phases, have been used to derive the left hand side, and the relative velocity $\overline{\boldsymbol{u}_R}$, together with ρ_c and ρ_d, are assumed to be constant. The relations $\overline{P_c}^c \boldsymbol{I} + \overline{\boldsymbol{\tau}_c}^c \approx \overline{P_d}^d \boldsymbol{I} + \overline{\boldsymbol{\tau}_d}^d$ and $\overline{\boldsymbol{u}_d}^d \approx \overline{\boldsymbol{u}_c}^c$ are assumed for the sake of simplicity, based on the fact that $\boldsymbol{n} \cdot (P_c \boldsymbol{I} + \boldsymbol{\tau}_c) = \boldsymbol{n} \cdot (P_d \boldsymbol{I} + \boldsymbol{\tau}_d)$ and $\boldsymbol{u}_d = \boldsymbol{u}_c$ at the interfaces that abound in the two-phase flow field.