Online supplementary material to the paper "Bend theory of river meanders with spatial width variations"

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Editorial

Appendix A

The governing system (3.19, . . . , 3.22) can be put in the following form:

$$\left(\frac{\partial}{\partial s}U\right)U+\left(\frac{\partial}{\partial n}U\right)V+\frac{\partial}{\partial s}H+\frac{\beta\,\tau_s}{D}+f_\alpha=\nu\,f_{10}+\delta\,f_{01}+\nu\,\delta\,f_{11}$$

$$\left(\frac{\partial}{\partial n}V\right)V+\left(\frac{\partial}{\partial s}V\right)U+\frac{\partial}{\partial n}H+\frac{\beta\,\tau_n}{D}+g_\alpha=\nu\,g_{10}+\delta\,g_{01}+\nu\,\delta\,g_{11}$$

$$\frac{\partial}{\partial s}(DU)+\frac{\partial}{\partial n}(DV)=\nu\,m_{10}+\delta\,m_{01}$$

$$\frac{\partial}{\partial s}q_s+\frac{\partial}{\partial n}q_n=\nu\,p_{10}+\delta\,p_{01}$$

where the terms $f_{ij}, g_{ij}, m_{ij}, p_{ij}$, $(j=0,1)$, and f_α, g_α can be written as:

$$f_\alpha=-K_0\Gamma_{L0\_b}-K_1\Gamma_{L1\_b}-K_2\Gamma_{L2\_b}$$

$$f_{10}=\left(-UV-\left(\frac{\partial}{\partial n}U\right)Vn-\frac{\beta\,\tau_s n}{D}\right)C+K_0\left(\Gamma_{L0}+\Gamma_{L0\_b}Cn+\Gamma_{L0\_a}\right)+$$
$$+K_1\left(\Gamma_{L1}+\Gamma_{L1\_b}Cn+\Gamma_{L1\_a}\right)+K_2\left(\Gamma_{L2}+\Gamma_{L2\_b}Cn+\Gamma_{L2\_a}\right)$$

$$f_{01}=n\left(\frac{d}{ds}B\right)\frac{\partial}{\partial n}H+B\left(\frac{\partial}{\partial n}U\right)V+n\left(\frac{d}{ds}B\right)\left(\frac{\partial}{\partial n}U\right)U+$$
$$+K_0\left(-B\Gamma_{L0\_b}+\Gamma_{L0\_d}n+1/3\,\Gamma_{L0\_c}\right)+K_1\left(-B\Gamma_{L1\_b}+\Gamma_{L1\_d}n+1/3\,\Gamma_{L1\_c}\right)+$$
$$+K_2\left(-B\Gamma_{L2\_b}+\Gamma_{L2\_d}n+1/3\,\Gamma_{L2\_c}\right)$$

$$f_{11}=-\frac{\beta\,\tau_s nBC}{D}+K_0\left(-B\Gamma_{L0}+n\Gamma_{L0\_c}C+n^2\Gamma_{L0\_d}C\right)+$$
$$+K_1\left(-B\Gamma_{L1}+n\Gamma_{L1\_c}C+n^2\Gamma_{L1\_d}C\right)+$$
$$+K_2\left(-B\Gamma_{L2}+n\Gamma_{L2\_c}C+n^2\Gamma_{L2\_d}C\right)$$

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$$
\begin{aligned}
g_\alpha = {} & K_0\left(-\Gamma_{T0\_b}C+\Gamma_{T0\_d}\right)+K_1\left(-\Gamma_{T1\_b}C+\Gamma_{T1\_d}\right)+K_2\left(-\Gamma_{T2\_b}C+\Gamma_{T2\_d}\right)+\\
& -K_6\Gamma_{T6\_a}C-K_7\Gamma_{T7\_a}C-K_8\Gamma_{T8\_a}C-K_9\Gamma_{T9\_a}C-K_{10}\Gamma_{T10\_a}C-K_{11}\Gamma_{T11\_a}C\\
g_{10} = {} & -nC\frac{\partial}{\partial n}H-nC\left(\frac{\partial}{\partial n}V\right)V-\frac{nC\beta\,\tau_n}{D}+CU^2+K_0\left(\Gamma_{T0\_a}+\Gamma_{T0}+\Gamma_{T0\_b}Cn+\Gamma_{T0\_c}\right)+\\
& +K_1\left(\Gamma_{T1\_a}+\Gamma_{T1}+\Gamma_{T1\_b}Cn+\Gamma_{T1\_c}\right)+K_2\left(\Gamma_{T2\_a}+\Gamma_{T2}+\Gamma_{T2\_b}Cn+\Gamma_{T2\_c}\right)+\\
& +K_6\left(\Gamma_{T6\_a}Cn+\Gamma_{T6}\right)+K_7\left(\Gamma_{T7\_a}Cn+\Gamma_{T7}\right)+K_8\left(\Gamma_{T8\_a}Cn+\Gamma_{T8}\right)+K_9\left(\Gamma_{T9\_a}Cn+\Gamma_{T9}\right)+\\
& +K_{10}\left(\Gamma_{T10\_a}Cn+\Gamma_{T10}\right)+K_{11}\left(\Gamma_{T11\_a}Cn+\Gamma_{T11}\right)\\
g_{01} = {} & B\left(\frac{\partial}{\partial n}V\right)V+B\frac{\partial}{\partial n}H+n\left(\frac{d}{ds}B\right)\left(\frac{\partial}{\partial n}V\right)U+\\
& +K_0\left(-\Gamma_{T0\_b}B+\Gamma_{T0\_g}n+1/2\,\Gamma_{T0\_f}+n\Gamma_{T0\_h}+n\Gamma_{T0\_i}\right)+\\
& +K_1\left(-\Gamma_{T1\_b}B+\Gamma_{T1\_g}n+1/2\,\Gamma_{T1\_f}+n\Gamma_{T1\_h}+n\Gamma_{T1\_i}\right)+\\
& +K_2\left(-\Gamma_{T2\_b}B+\Gamma_{T2\_g}n+1/2\,\Gamma_{T2\_f}+n\Gamma_{T2\_h}+n\Gamma_{T2\_i}\right)+\\
& +K_6\left(-B\Gamma_{T6\_a}+\Gamma_{T6\_e}n+\Gamma_{T6\_f}\right)+K_7\left(-B\Gamma_{T7\_a}+\Gamma_{T7\_e}n+\Gamma_{T7\_f}\right)+\\
& +K_8\left(-B\Gamma_{T8\_a}+\Gamma_{T8\_e}n+\Gamma_{T8\_f}\right)+K_9\left(-B\Gamma_{T9\_a}+\Gamma_{T9\_e}n+\Gamma_{T9\_f}\right)+\\
& +K_{10}\left(-B\Gamma_{T10\_a}+\Gamma_{T10\_e}n+\Gamma_{T10\_f}\right)+K_{11}\left(-B\Gamma_{T11\_a}+\Gamma_{T11\_e}n+\Gamma_{T11\_f}\right)\\
g_{11} = {} & -\frac{nBC\beta\,\tau_n}{D}+K_0\left(-B\Gamma_{T0}+\Gamma_{T0\_e}n+\Gamma_{T0\_f}Cn+\Gamma_{T0\_g}Cn^2\right)+\\
& +K_1\left(-B\Gamma_{T1}+\Gamma_{T1\_e}n+\Gamma_{T1\_f}Cn+\Gamma_{T1\_g}Cn^2\right)+\\
& +K_2\left(-B\Gamma_{T2}+\Gamma_{T2\_e}n+\Gamma_{T2\_f}Cn+\Gamma_{T2\_g}Cn^2\right)+K_6\left(\Gamma_{T6\_b}+\Gamma_{T6\_c}+\Gamma_{T6\_d}n+\Gamma_{T6\_e}Cn^2\right)+\\
& +K_7\left(\Gamma_{T7\_b}+\Gamma_{T7\_c}+\Gamma_{T7\_d}n+\Gamma_{T7\_e}Cn^2\right)+K_8\left(\Gamma_{T8\_b}+\Gamma_{T8\_c}+\Gamma_{T8\_d}n+\Gamma_{T8\_e}Cn^2\right)+\\
& +K_9\left(\Gamma_{T9\_b}+\Gamma_{T9\_c}+\Gamma_{T9\_d}n+\Gamma_{T9\_e}Cn^2\right)+K_{10}\left(\Gamma_{T10\_b}+\Gamma_{T10\_c}+\Gamma_{T10\_d}n+\Gamma_{T10\_e}Cn^2\right)+\\
& +K_{11}\left(\Gamma_{T11\_b}+\Gamma_{T11\_c}+\Gamma_{T11\_d}n+\Gamma_{T11\_e}Cn^2\right)
\end{aligned}
$$

$$
\begin{aligned}
m_{10} &= -nC\frac{\partial}{\partial n}(DV)-CVD\\
m_{01} &= n\left(\frac{d}{ds}B\right)\frac{\partial}{\partial n}(DU)+B\frac{\partial}{\partial n}(DV)
\end{aligned}
$$

$$
\begin{aligned}
p_{10} &= -nC\frac{\partial}{\partial n}q_n-Cq_n\\
p_{01} &= n\left(\frac{d}{ds}B\right)\frac{\partial}{\partial n}q_s+B\frac{\partial}{\partial n}q_n
\end{aligned}
$$

The Γ-coefficients result from the expressions below:

$$
\Gamma_{L0} = -\frac{\left(U^2\frac{\partial}{\partial n}D+\frac{\partial}{\partial n}DU^2\right)C}{b_c}
$$

$$\Gamma_{L0\_a} = 2\,\frac{DU^2C\alpha_s}{b_c}$$

$$\Gamma_{L0\_b} = \frac{U^2\left(\frac{\partial}{\partial n}D\right)\alpha_s + \frac{\partial}{\partial n}\left(DU^2\alpha_s\right)}{b_c}$$

$$\Gamma_{L0\_c} = -3\,\frac{DU^2\alpha_n\frac{d}{ds}B}{b_c}$$

$$\Gamma_{L0\_d} = -\frac{\left(\frac{d}{ds}B\right)\left(\frac{\partial}{\partial n}\left(DU^2\alpha_n\right) + U^2\left(\frac{\partial}{\partial n}D\right)\alpha_n\right)}{b_c}$$

$$\Gamma_{L1} = -\frac{U\left(\frac{\partial}{\partial n}D\right)D\frac{\partial}{\partial s}\left(UC\right) + \frac{\partial}{\partial n}\left(D^2U\frac{\partial}{\partial s}\left(UC\right)\right)}{{b_c}^2}$$

$$\Gamma_{L1\_a} = 2\,\frac{UCD^2\frac{\partial}{\partial s}\left(U\alpha_s\right)}{{b_c}^2}$$

$$\Gamma_{L1\_b} = \frac{\frac{\partial}{\partial n}\left(UD^2\frac{\partial}{\partial s}\left(U\alpha_s\right)\right) + U\left(\frac{\partial}{\partial n}D\right)D\frac{\partial}{\partial s}\left(U\alpha_s\right)}{{b_c}^2}$$

$$\Gamma_{L1\_c} = -3\,\frac{D^2U\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)}{{b_c}^2}$$

$$\Gamma_{L1\_d} = -\frac{\frac{\partial}{\partial n}\left(D^2U\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)\right) + U\left(\frac{\partial}{\partial n}D\right)D\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)}{{b_c}^2}$$

$$\Gamma_{L2} = -\frac{\left(U^2\left(\frac{\partial}{\partial n}D\right)\frac{\partial}{\partial s}D + \frac{\partial}{\partial n}\left(DU^2\frac{\partial}{\partial s}D\right)\right)C}{{b_c}^2}$$

$$\Gamma_{L2\_a} = 2\,\frac{CDU^2\left(\frac{\partial}{\partial s}D\right)\alpha_s}{{b_c}^2}$$

$$\Gamma_{L2\_b} = \frac{\frac{\partial}{\partial n}\left(DU^2\alpha_s\frac{\partial}{\partial s}D\right) + U^2\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}D\right)\alpha_s}{{b_c}^2}$$

$$\Gamma_{L2\_c} = -3\,\frac{DU^2\left(\frac{\partial}{\partial s}D\right)\alpha_n\frac{d}{ds}B}{{b_c}^2}$$

$$\Gamma_{L2\_d} = -\frac{\left(U^2\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}D\right)\alpha_n + \frac{\partial}{\partial n}\left(U^2D\alpha_n\frac{\partial}{\partial s}D\right)\right)\frac{d}{ds}B}{{b_c}^2}$$

$$\Gamma_{T0} = -2\,\frac{C\left(V\left(\frac{\partial}{\partial n}D\right)U + \frac{\partial}{\partial n}\left(DUV\right)\right)}{b_c}$$

$$\Gamma_{T0\_a} = -\frac{U^2\left(\frac{\partial}{\partial s}D\right)C + \frac{\partial}{\partial s}\left(DU^2C\right)}{b_c}$$

$$\Gamma_{T0\_b} = 2\,\frac{\frac{\partial}{\partial n}\left(DVU\alpha_s\right) + V\left(\frac{\partial}{\partial n}D\right)U\alpha_s}{b_c}$$

$$\Gamma_{T0\_c} = 2\,\frac{CVDU\alpha_s}{b_c}$$

$$\Gamma_{T0\_d} = -\frac{\frac{\partial}{\partial s}\left(DU^2\alpha_s\right) + U^2\left(\frac{\partial}{\partial s}D\right)\alpha_s}{b_c}$$

$$\Gamma_{T0\_e} = \frac{\left(\frac{d}{ds}B\right)C\left(U^2\frac{\partial}{\partial n}D + \frac{\partial}{\partial n}\left(DU^2\right)\right)}{b_c}$$

$$\Gamma_{T0\_f} = -4\,\frac{DUV\alpha_n \frac{d}{ds}B}{b_c}$$

$$\Gamma_{T0\_g} = -2\,\frac{\left(\frac{d}{ds}B\right)\left(\frac{\partial}{\partial n}\left(DUV\alpha_n\right) + V\left(\frac{\partial}{\partial n}D\right)U\alpha_n\right)}{b_c}$$

$$\Gamma_{T0\_h} = -\frac{\left(\frac{\partial}{\partial n}\left(DU^2\alpha_s\right) + U^2\left(\frac{\partial}{\partial n}D\right)\alpha_s\right)\frac{d}{ds}B}{b_c}$$

$$\Gamma_{T0\_i} = -\frac{U^2\left(\frac{\partial}{\partial s}D\right)\alpha_n\frac{d}{ds}B + \frac{\partial}{\partial s}\left(DU^2\alpha_n\frac{d}{ds}B\right)}{b_c}$$

$$\Gamma_{T1} = -2\,\frac{\frac{\partial}{\partial n}\left(D^2V\frac{\partial}{\partial s}\left(UC\right)\right) + V\left(\frac{\partial}{\partial n}D\right)D\frac{\partial}{\partial s}\left(UC\right)}{{b_c}^2}$$

$$\Gamma_{T1\_a} = -\frac{\frac{\partial}{\partial s}\left(D^2U\frac{\partial}{\partial s}\left(UC\right)\right) + U\left(\frac{\partial}{\partial s}D\right)D\frac{\partial}{\partial s}\left(UC\right)}{{b_c}^2}$$

$$\Gamma_{T1\_b} = 2\,\frac{V\left(\frac{\partial}{\partial n}D\right)D\frac{\partial}{\partial s}\left(U\alpha_s\right) + \frac{\partial}{\partial n}\left(D^2V\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)}{{b_c}^2}$$

$$\Gamma_{T1\_c} = 2\,\frac{CVD^2\frac{\partial}{\partial s}\left(U\alpha_s\right)}{{b_c}^2}$$

$$\Gamma_{T1\_d} = -\frac{U\left(\frac{\partial}{\partial s}D\right)D\frac{\partial}{\partial s}\left(U\alpha_s\right) + \frac{\partial}{\partial s}\left(D^2U\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)}{{b_c}^2}$$

$$\Gamma_{T1\_e} = \frac{\left(\frac{d}{ds}B\right)\left(\frac{\partial}{\partial n}\left(D^2U\frac{\partial}{\partial s}\left(UC\right)\right) + U\left(\frac{\partial}{\partial n}D\right)D\frac{\partial}{\partial s}\left(UC\right)\right)}{{b_c}^2}$$

$$\Gamma_{T1\_f} = -4\,\frac{D^2V\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)}{{b_c}^2}$$

$$\Gamma_{T1\_g} = -2\,\frac{V\left(\frac{\partial}{\partial n}D\right)D\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right) + \frac{\partial}{\partial n}\left(D^2V\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)\right)}{{b_c}^2}$$

$$\Gamma_{T1\_h} = -\frac{\left(\frac{d}{ds}B\right)\left(U\left(\frac{\partial}{\partial n}D\right)D\frac{\partial}{\partial s}\left(U\alpha_s\right) + \frac{\partial}{\partial n}\left(D^2U\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)\right)}{{b_c}^2}$$

$$\Gamma_{T1\_i} = \frac{-\frac{\partial}{\partial s}\left(D^2U\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)\right) - U\left(\frac{\partial}{\partial s}D\right)D\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)}{{b_c}^2}$$

$$\Gamma_{T2} = -2\,\frac{C\left(\frac{\partial}{\partial n}\left(DUV\frac{\partial}{\partial s}D\right) + V\left(\frac{\partial}{\partial n}D\right)U\frac{\partial}{\partial s}D\right)}{{b_c}^2}$$

$$\Gamma_{T2\_a} = -\frac{\frac{\partial}{\partial s}\left(CDU^2\frac{\partial}{\partial s}D\right) + U^2\left(\frac{\partial}{\partial s}D\right)^2C}{{b_c}^2}$$

$$\Gamma_{T2\_b} = 2\,\frac{\frac{\partial}{\partial n}\left(DVU\alpha_s\frac{\partial}{\partial s}D\right) + V\left(\frac{\partial}{\partial n}D\right)U\left(\frac{\partial}{\partial s}D\right)\alpha_s}{{b_c}^2}$$

$$\Gamma_{T2\_c} = 2\,\frac{CVDU\left(\frac{\partial}{\partial s}D\right)\alpha_s}{{b_c}^2}$$

$$\Gamma_{T2\_d} = -\frac{U^2\alpha_s\left(\frac{\partial}{\partial s}D\right)^2 + \frac{\partial}{\partial s}\left(\alpha_sDU^2\frac{\partial}{\partial s}D\right)}{{b_c}^2}$$

$$\Gamma_{T2\_e} = \frac{\left(\frac{d}{ds}B\right)C\left(\frac{\partial}{\partial n}\left(DU^2\frac{\partial}{\partial s}D\right) + U^2\left(\frac{\partial}{\partial n}D\right)\frac{\partial}{\partial s}D\right)}{{b_c}^2}$$

$$\Gamma_{T2\_f} = -4\,\frac{DUV\left(\frac{\partial}{\partial s}D\right)\alpha_n\frac{d}{ds}B}{{b_c}^2}$$

$$\Gamma_{T2\_g} = -2\,\frac{\left(V\left(\frac{\partial}{\partial n}D\right)U\left(\frac{\partial}{\partial s}D\right)\alpha_n - DV\frac{\partial}{\partial n}\left(-U\left(\frac{\partial}{\partial s}D\right)\alpha_n\right) + \left(\frac{\partial}{\partial n}\left(DV\right)\right)U\left(\frac{\partial}{\partial s}D\right)\alpha_n\right)\frac{d}{ds}B}{{b_c}^2}$$

$$\Gamma_{T2\_h} = -\frac{\left(\frac{d}{ds}B\right)\left(U^2\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}D\right)\alpha_s + \frac{\partial}{\partial n}\left(\alpha_s DU^2\frac{\partial}{\partial s}D\right)\right)}{{b_c}^2}$$

$$\Gamma_{T2\_i} = -\frac{U^2\left(\frac{\partial}{\partial s}D\right)^2\alpha_n\frac{d}{ds}B + \frac{\partial}{\partial s}\left(DU^2\left(\frac{\partial}{\partial s}D\right)\alpha_n\frac{d}{ds}B\right)}{{b_c}^2}$$

$$\Gamma_{T6} = -\frac{\left(D^2U^2{\alpha_s}^2 - 2\,D\left(\frac{\partial}{\partial n}D\right)U^2\alpha_s - 2\,\frac{\partial}{\partial n}\left(D^2U^2\alpha_s\right)\right)C}{{b_c}^2}$$

$$\Gamma_{T6\_a} = -\frac{\frac{\partial}{\partial n}\left(D^2U^2{\alpha_s}^2\right) + D\left(\frac{\partial}{\partial n}D\right)U^2{\alpha_s}^2}{{b_c}^2}$$

$$\Gamma_{T6\_b} = -2\,\frac{BC\left(\frac{\partial}{\partial n}\left(D^2U^2\alpha_s\right) + D\left(\frac{\partial}{\partial n}D\right)U^2\alpha_s\right)}{{b_c}^2}$$

$$\Gamma_{T6\_c} = -2\,\frac{D^2U^2C\alpha_n\frac{d}{ds}B}{{b_c}^2}$$

$$\Gamma_{T6\_d} = 2\,\frac{\left(\frac{d}{ds}B\right)C\left(-\frac{\partial}{\partial n}\left(D^2U^2\alpha_n\right) - D\left(\frac{\partial}{\partial n}D\right)U^2\alpha_n + 2\,D^2U^2\alpha_s\alpha_n\right)}{{b_c}^2}$$

$$\Gamma_{T6\_e} = \frac{\left(\frac{d}{ds}B\right)\left(2\,D\left(\frac{\partial}{\partial n}D\right)U^2\alpha_n\alpha_s + 2\,\frac{\partial}{\partial n}\left(D^2U^2\alpha_s\alpha_n\right)\right)}{{b_c}^2}$$

$$\Gamma_{T6\_f} = 2\,\frac{D^2U^2\alpha_s\alpha_n\frac{d}{ds}B}{{b_c}^2}$$

$$\Gamma_{T7} = -2\,\frac{C\left(-2\,\frac{\partial}{\partial n}\left(D^2U^2\left(\frac{\partial}{\partial s}D\right)\alpha_s\right) - 2\,D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right)\alpha_s + D^2U^2{\alpha_s}^2\frac{\partial}{\partial s}D\right)}{{b_c}^3}$$

$$\Gamma_{T7\_a} = -2\,\frac{D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right){\alpha_s}^2 + \frac{\partial}{\partial n}\left(D^2U^2{\alpha_s}^2\frac{\partial}{\partial s}D\right)}{{b_c}^3}$$

$$\Gamma_{T7\_b} = -4\,\frac{BC\left(\frac{\partial}{\partial n}\left(D^2U^2\left(\frac{\partial}{\partial s}D\right)\alpha_s\right) + D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right)\alpha_s\right)}{{b_c}^3}$$

$$\Gamma_{T7\_c} = -4\,\frac{D^2U^2C\left(\frac{\partial}{\partial s}D\right)\alpha_n\frac{d}{ds}B}{{b_c}^3}$$

$$\Gamma_{T7\_d} = 4\,\frac{\left(\frac{d}{ds}B\right)C\left(\frac{\partial}{\partial n}\left(-D^2U^2\left(\frac{\partial}{\partial s}D\right)\alpha_n\right) - D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right)\alpha_n + 2\,D^2U^2\left(\frac{\partial}{\partial s}D\right)\alpha_s\alpha_n\right)}{{b_c}^3}$$

$$\Gamma_{T7\_e} = 2\,\frac{\left(\frac{d}{ds}B\right)\left(2\,D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right)\alpha_n\alpha_s + 2\,\frac{\partial}{\partial n}\left(D^2U^2\left(\frac{\partial}{\partial s}D\right)\alpha_s\alpha_n\right)\right)}{{b_c}^3}$$

$$\Gamma_{T7\_f} = 4\,\frac{D^2U^2\left(\frac{\partial}{\partial s}D\right)\alpha_s\alpha_n\frac{d}{ds}B}{{b_c}^3}$$

$$\Gamma_{T8} = -\frac{\left(-2\,\frac{\partial}{\partial n}\left(D^2U^2\alpha_s\left(\frac{\partial}{\partial s}D\right)^2\right) + D^2U^2{\alpha_s}^2\left(\frac{\partial}{\partial s}D\right)^2 - 2\,D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right)^2\alpha_s\right)C}{{b_c}^4}$$

$$\Gamma_{T8\_a} = -\frac{D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right)^2{\alpha_s}^2 + \frac{\partial}{\partial n}\left(D^2U^2{\alpha_s}^2\left(\frac{\partial}{\partial s}D\right)^2\right)}{{b_c}^4}$$

$$\Gamma_{T8\_b} = -2\,\frac{BC\left(D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right)^2\alpha_s + \frac{\partial}{\partial n}\left(D^2U^2\alpha_s\left(\frac{\partial}{\partial s}D\right)^2\right)\right)}{{b_c}^4}$$

$$\Gamma_{T8\_c} = -2\,\frac{D^2U^2C\left(\frac{\partial}{\partial s}D\right)^2\alpha_n\frac{d}{ds}B}{{b_c}^4}$$

$$\Gamma_{T8\_d} = 2\,\frac{C\left(\frac{d}{ds}B\right)\left(2\,U^2D^2\left(\frac{\partial}{\partial s}D\right)^2\alpha_s\alpha_n - D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right)^2\alpha_n - \frac{\partial}{\partial n}\left(D^2U^2\left(\frac{\partial}{\partial s}D\right)^2\alpha_n\right)\right)}{{b_c}^4}$$

$$\Gamma_{T8\_e} = 2\,\frac{\left(\frac{d}{ds}B\right)\left(\frac{\partial}{\partial n}\left(U^2D^2\left(\frac{\partial}{\partial s}D\right)^2\alpha_s\alpha_n\right) + D\left(\frac{\partial}{\partial n}D\right)U^2\left(\frac{\partial}{\partial s}D\right)^2\alpha_n\alpha_s\right)}{{b_c}^4}$$

$$\Gamma_{T8\_f} = 2\,\frac{U^2D^2\left(\frac{\partial}{\partial s}D\right)^2\alpha_s\alpha_n\frac{d}{ds}B}{{b_c}^4}$$

$$\Gamma_{T9} = -\frac{D\left(D^2C\frac{\partial}{\partial s}\left(U^2{\alpha_s}^2\right) - 2\,D\left(\frac{\partial}{\partial n}D\right)\frac{\partial}{\partial s}\left(U^2C\alpha_s\right) - 2\,\left(\frac{\partial}{\partial n}\left(D^2\right)\right)\frac{\partial}{\partial s}\left(U^2C\alpha_s\right) - 2\,D\frac{\partial}{\partial n}\left(D\frac{\partial}{\partial s}\left(U^2C\alpha_s\right)\right)\right)}{{b_c}^3}$$

$$\Gamma_{T9\_a} = -\frac{\frac{\partial}{\partial n}\left(D^3\frac{\partial}{\partial s}\left(U^2{\alpha_s}^2\right)\right) + D^2\left(\frac{\partial}{\partial n}D\right)\frac{\partial}{\partial s}\left(U^2{\alpha_s}^2\right)}{{b_c}^3}$$

$$\Gamma_{T9\_b} = -2\,\frac{B\left(\frac{\partial}{\partial n}\left(D^3\frac{\partial}{\partial s}\left(U^2C\alpha_s\right)\right) + D^2\left(\frac{\partial}{\partial n}D\right)\frac{\partial}{\partial s}\left(U^2C\alpha_s\right)\right)}{{b_c}^3}$$

$$\Gamma_{T9\_c} = -2\,\frac{D^3\frac{\partial}{\partial s}\left(U^2C\alpha_n\frac{d}{ds}B\right)}{{b_c}^3}$$

$$\Gamma_{T9\_d} = \frac{-2\,\frac{\partial}{\partial n}\left(D^3\frac{\partial}{\partial s}\left(U^2C\alpha_n\frac{d}{ds}B\right)\right) - 2\,D^2\left(\frac{\partial}{\partial n}D\right)\frac{\partial}{\partial s}\left(U^2C\alpha_n\frac{d}{ds}B\right) + 4\,CD^3\frac{\partial}{\partial s}\left(U^2\alpha_s\alpha_n\frac{d}{ds}B\right)}{{b_c}^3}$$

$$\Gamma_{T9\_e} = 2\,\frac{\frac{\partial}{\partial n}\left(D^3\frac{\partial}{\partial s}\left(U^2\alpha_s\alpha_n\frac{d}{ds}B\right)\right)D^2\left(\frac{\partial}{\partial n}D\right)\frac{\partial}{\partial s}\left(U^2\alpha_s\alpha_n\frac{d}{ds}B\right)}{{b_c}^3}$$

$$\Gamma_{T9\_f} = 2\,\frac{D^3\frac{\partial}{\partial s}\left(U^2\alpha_s\alpha_n\frac{d}{ds}B\right)}{{b_c}^3}$$

$$\Gamma_{T10} = \frac{2\,\frac{\partial}{\partial n}\left(D^4\left(\frac{\partial}{\partial s}\left(UC\right)\right)\frac{\partial}{\partial s}\left(U\alpha_s\right)\right) + 2\,D^3\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}\left(UC\right)\right)\frac{\partial}{\partial s}\left(U\alpha_s\right) - D^4C\left(\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)^2}{{b_c}^4}$$

$$\Gamma_{T10\_a} = -\frac{D^3\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)^2 + \frac{\partial}{\partial n}\left(D^4\left(\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)^2\right)}{{b_c}^4}$$

$$\Gamma_{T10\_b} = -2\,\frac{\left(\frac{\partial}{\partial n}\left(D^4\left(\frac{\partial}{\partial s}\left(UC\right)\right)\frac{\partial}{\partial s}\left(U\alpha_s\right)\right) + D^3\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}\left(UC\right)\right)\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)B}{{b_c}^4}$$

$$\Gamma_{T10\_c} = -2\,\frac{D^4\left(\frac{\partial}{\partial s}\left(UC\right)\right)\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)}{{b_c}^4}$$

$$\Gamma_{T10\_d} = -2\,\frac{D^3\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}\left(UC\right)\right)\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right) + \frac{\partial}{\partial n}\left(D^4\left(\frac{\partial}{\partial s}\left(UC\right)\right)\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)\right)}{{b_c}^4} +$$

$$+4\,\frac{CD^4\left(\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)}{{b_c}^4}$$

$$\Gamma_{T10\_e} = 2\,\frac{\frac{\partial}{\partial n}\left(D^4\left(\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)\right) + D^3\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)}{{b_c}^4}$$

$$\Gamma_{T10\_f} = 2\,\frac{D^4\left(\frac{\partial}{\partial s}\left(U\alpha_s\right)\right)\frac{\partial}{\partial s}\left(U\alpha_n\frac{d}{ds}B\right)}{{b_c}^4}$$

$$\Gamma_{T11} = \frac{2\,D^2\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}D\right)\frac{\partial}{\partial s}\left(U^2C\alpha_s\right) - D^3C\left(\frac{\partial}{\partial s}D\right)\frac{\partial}{\partial s}\left(U^2{\alpha_s}^2\right) + 2\,\frac{\partial}{\partial n}\left(D^3\left(\frac{\partial}{\partial s}\left(U^2C\alpha_s\right)\right)\frac{\partial}{\partial s}D\right)}{{b_c}^4}$$

$$\Gamma_{T11\_a} = -\frac{\frac{\partial}{\partial n}\left(D^3\left(\frac{\partial}{\partial s}\left(U^2{\alpha_s}^2\right)\right)\frac{\partial}{\partial s}D\right) + D^2\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}D\right)\frac{\partial}{\partial s}\left(U^2{\alpha_s}^2\right)}{{b_c}^4}$$

$$\Gamma_{T11\_b} = -2\,\frac{B\left(\frac{\partial}{\partial n}\left(D^3\left(\frac{\partial}{\partial s}\left(U^2C\alpha_s\right)\right)\frac{\partial}{\partial s}D\right) + D^2\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}D\right)\frac{\partial}{\partial s}\left(U^2C\alpha_s\right)\right)}{{b_c}^4}$$

$$\Gamma_{T11\_c} = -2\,\frac{D^3\left(\frac{\partial}{\partial s}\left(U^2C\alpha_n\frac{d}{ds}B\right)\right)\frac{\partial}{\partial s}D}{{b_c}^4}$$

$$\Gamma_{T11\_d} = -2\,\frac{\frac{\partial}{\partial n}\left(D^3\left(\frac{\partial}{\partial s}\left(U^2C\alpha_n\frac{d}{ds}B\right)\right)\frac{\partial}{\partial s}D\right) - 2\,CD^3\left(\frac{\partial}{\partial s}D\right)\frac{\partial}{\partial s}\left(U^2\alpha_s\alpha_n\frac{d}{ds}B\right)}{{b_c}^4} + \\ -2\,\frac{D^2\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}D\right)\frac{\partial}{\partial s}\left(U^2C\alpha_n\frac{d}{ds}B\right)}{{b_c}^4}$$

$$\Gamma_{T11\_e} = 2\,\frac{D^2\left(\frac{\partial}{\partial n}D\right)\left(\frac{\partial}{\partial s}D\right)\frac{\partial}{\partial s}\left(U^2\alpha_s\alpha_n\frac{d}{ds}B\right) + \frac{\partial}{\partial n}\left(D^3\left(\frac{\partial}{\partial s}D\right)\frac{\partial}{\partial s}\left(U^2\alpha_s\alpha_n\frac{d}{ds}B\right)\right)}{{b_c}^4}$$

$$\Gamma_{T11\_f} = 2\,\frac{D^3\left(\frac{\partial}{\partial s}D\right)\frac{\partial}{\partial s}\left(U^2\alpha_s\alpha_n\frac{d}{ds}B\right)}{{b_c}^4}$$

Note that:

$$\alpha_s = \frac{U\left(\frac{\partial}{\partial s}V\right) - V\left(\frac{\partial}{\partial s}U\right)}{U^2+V^2}; \quad \alpha_n = \frac{U\left(\frac{\partial}{\partial n}V\right) - V\left(\frac{\partial}{\partial n}U\right)}{U^2+V^2}; \quad b_c = \beta\sqrt{C_{f0}};$$

where α_s and α_n represent the local deviation of streamlines from the channel axis.