Modelling photo-gyrotactic bioconvection in suspensions of swimming micro-organisms: asymptotic analysis for Model B

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1. Asymptotic analysis for Model B

An unabridged version of section 6.2 in the full paper.

Since for Model B none of the K_i, J_i, P_i or A_i are constants, the asymptotic analysis involves much expanding. We write each expansion in the same way, for example for K_i

$$K_i = K_{(i,0)} + d^{-1}K_{(i,-1)} + d^{-2}(\kappa m^0_{-1}K_{(i,-1)} + K_{(i,-2)}) \\ +d^{-3}\left(K_{(i,-1)}\kappa\left(m^0_{-2} + \frac{\kappa(m^0_{-1})^2}{2}\right) + K_{(i,-3)} + 2\kappa m_{-1}K_{(i,-2)}\right), \tag{1.1}$$

and similarly for all J_i, P_i and A_i. Each component $K_{(i,j)}$ (and $J_{(i,j)}$, $P_{(i,j)}$ $A_{(i,j)}$) can be calculated directly using Taylor series to expand the expressions for K_i (and J_i, P_i and A_i) for $d \ll 1$. Note that the leading order component of K_i and J_i do not depend on χ and are constants that depend on λ, as in Model A where $\Lambda = \lambda$. For example, $K_{(1,0)} = \bar{K}_1$. The definitions of the relevant components for use in the asymptotic solution are shown in table 1.

1.0.1. *Equilibrium solution*

Multiplying the equilibrium solution in equation (5.2) by d^{-1} gives

$$d^{-1}\frac{d^2m}{dz^2} - \frac{\bar{K}_2}{\bar{K}_1}\frac{K_1(\Lambda)}{K_2(\Lambda)}\frac{dm}{dz} = 0, \tag{1.2}$$

with boundary conditions $m = 0$ at $z = 0$, and $m = d^{-1}(e^{-d} - 1)$ at $z = -1$. If we expand K_1, K_2 and then $\frac{K_1}{K_2}$ for small d, using the notation as explained above, then expanding m in powers of d^{-1} and expanding the exponential in powers of d and substituting all this into equation 1.2 gives

$$d^{-1}\frac{d^2(m_0 d^{-1}m_{-1} + d^{-2}m_{-2} + ...)}{dz^2} - \frac{\bar{K}_2}{\bar{K}_1}\left[K_{(1/2,0)} + d^{-1}e^{\kappa m_0}K_{(1/2,-1)}\right. \\ \left.+d^{-2}(e^{2\kappa m_0}K_{(1/2,-2)} + e^{\kappa m_0}\kappa m_{-1}K_{(1/2,-1)}) + O(d^{-3})\right] \\ \times\frac{d(m_0 + d^{-1}m_{-1} + d^{-2}m_{-2} + ..)}{dz} = 0. \tag{1.3}$$

For the **outer** solution, we find m_0 =constant. Every subsequent m_{-n} will also be constant. The boundary condition at $z = -1$ gives that $m_0 = m_{-2} = ...m_{-n} = 0$ and $m_{-1} = -1$.

For the **inner** solution we scale $z_I = dz$ and find the solution at leading order is

$$m_0 = A_0(e^{z_I} - 1). \tag{1.4}$$

At next order we have

$$\frac{d^2 m_{-1}}{d{z_I}^2} - \frac{dm_{-1}}{dz_I} - \frac{\bar{K_2}}{\bar{K_1}} K_{(1/2,-1)} e^{\kappa m_0} \frac{dm_0}{dz_I} = 0. \tag{1.5}$$

In order to match with the outer solution we require A_0=0. Solving equation 1.5 then gives

$$m_{-1} = A_1(e^{z_I} - 1). \tag{1.6}$$

The matching as z_I tends to $-\infty$ provides $A_1 = 1$. Solving the equation at second order, substituting in m_{-1}, gives

$$m_{-2} = \frac{\bar{K_2}}{\bar{K_1}} K_{(1/2,-1)}(z_I e^{z_I} + 1) - A_2 + B_2 e^{z_I}. \tag{1.7}$$

On applying the boundary condition at $z_I = 0$ and using the matching, we find $A_2 = \frac{\bar{K_2}}{\bar{K_1}} K_{(1/2,-1)}$ and $B_2 = 0$. Thus,

$$m_{-2} = \frac{\bar{K_2}}{\bar{K_1}} K_{(1/2,-1)} z_I e^{z_I}. \tag{1.8}$$

1.0.2. *Linear stability analysis*

The asymptotic linear stability theory is performed on the Navier-Stokes equation,

$$(D^2 - k^2)^2 U = -k^2 d^{-1} R\Phi, \tag{1.9}$$

and the cell conservation equation (5.22), with $P_i(z)$ and $K_i(z)$ defined in table 2 and §5.3.2. As in linear stability analysis in §5.3, the equilibrium components are now denoted with a superscript 0.

For the **outer** solution, the solutions for Φ and U are the same as for Model A, as shown in §6.1.2 and equation (6.5).

For the **inner** solution we re-scale equation (5.22) and the Navier-Stokes equation (1.9) using $z_I = dz$, so that

$$(D_I^2 - d^{-2}k^2)^2 U = -k^2 d^{-5} R\Phi, \tag{1.10}$$

$$\begin{aligned}&\left\{P_V(z_I)\frac{d^2}{dz_I^2} - \frac{\bar{K_2}}{\bar{K_1}} K_1(z_I)\frac{d}{dz_I} - d^{-2}P_H(z_I)k^2 - d^{-2}\sigma - \frac{\bar{K_2}}{\bar{K_1}}\frac{dK_1}{dz_I} + \frac{dP_V(z_I)}{dz_I}\frac{d}{dz_I}\right.\\ &\left. + d^{-2}\lambda\chi_{-1}\kappa e^{\kappa m^0} P_R(z_I)\right\}\Phi + d^{-1}\lambda\chi_{-1}\kappa e^{\kappa m^0} P_M(z_I) = \left\{d^{-1}\frac{dn^0}{dz_I}\right.\\ &\left. - d\eta P_5(z_I)\frac{d^2}{dz_I^2} - d\eta P_6(z_I)\frac{d}{dz_I} + d^{-1}\eta P_7(z_I)k^2\right\} U,\end{aligned} \tag{1.11}$$

where $P_i(z_I)$ can be calculated directly from the expression for $P_i(z)$ in §5.3.2, with boundary conditions

$$U = 0 \quad \text{and} \quad D_I U = 0 \quad \text{on} \quad z_I = 0, \tag{1.12}$$

$$\text{and} \quad K_2(z_I) D_I \Phi - \frac{K_{(2,0)}}{K_{(1,0)}} K_1(z_I)\Phi = 0 \quad \text{on} \quad z_I = 0. \tag{1.13}$$

The terms on the right hand side of equation 1.11 complicate the expression and so we consider the case in which they do not appear at first order (as in Model A). This requires $U \leqslant O(1)$ and $\eta U \leqslant O(d^{-2})$. For a non-trivial solution we need $R \sim d^5 U$ and we follow the logic outlined in §6.1.2 and consider expanding using equations (6.8) and (6.9). At first order the cell conservation and Navier-Stokes equations are the same as for Model A, hence Φ_0 and U_{-n} are given in equation (6.10).

The expressions at second order are calculated in the same was as for Model A, thus we obtain

$$\Phi_{-1} = -A_K z_I e^{z_I} + C_K e^{z_I}, \tag{1.14}$$

where C_K is a constant of integration and

$$A_K = \frac{K_{(2,-1)} - \frac{K_{(2,0)}}{K_{(1,0)}} K_{(1,-1)}}{K_{(2,0)}}, \tag{1.15}$$

and the Navier-Stokes equation yields

$$\begin{aligned} U_{-n-1} &= a_{-n-1} z_I^3 + b_{-n-1} z_I^2 + k^2 R_{5-n-1} \left(z_I + 1 - e^{z_I} \right) \\ &\quad + k^2 R_{5-n} \left(A_K (z_I e^{z_I} - 4e^{z_I}) - C_K e^{z_I} + (3A_K + C_K) z_I + 4A_K + C_K \right). \end{aligned} \tag{1.16}$$

Solutions are matched up to second order in the usual way, as for Model A, and we find that the terms due to phototaxis do not appear at this order. If we look in the region of parameter space where $\eta \sim d^{-2}$ and $n = 1$, as for Model A, then A_0, B_0 and b_{-2} are given by (6.13). The cell conservation equation at third order is then integrated from $-\infty$ to 0 to obtain the solvability condition:

$$R_4 = \frac{2P_{(H,0)}}{(1 - \eta_{-2}(P_{(5,0)} - P_{(6,0)}))}. \tag{1.17}$$

Integrating the cell conservation equation between 0 and $-\infty$ at fourth order gives

$$\begin{aligned} R_3 &= 4b_{-2} + \frac{2(P_{(H,0)}(A_K + C_K) + P_{(H,-1)})}{(1 - \eta_{-2}(P_{(5,0)} - P_{(6,0)}))} \\ &\quad + \frac{2R_4}{(1 - \eta_{-2}(P_{(5,0)} - P_{(6,0)}))} \left[\frac{N_K}{4} - \frac{5A_K}{4} - \frac{C_K}{2} + \eta_{-2} \left\{ \left(\frac{5A_K}{4} + \frac{C_K}{2} \right) \right. \right. \\ &\quad \times (P_{(5,0)} - P_{(6,0)}) + \frac{3}{4} A_{(1,0)} N_K + \frac{1}{2} A_{(1,-1)} + \frac{A_{(4,-1)}}{2} + \frac{3N_K A_{(4,0)}}{4} + \\ &\quad \left. \left. \frac{K_{(2,0)}}{K_{(1,0)}} \left(\frac{1}{2} A_{(2,-1)} + \frac{N_K}{4} A_{(2,0)} - \frac{1}{2} A_{(3,-1)} - \frac{N_K}{4} A_{(3,0)} \right) \right\} \right], \end{aligned} \tag{1.18}$$

where $N_k = \frac{K_{(2,0)}}{K_{(1,0)}} K_{(1/2,-1)}$ is a constant and the definitions of $A_{(i,j)}$, $P_{(i,j)}$ and all other constants can be found in table 1, which also shows the dependance of the fourth order term on χ. Note that the expression for the Raleigh number as a function of the wavenumber, $R(k)$, to third order is the same as the expression to third order for Model A and Bees & Hill (1998), since $P_{(H,0)} = P_H$, $P_{(5,0)} = P_5$ and $P_{(6,0)} = P_6$, where P_H, P_5 and P_6 constants. Thus the effects of phototaxis only comes in at fourth order.

2. Table of constants

Table 1 summarizes the definitions of parameters that are needed for the asymptotic analysis of Model B. Values are calculated using the standard values $\lambda = 2.2$ and $\alpha_0 = 0.2$.

Parameter	Definition	Value
$K_{(1,0)}$	$\coth \lambda_0 - \frac{1}{\lambda_0}$	0.570

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Table 1 – Continued

Parameter	Definition	Value
$K_{(1,-1)}$	$\frac{\lambda_1\left(\cosh^2\lambda_0-1-\lambda_0^2\right)}{\lambda_0^2\sinh^2\lambda_0}$	-0.344
$K_{(2,0)}$	$1-\coth^2\lambda_0+\frac{1}{\lambda_0^2}$	0.156
$K_{(2,-1)}$	$\frac{-2\lambda_1\left(\sinh\lambda_0\cosh^2\lambda_0-\sinh\lambda_0-\lambda_0^3\cosh\lambda_0\right)}{\lambda_0^3\sinh^3\lambda_0}$	0.186
$K_{(4,0)}$	$K_{(2,0)}-\frac{K_{(1,0)}}{\lambda_0}$	-0.103
$K_{(4,-1)}$	$K_{(2,-1)}-\frac{K_{(1,-1)}}{\lambda_0}+\frac{K_{(1,0)}\lambda_1}{\lambda_0^2}$	0.0833
$K_{(5,0)}$	$-\frac{2}{\lambda_0}\left[1+K_{(2,0)}-\frac{4K_{(1,0)}}{\lambda_0}\right]$	-0.108
$K_{(5,-1)}$	$-\frac{2}{\lambda_0}\left(K_{(2,-1)}-\frac{4}{\lambda_0}\left(K_{(1,-1)}-\frac{K_{(1,0)}\lambda_1}{\lambda_0}\right)\right)$ $+\frac{2\lambda_1}{\lambda_0^2}\left(1+K_{(2,0)}-\frac{4K_{(1,0)}}{\lambda_0}\right)$	0.0966
$J_{(1,0)}$	$\frac{\lambda_0^2}{3\sinh(\lambda_0)}\sum_{l=0}^{\infty}\lambda_0^{2l+1}(z)a_{2l+1,1}$	0.452
$J_{(1,-1)}$	$\frac{\lambda_0}{3}\left((\lambda_1\text{cosech}(\lambda_0)-\lambda_0\lambda_1\coth(\lambda_0)\text{cosech}(\lambda_0))\times\sum_{l=0}^{\infty}\lambda_0^{2l+1}(z)a_{2l+1,1}+\frac{\lambda_0}{\sinh(\lambda_0)}\sum_{l=0}^{\infty}\lambda_1^{2l+1}(z)a_{2l+1,1}\right)$	-0.0225
$J_{(2,0)}$	$\frac{\lambda_0^2}{5\sinh(\lambda_0)}\sum_{l=1}^{\infty}\lambda_0^{2l}(z)a_{2l,2}$	0.159
$J_{(2,-1)}$	$\frac{\lambda_0}{5}\left((\lambda_1\text{cosech}(\lambda_0)-\lambda_0\lambda_1\coth(\lambda_0)\text{cosech}(\lambda_0))\times\sum_{l=1}^{\infty}\lambda_0^{2l}(z)a_{2l,2}+\frac{\lambda_0}{\sinh(\lambda_0)}\sum_{l=1}^{\infty}\lambda_1^{2l}(z)a_{2l,2}\right)$	-0.163
$J_{(4,0)}$	$\frac{\lambda_0^2}{3\sinh(\lambda_0)}\sum_{l=0}^{\infty}\lambda_0^{2l+1}(z)\tilde{a}_{2l+1,1}$	-0.227
$J_{(4,-1)}$	$\frac{\lambda_0}{3}\left((\lambda_1\text{cosech}(\lambda_0)-\lambda_0\lambda_1\coth(\lambda_0)\text{cosech}(\lambda_0))\times\sum_{l=0}^{\infty}\lambda_0^{2l+1}(z)\tilde{a}_{2l+1,1}+\frac{\lambda_0}{\sinh(\lambda_0)}\sum_{l=0}^{\infty}\lambda_1^{2l+1}(z)\tilde{a}_{2l+1,1}\right)$	0.114
$J_{(5,0)}$	$\frac{\lambda_0^2}{5\sinh(\lambda_0)}\sum_{l=0}^{\infty}\lambda_0^{2l}(z)\tilde{a}_{2l,2}$	-0.166
$J_{(5,-1)}$	$\frac{\lambda_0}{5}\left((\lambda_1\text{cosech}(\lambda_0)-\lambda_0\lambda_1\coth(\lambda_0)\text{cosech}(\lambda_0))\times\sum_{l=0}^{\infty}\lambda_0^{2l}(z)\tilde{a}_{2l,2}+\frac{\lambda_0}{\sinh(\lambda_0)}\sum_{l=0}^{\infty}\lambda_1^{2l}(z)\tilde{a}_{2l,2}\right)$	0.0195
$A_{(1,0)}$	$J_{(1,0)}K_{(1,0)}-J_{(2,0)}+\alpha_0(J_{(5,0)}-K_{(1,0)}J_{(4,0)}-3(K_{(5,0)}-2K_{(1,0)}K_{(4,0)}))$	0.0862

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Table 1 – Continued

Parameter	Definition	Value
$A_{(1,-1)}$	$J_{(1,0)}K_{(1,-1)} + J_{(1,-1)}K_{(1,0)} - J_{(2,-1)} + \alpha_0(J_{(5,-1)} - K_{(1,0)}J_{(4,-1)} - K_{(1,-1)}J_{(4,0)} - 3(K_{(5,-1)} - 2K_{(1,0)}K_{(4,-1)} - 2K_{(1,-1)}K_{(4,0)}))$	0.0114
$A_{(2,0)}$	$J_{(1,0)} - \alpha_0(J_{(4,0)} - 3K_{(4,0)})$	0.436
$A_{(2,-1)}$	$J_{(1,-1)} - \alpha_0(J_{(4,-1)} - 3K_{(4,-1)})$	0.00453
$A_{(3,0)}$	$3\alpha_0 K_{(4,0)}$	-0.0618
$A_{(3,-1)}$	$3\alpha_0 K_{(4,-1)}$	0.0500
$A_{(4,0)}$	$3\alpha_0(K_{(5,0)} - 2K_{(1,0)}K_{(4,0)})$	0.00537
$A_{(4,-1)}$	$3\alpha_0(K_{(5,-1)} - 2K_{(1,0)}K_{(4,-1)} - 2K_{(1,-1)}K_{(4,0)})$	-0.0415
$P_{(H,0)}$	$\frac{K_{(1,0)}}{\lambda_0}$	0.259
$P_{(H,-1)}$	$\frac{K_{(1,-1)}}{\lambda_0} - \frac{K_{(1,0)}\lambda_1}{\lambda_0^2}$	0.103
$P_{(5,0)}$	$A_{(1,0)} + \frac{K_{(2,0)}A_{(2,0)}}{K_{(1,0)}}$	0.205
$P_{(6,0)}$	$\frac{K_{(2,0)}A_{(3,0)}}{K_{(1,0)}} - A_{(4,0)}$	-0.0223
A_K	$\frac{1}{K_{(2,0)}}\left(K_{(2,-1)} - \frac{K_{(2,0)}K_{(1,-1)}}{K_{(1,0)}}\right)$	1.79
C_K	1.0	1.0
N_K	$\frac{1}{K_{(1,0)}}\left(K_{(1,-1)} - \frac{K_{(1,0)}K_{(2,-1)}}{K_{(2,0)}}\right)$	-1.79

Table 1: Summary of constants needed to compute the asymptotic solution for Model B, where $\alpha_0 = 0.2$. Here, $\lambda_0 = \lambda$ and $\lambda_1 = -\chi_{-1}\lambda$, where $\lambda = 2.2$ and $\chi_{-1} = 1$. $K_{(i,0)}$ and $J_{(i,0)}$ are equivalent to the values of K_i and J_i when $\Lambda(z) = \lambda$, i.e. the values of K_i and J_i when $\chi = 0$. These are the (corrected) values of K_i and J_i used in Bees & Hill (1998).