Appendix to "The rapid advance and slow retreat of a mushy zone"

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Appendix. Liquid fraction at the solid-mush interface

We have claimed that the liquid fraction at the solid-mush interface is zero for the similarity solution discussed in Worster (1986) under the assumptions of negligible latent heat and uniform thermal properties. Also, the numerical scheme incorporates the similarity solution at the initial time, so the liquid fraction at that interface needs to be known. The differential equation governing the liquid fraction in the mush, in terms of the similarity variable, develops a singularity under the present assumptions. This disrupts numerical integration of the similarity solution. However, as we will demonstrate, we can analytically derive that the similarity solution has zero liquid fraction at the solid-mush interface.

The system for the similarity solution is similar to the system presented in this paper. However, since the similarity solution uses an infinite domain, the conditions that we use at $z = 1$ for the finite domain are replaced with the farfield ($z \to \infty$) conditions,

$$\theta \to 1,\ C \to C_0.$$

As in the finite-domain case, the temperature field decouples from the rest of the system under our assumptions. In this case, the temperature field is given by

$$\theta(z,t) = \mathrm{erf}\left(\frac{z}{2\sqrt{t}}\right),$$

where the error function erf(x) is defined to be

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(\psi^2)\ d\psi.$$

Equivalently, this is

$$\theta(\eta) = \mathrm{erf}(\eta),$$

where

$$\eta = \frac{z}{2\sqrt{t}},$$

is a similarity variable. Under this similarity transformation, the equation governing the liquid fraction in the mush becomes

$$(2\eta C + \epsilon C')\,\chi' = -\,(2\eta C' + \epsilon C'')\,\chi. \tag{A 1}$$

Furthermore, due to the liquidus relationship, the concentration and its derivatives in the mush are given by

$$C(\eta) = C_B - \frac{1}{\hat{\Gamma}}\mathrm{erf}(\eta),$$

$$C'(\eta) = -\frac{2}{\hat{\Gamma}\sqrt{\pi}}\exp\left(-\eta^2\right),$$

$$C''(\eta) = \frac{4\eta}{\hat{\Gamma}\sqrt{\pi}}\exp\left(-\eta^2\right).$$

We can solve for the concentration C_b at the mush-liquid interface $\eta = \lambda_b$ (see Worster (1986) for details), and as a result we can integrate (A 1) to solve for the liquid fraction throughout the mush,

$$\chi(\eta) = \chi_b \exp\left(\int_\eta^{\lambda_b} \frac{(2\xi C'(\xi) + \epsilon C''(\xi))}{2\xi C(\xi) + \epsilon C'(\xi)}\, d\xi\right).$$

Due to the condition of marginal equilibrium in the special case of uniform thermal properties in all phases, we have that $\chi_b = 1$, so

$$\chi(\eta) = \exp\left(\int_\eta^{\lambda_b} \frac{(2\xi C'(\xi) + \epsilon C''(\xi))}{2\xi C(\xi) + \epsilon C'(\xi)}\, d\xi\right).$$

The free-interface condition for the solid-mush interface (2.8b) transforms to

$$(2C_a\lambda_a + \epsilon C_a')\,\chi_a = 0,$$

where $\eta = \lambda_a$ denotes the interface position in terms of the similarity variable. This requires either $\chi_a = 0$ or $2\lambda_a C_a + \epsilon C_a' = 0$. In the latter case, we will show that the interface is located at a regular singular point of (A 1).

Equation (A 1) admits $\chi = 0$ as a solution everywhere. However, this is inconsistent with $\chi_b = 1$, so we need to focus our attention to the behavior of the solution near the singularity. We can rewrite (A 1) as

$$\chi' + \frac{2\eta C' + \epsilon C''}{2\eta C + \epsilon C'}\chi = 0.$$

If we can show that the function

$$p(\eta) = \frac{2\eta C' + \epsilon C''}{2\eta C + \epsilon C'},$$

has a simple pole at the singularity, that is when $2\eta C + \epsilon C' = 0$, then we can examine the series solution to determine the behavior of the solution near the singularity. We denote the location of the singularity as ξ_s.

Consider the function $f(\eta) = 2\eta C + \epsilon C'$. We have already solved for $C(\eta)$ and $C'(\eta)$, so

$$f(\eta) = 2\eta C_B - \frac{2\eta}{\hat{\Gamma}}\mathrm{erf}(\eta) - \frac{2\epsilon}{\hat{\Gamma}\sqrt{\pi}}\exp(-\eta^2).$$

If we can show that $f'(\xi_s) \neq 0$, then $p(\eta)$ has a simple pole at $\eta = \xi_s$. Now,

$$f'(\eta) = 2C_B - \frac{2}{\hat{\Gamma}}\left[\mathrm{erf}(\eta) + \frac{2(1-\epsilon)\eta}{\sqrt{\pi}}\exp\left(-\eta^2\right)\right],$$

which is zero only if the bracketed expression is equal to $\hat{\Gamma}C_B$. We will now focus our attention on the bracketed expression, which we will now call $g(\eta)$. First, we note that $g(0) = 0$. Then, we see that

$$g'(\eta) = \frac{-2}{\sqrt{\pi}}\exp\left(-\eta^2\right)\left[\epsilon + (1-\epsilon)\eta\right],$$

which is negative for all $\eta \geq 0$. As a result, $g(\eta) \leq 0$ for all $\eta \geq 0$, so $g(\eta) \neq \hat{\Gamma}C_B$ for any $\eta \geq 0$.

From the above, we see that $f'(\eta) \neq 0$ for $\eta \geq 0$, so $f(\xi_s) \neq 0$. Therefore, $p(\eta)$ has a simple pole at $\eta = \xi_s$. There exists a regular series solution to (A 1) near $\eta = \xi_s$ of the form

$$\chi(\eta) = \sum_{n=0}^{\infty} a_n \left(\eta - \xi_s\right)^{n+r},$$

with

$$r = -\lim_{\eta \to \xi_s} (\eta - \xi_s) p(\eta) = \frac{-2(1-\epsilon)\xi_s^2}{2(1-\epsilon)\xi_s^2 - \epsilon}.$$

For the parameters described in the main text of the paper, we have $r \approx 0.011$.

From the above, we see that

$$\lim_{\eta \to \xi_s^+} \chi(\eta) = 0,$$

so we conclude that $\lambda_a = \xi_s$, and that both conditions $\chi_a = 0$ and $2\lambda_a C_a + \epsilon C_a' = 0$ are satisfied at the interface. Furthermore, the series solution above indicates very rapid growth of the liquid fraction near the solid-mush interface.

REFERENCES

WORSTER, M. G. 1986 Solidification of an alloy from a cooled boundary. *J. Fluid Mech.* **167**, 481–501.