Appendix to Modulational instability of Rossby and drift waves and generation of zonal jets

By Colm P. Connaughton, Balasubramanya T. Nadiga , Sergey V. Nazarenko & Brenda E. Quinn

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Appendix A. Details of linear stability analysis

The decay instability is an instability of a primary wave involving a pair of other modes (i.e. the primary wave decays into two secondary waves, see e.g. (Sagdeev & Galeev 1969)). We shall derive this instability from the 3MT, Eqs. (3.1). Introducing the vector notation $\mathbf{\Psi} = (\Psi_{\mathbf{p}}, \Psi_{\mathbf{q}}, \Psi_{\mathbf{p}_-})$, a monochromatic primary wave is given by $\mathbf{\Psi}_0 = (\Psi_0, 0, 0)$ where Ψ_0 is a complex constant representing the amplitude of the initial primary wave. This is an exact solution of Eqs. (3.1). We consider the stability of this solution to small perturbations involving the modes $\mathbf{q}$ and $\mathbf{p}_-$ by taking $\mathbf{\Psi} = \mathbf{\Psi}_0 + \epsilon \mathbf{\Psi}_1$ with the perturbation given by $\mathbf{\Psi}_1 = (0, \widetilde{\psi}_{\mathbf{q}}, \widetilde{\psi}_{\mathbf{p}_-})$. Linearisation yields the following equations at first order in ϵ:

$$\begin{aligned} \partial_t \widetilde{\psi}_{\mathbf{q}} &= T(\mathbf{q}, \mathbf{p}, -\mathbf{p}_-)\, \Psi_0\, \overline{\widetilde{\psi}}_{\mathbf{p}_-}\, \mathrm{e}^{-i\,\Delta_-\,t} \\ \partial_t \overline{\widetilde{\psi}}_{\mathbf{p}_-} &= T(\mathbf{p}_-, \mathbf{p}, -\mathbf{q})\, \overline{\Psi}_0\, \widetilde{\psi}_{\mathbf{q}}\, \mathrm{e}^{i\,\Delta_-\,t}. \end{aligned} \tag{A 1}$$

We now seek solutions in form:

$$\begin{aligned} \widetilde{\psi}_{\mathbf{q}}(t) &= A_{\mathbf{q}} \mathrm{e}^{-i\,\Omega_{\mathbf{q}}\,t} \\ \widetilde{\psi}_{\mathbf{p}_-}(t) &= A_{\mathbf{p}_-} \mathrm{e}^{-i\,\Omega_{\mathbf{p}_-}\,t}. \end{aligned}$$

This requires $\overline{\Omega}_{\mathbf{p}_-} = -\Omega_{\mathbf{q}} + \Delta_-$. Solving Eqs. (A 1) then reduces to finding solutions of the linear system

$$A \begin{pmatrix} A_{\mathbf{q}} \\ \overline{A}_{\mathbf{p}_-} \end{pmatrix} = 0$$

where

$$A = \begin{pmatrix} -i\Omega_{\mathbf{q}} & T(\mathbf{q}, \mathbf{p}, -\mathbf{p}_-)\, \Psi_0 \\ T(\mathbf{p}_-, \mathbf{p}, -\mathbf{q})\, \overline{\Psi}_0 & i(-\Omega_{\mathbf{q}} + \Delta_-). \end{pmatrix} \tag{A 2}$$

To obtain non-trivial solutions, we require $\det A = 0$, which yields the dispersion relation:

$$\Omega_{\mathbf{q}}(-\Omega_{\mathbf{q}} + \Delta_-) - T(\mathbf{q}, \mathbf{p}, -\mathbf{p}_-)\, T(\mathbf{p}_-, \mathbf{p}, -\mathbf{q})\; |\Psi_0|^2 = 0. \tag{A 3}$$

This has two roots, $\Omega^{\pm}_{\mathbf{q}}$ with corresponding eigenvectors:

$$\begin{pmatrix} A_{\mathbf{q}} \\ A_{\mathbf{p}_-} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{T(\mathbf{p}_-, \mathbf{p}, -\mathbf{q})\, \Psi_0}{i\,(\overline{\Omega}_{\mathbf{q}} - \Delta_-)} \end{pmatrix}. \tag{A 4}$$

Instability occurs when $\Omega_{\mathbf{q}}$ has a non-zero imaginary part. For an exactly resonant triad, $\Delta_- = 0$. For resonant triads, using Eq. (2.5) the roots of Eq. (4.1) are

$$\Omega_{\mathbf{q}} = \pm i \frac{|\Psi_0|\, |\mathbf{p} \times \mathbf{q}|}{\sqrt{(q^2 + F)(p_-^2 + F)}} \sqrt{(p^2 - q^2)(p_-^2 - p^2)}. \tag{A 5}$$

In this case, instability occurs if $q < p < p_-$.

Let us now derive the modulational instability in the same way as we have done for the decay instability. The modulational instability is studied using the 4MT. We begin by linearising Eqs. (3.3) about the pure primary wave solution, $\boldsymbol{\Psi}_0 = (\Psi_0, 0, 0, 0)$ where Ψ_0 is a complex constant representing the amplitude of the initial primary wave. We consider the stability of this solution to small perturbations involving the 3 modes $\mathbf{q}$, $\mathbf{p}_-$ and $\mathbf{p}_+$ by taking $\boldsymbol{\Psi} = \boldsymbol{\Psi}_0 + \epsilon\boldsymbol{\Psi}_1$ with the perturbation given by $\boldsymbol{\Psi}_1 = (0, \widetilde{\psi}_{\mathbf{q}}, \widetilde{\psi}_{\mathbf{p}_+}, \widetilde{\psi}_{\mathbf{p}_-})$. Linearisation yields the following equations at first order in ϵ:

$$
\begin{aligned}
\partial_t \widetilde{\psi}_{\mathbf{q}} &= T(\mathbf{q}, \mathbf{p}, -\mathbf{p}_-)\, \Psi_0\, \overline{\widetilde{\psi}}_{\mathbf{p}_-}\, \mathrm{e}^{-i\,\Delta_-\,t} \\
&\qquad + T(\mathbf{q}, -\mathbf{p}, \mathbf{p}_+)\, \overline{\Psi_0}\, \widetilde{\psi}_{\mathbf{p}_+} \mathrm{e}^{i\,\Delta_+\,t} \\
\partial_t \overline{\widetilde{\psi}}_{\mathbf{p}_+} &= T(\mathbf{p}_+, \mathbf{p}, \mathbf{q})\, \Psi_0\, \widetilde{\psi}_{\mathbf{q}}\, \mathrm{e}^{-i\,\Delta_+\,t} \\
\partial_t \overline{\widetilde{\psi}}_{\mathbf{p}_-} &= T(\mathbf{p}_-, \mathbf{p}, -\mathbf{q})\, \overline{\Psi}_0\, \widetilde{\psi}_{\mathbf{q}}\, \mathrm{e}^{i\,\Delta_-\,t}.
\end{aligned}
\tag{A 6}
$$

We again seek solutions of the form:

$$
\begin{aligned}
\widetilde{\psi}_{\mathbf{q}}(t) &= A_{\mathbf{q}} \mathrm{e}^{-i\,\Omega_{\mathbf{q}}\,t} \\
\widetilde{\psi}_{\mathbf{p}_+}(t) &= A_{\mathbf{p}_+} \mathrm{e}^{-i\,\Omega_{\mathbf{p}_+}\,t} \\
\widetilde{\psi}_{\mathbf{p}_-}(t) &= A_{\mathbf{p}_-} \mathrm{e}^{-i\,\Omega_{\mathbf{p}_-}\,t}.
\end{aligned}
$$

This requires requires $\Omega_{\mathbf{p}_+} = \Omega_{\mathbf{q}} + \Delta_+$ and $\overline{\Omega}_{\mathbf{p}_-} = -\Omega_{\mathbf{q}} + \Delta_-$. Solving Eqs. (A 6) then reduces to finding solutions of the linear system

$$
A \begin{pmatrix} A_{\mathbf{q}} \\ A_{\mathbf{p}_+} \\ \overline{A}_{\mathbf{p}_-} \end{pmatrix} = 0
$$

where

$$
A = \begin{pmatrix} i\Omega_{\mathbf{q}} & T(\mathbf{q}, -\mathbf{p}, \mathbf{p}_+)\, \overline{\Psi}_0 & T(\mathbf{q}, \mathbf{p}, -\mathbf{p}_-)\, \Psi_0 \\ T(\mathbf{p}_+, \mathbf{p}, \mathbf{q})\, \Psi_0 & i(\Omega_{\mathbf{q}} + \Delta_+) & 0 \\ T(\mathbf{p}_-, \mathbf{p}, -\mathbf{q})\, \overline{\Psi}_0 & 0 & -i(-\Omega_{\mathbf{q}} + \Delta_-) \end{pmatrix}. \tag{A 7}
$$

Setting $\det A = 0$ yields a cubic dispersion relation:

$$
\begin{aligned}
&\Omega_{\mathbf{q}}(\Omega_{\mathbf{q}} + \Delta_+)(-\Omega_{\mathbf{q}} + \Delta_-) \\
&+ T(\mathbf{q}, -\mathbf{p}, \mathbf{p}_+)\, T(\mathbf{p}_+, \mathbf{p}, \mathbf{q})\, |\Psi_0|^2\, (-\Omega_{\mathbf{q}} + \Delta_-) \\
&- T(\mathbf{q}, \mathbf{p}, -\mathbf{p}_-)\, T(\mathbf{p}_-, \mathbf{p}, -\mathbf{q})\, |\Psi_0|^2\, (\Omega_{\mathbf{q}} + \Delta_+) = 0.
\end{aligned}
\tag{A 8}
$$

The corresponding eigenvectors are given by

$$
\begin{pmatrix} A_{\mathbf{q}} \\ A_{\mathbf{p}_+} \\ A_{\mathbf{p}_-} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{T(\mathbf{p}_+, \mathbf{p}, \mathbf{q})\, \Psi_0}{-i\,(\Omega_{\mathbf{q}} + \Delta_+)} \\ \frac{T(\mathbf{p}_-, \mathbf{p}, -\mathbf{q})\, \Psi_0}{i\,(\overline{\Omega}_{\mathbf{q}} - \Delta_-)} \end{pmatrix}. \tag{A 9}
$$

This derivation holds for any system with a quadratic nonlinearity. Using Eq. (2.5) and performing some algebra we recover the usual form of the dispersion relation specific to the CHM equation (Gill 1974) (see also (Lorentz 1972; Manin & Nazarenko 1994; Smolyakov *et al.* 2000; Onishchenko *et al.* 2004)):

$$
(q^2 + F)\Omega + \beta q_x + |\Psi_0|^2\, |\mathbf{p} \times \mathbf{q}|^2\, (p^2 - q^2) \left[\frac{p_+^2 - p^2}{(p_+^2 + F)(\Omega + \omega) + \beta p_{+x}} - \frac{p_-^2 - p^2}{(p_-^2 + F)(\Omega - \omega) + \beta p_{-x}} \right] = 0. \tag{A 10}
$$

This can be solved numerically, and sometimes analytically, for a given set of parameters to determine Ω.

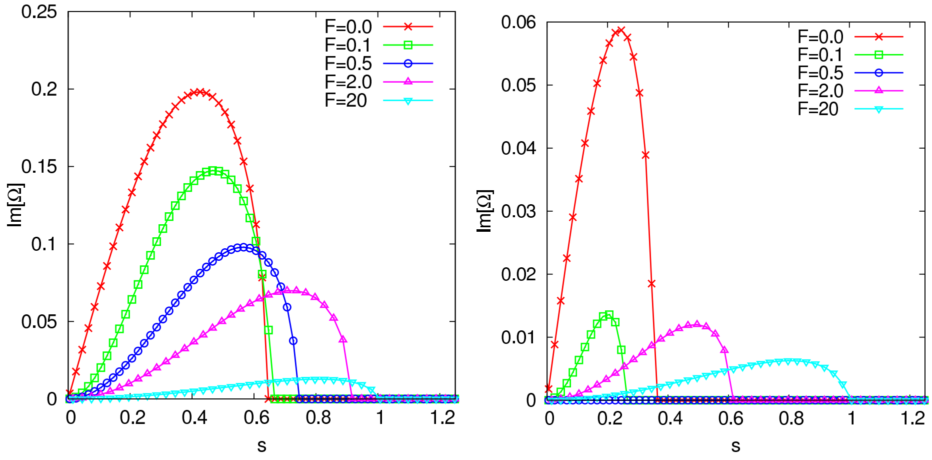

FIGURE 18. Instability growthrate for purely meridional perturbations with $M = 0.5 > \sqrt{2/27}$ for different values of the deformation radius.

Appendix B. Effects of Finite Deformation Radius

We now consider the dependence of MI on the deformation or Larmor radius, noting that a finite deformation radius is obtained in the QG system under a reduced gravity approximation. When F is finite, there are two regimes, depending on the value of M. For an interval of instability to exist, we require $s^2_{\max} > 0$. Referring to Eq. (7.4), this requires that

$$p(F) = 2M^2(1+F)^3 - F > 0. \tag{B 1}$$

The discriminant of the corresponding cubic, $p(F) = 0$, is $\Delta = -4(-2M^2 + 27M^4)$. Since we are only interested in $F > 0$ we can identify two regimes.

- Regime 1: $M > \sqrt{\frac{2}{27}}$

In this case, $\Delta < 0$ so $p(F) = 0$ has one real root, F_1 (which is negative) and $p(F) > 0$ when $F > F_1$. Then for any positive value of F there exists a finite range of s, $s \in (0, s_{\max})$, for which the perturbation is unstable. $s_{\max}$ is given by Eq. (7.4). In this regime, finite deformation radius tends to reduce the growth rate of the instability but cannot suppress it. See Fig. 18.

- Regime 2: $M \leq \sqrt{\frac{2}{27}}$

In this case, $\Delta > 0$ and $p(F) = 0$ has three real roots, F_1, F_2 and F_3. F_1 is always negative and F_2 and F_3 are always positive. $p(F) < 0$ in the range (F_2, F_3). In this regime, there are critical values of F, F_2 and F_3 such that the range $s \in (0, s_{\max})$ of unstable pertubations only exists if $F < F_2$ or $F > F_3$. F_2 and F_3 are obtained by finding the positive roots of Eq. (B 1) and $s_{\max}$ is again given by Eq. (7.4). In this regime, there is a range of intermediate deformation radii which completely suppresses the instability. See Fig. 18.

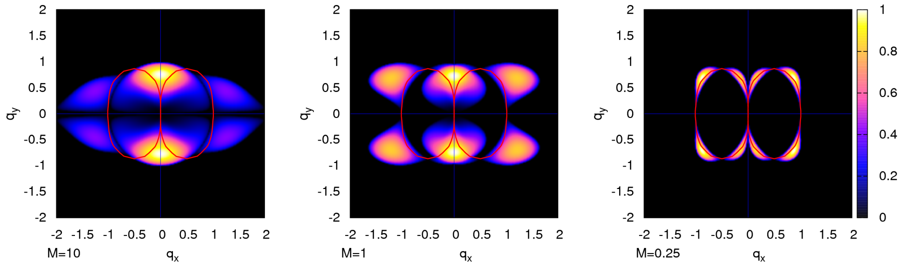

FIGURE 19. Same as in Fig. 2 but now for a finite deformation radius, $F = 2$.

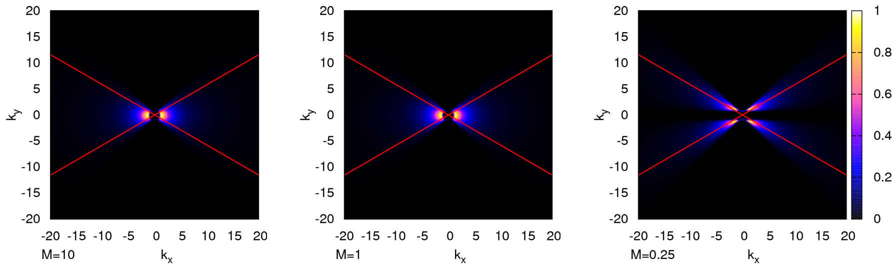

FIGURE 20. Same as in Fig. 6 but now for a finite deformation radius, $F = 1$.

Appendix C. Case of Meridional primary Wave and Off-Zonal Modulation.

Above we considered the case when the primary wave is purely meridional and the modulation is purely zonal. This geometry is important considering that both the baroclinic instability in GFD and the drift-wave instabilities in plasmas typically have most unstable modes being in the meridional direction. These modes can be considered as an initial condition for the secondary modulational instabilities as it is done in the present paper. At the same time, we have established above that the most unstable modulations for $M > 0.53$ are zonal.

On the other hand, for low M the most unstable modulations are off-zonal. This, in our opinion, is the reason why the final statistical state in the system in the $M = 0.1$ simulation showed the presence of off-zonal anisotropic flows even though the initial modulation was purely zonal. Moreover, it is quite likely that in such weakly-nonlinear cases the system will pick the modulation which is off-zonal already at the initial moments.

Thus, here we will consider a case with $M = 0.1$ where we start with purely meridional primary wave, $\mathbf{p} = (10, 0)$ and with the modulation wavevector corresponding to the fastest growing mode in this case, namely $\mathbf{q} = (9, 6)$. Corresponding numerical results for this case are shown in Fig. (21) (vorticity snapshots) and Fig. (4) (evolution of the q-mode amplitude $|\psi_q|$ and respective results obtained from simulating the 4MT and 3MT models).

First of all, as in all previous cases, we see good agreement of the initial evolution with predictions for the linear instability obtained based on the 4MT and the 3MT models. Moreover, we see that the 4MT and the 3MT in this case qualitatively describe the nonlinear behavior too. Namely, like in the four-mode system, we see oscillatory behavior, even though the oscillations appear to be irregular. However, these irregular oscillations are clearly non-turbulent, as one can see from the vorticity frames in Fig. (21) which shows quite a regular pattern even at $t = 100$ (in units of the inverse instability

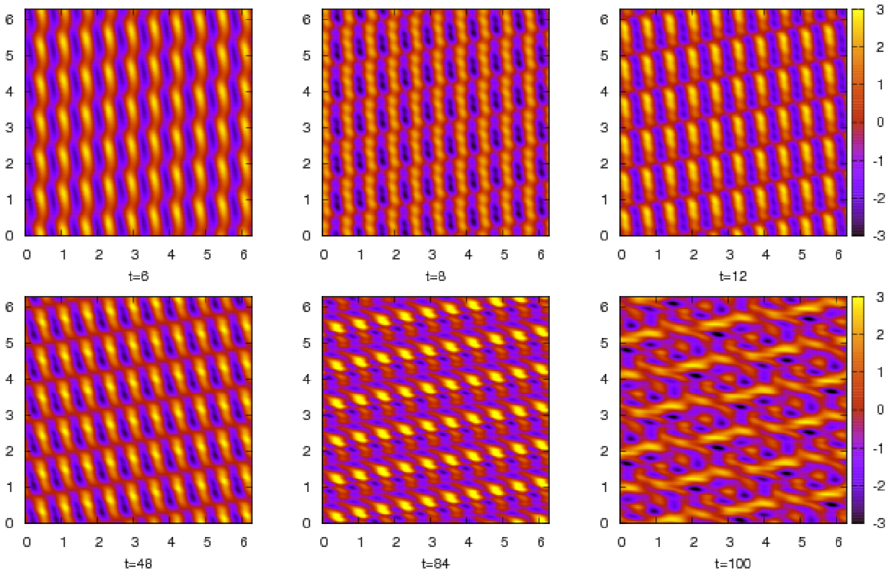

FIGURE 21. Vorticity snapshots showing the growth, saturation and transition to turbulence of an off-zonal perturbation of a meridional primary wave having $M = 0.1$.

growthrate), by which time the respective $M = 0.1$ system with zonal $\mathbf{q}$ is completely turbulent, see Fig. 10. A transition to turbulence does eventually occur after a very long time, and the turbulent state does exhibit off-zonal striations similar to the respective $M = 0.1$ system with zonal initial modulations $\mathbf{q}$.