

## Appendix to “Finite-amplitude river dunes”

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*Journal of Fluid Mechanics*, vol. 611 (2008), pp. 283–306

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### Appendix B

In this Appendix the inhomogeneous terms of the differential systems at  $O(\epsilon^2)$  and  $O(\epsilon^3)$  are specified.

The vector  $\mathbf{P}_{22}$  as it appears in (5.6) at the order  $O(\epsilon^2)$  can be expressed as:

$$(P_{22})_1 = -\frac{1}{2}(U_{11,\eta} + ikV_{11})U_{11,\eta}/U_{0,\eta} + \frac{1}{2}ikY_{11}V_{11,\eta} - 2k^2Y_{11}U_{11} - \frac{1}{2}ikD_{11}V_{11} + k^2U_{0,\eta}Y_{11}^2 \quad (1)$$

$$(P_{22})_2 = ikY_{11}U_{11,\eta} + D_{11}V_{11,\eta} - ikU_{0,\eta}Y_{11}D_{11} \quad (2)$$

$$(P_{22})_3 = D_{11}T_{t11,\eta} + ikY_{11}T_{n11,\eta} + (U_0^\Omega Y_{11} + V_{11} - 8k^2\nu_{t0}Y_{11})U_{11,\eta} + 8ik\nu_{t0}(U_{11,\eta}/U_{0,\eta} + D_{11})V_{11,\eta} + ik(U_{11} - U_{0,\eta}Y_{11}) + 4k^2\nu_{t0}D_{11})U_{11} - U_{0,\eta}D_{11}V_{11} + (D_{11} - U_0^\Omega U_{0,\eta}Y_{11} + 6k^2(1-\eta)Y_{11})D_{11} + 4k^2Y_{11}T_{t11} \quad (3)$$

$$(P_{22})_4 = -ikY_{11}T_{t11,\eta} + D_{11}T_{n11,\eta} + (V_{11} + U_0^\Omega Y_{11})V_{11,\eta} - 2ikD_{11}T_{t11} + (8ik^3\nu_{t0}Y_{11} + ikV_{11})U_{11} + (2ik(1-\eta)D_{11} - ikY_{11})D_{11} - 4ik^3(1-\eta)Y_{11}^2 \quad (4)$$

having defined the quantities  $Y_{11} = R_{11} + \eta D_{11}$  and  $U_0^\Omega = -ik(U_0 - \Omega)$ .

The vector  $\mathbf{P}_{20}$  as it appears in (5.14) at the order  $O(\epsilon^2)$  can be expressed as:

$$(P_{20})_1 = \left\{ -\frac{1}{2}(U_{11,\eta} + ikV_{11})U_{11,\eta}^*/U_{0,\eta} + \frac{1}{2}ikY_{11}V_{11,\eta}^* + 2k^2Y_{11}U_{11}^* + \frac{1}{2}ikD_{11}V_{11}^* - k^2U_{0,\eta}Y_{11}Y_{11}^* + c.c. \right\} \quad (5)$$

$$(P_{20})_2 = \{ ikY_{11}U_{11,\eta}^* + D_{11}V_{11,\eta}^* - ikU_{0,\eta}Y_{11}D_{11}^* + c.c. \} \quad (6)$$

$$(P_{20})_3 = \{ D_{11}T_{t11,\eta}^* + ikY_{11}T_{n11,\eta}^* + (U_0^\Omega Y_{11} + V_{11})U_{11,\eta}^* + (ikU_{11} - ikU_{0,\eta}Y_{11} - 4k^2\nu_{t0}D_{11})U_{11}^* - U_{0,\eta}D_{11}V_{11}^* + (D_{11} - U_0^\Omega U_{0,\eta}Y_{11} + 2k^2(1-\eta)Y_{11})D_{11}^* + c.c. \} \quad (7)$$

$$(P_{20})_4 = \{-ikY_{11}T_{t11,\eta}^* + D_{11}T_{n11,\eta}^* + (V_{11} + U_0^\Omega Y_{11})V_{11,\eta}^* - 2ikD_{11}T_{t11}^* - ikU_{11}V_{11}^* + (2ik(1-\eta)D_{11} - ikY_{11})D_{11}^* - 4ik^3(1-\eta)Y_{11}Y_{11}^* + c.c.\} \quad (8)$$

where a star superscript denotes the complex conjugate.

The three vectors  $\mathbf{P}_{31}^{(1,2,3)}$  that appear in the inhomogeneous term of the differential system (5.18) at the order  $O(\epsilon^3)$  are:

$$\left(P_{31}^{(1)}\right)_1 = \left(P_{31}^{(1)}\right)_2 = 0 \quad (9)$$

$$\left(P_{31}^{(1)}\right)_3 = U_{11} - U_{0,\eta}Y_{11} \quad (10)$$

$$\left(P_{31}^{(1)}\right)_4 = V_{11} \quad (11)$$

$$\left(P_{31}^{(2)}\right)_1 = -ikU_{11} + ikU_{0,\eta}Y_{11} \quad (12)$$

$$\left(P_{31}^{(2)}\right)_2 = -\frac{1}{2}ikV_{11} \quad (13)$$

$$\left(P_{31}^{(2)}\right)_3 = -ikT_{n11} - U_0^\Omega U_{11} + U_0^\Omega U_{0,\eta}Y_{11} + 8k^2\nu_{t0}U_{11} \quad (14)$$

$$\left(P_{31}^{(2)}\right)_4 = -ikT_{t11} - U_0^\Omega V_{11} + ikY_{11} - 2ik(1-\eta)D_{11} \quad (15)$$

$$\left(P_{31}^{(3)}\right)_1 = 4k^2U_{11}^*Y_{22} - 4k^2U_{0,\eta}Y_{11}^*Y_{22} + 4k^2Y_{11}^*U_{22} - \frac{U_{11,\eta}^*U_{22,\eta}}{U_{0,\eta}} - ik\frac{U_{11,\eta}^*V_{22}}{U_{0,\eta}} \quad (16)$$

$$-ikD_{11}^*V_{22} + \frac{ik}{2}V_{11}^*D_{22} + \frac{ik}{2}\frac{V_{11}^*U_{22,\eta}}{U_{0,\eta}} - \frac{ik}{2}Y_{11}^*V_{22,\eta} + ikV_{11,\eta}^*Y_{22}$$

$$- \frac{U_{11,\eta}U_{20,\eta}}{U_{0,\eta}} - \frac{ik}{2}\frac{V_{11}U_{20,\eta}}{U_{0,\eta}} + \frac{ik}{2}Y_{11}V_{20,\eta}$$

$$- \frac{k^2}{\nu_{t0}}T_{t11}^*Y_{11}^2 + 2\frac{k^2}{\nu_{t0}}T_{t11}Y_{11}^*Y_{11} + 4k^2U_{0,\eta}D_{11}Y_{11}^*Y_{11} - 2k^2U_{0,\eta}D_{11}^*Y_{11}^2 + 2k^2U_{11,\eta}^*Y_{11}^2$$

$$- 4k^2U_{11,\eta}Y_{11}^*Y_{11} - 2ikV_{11,\eta}D_{11}^*Y_{11} - 2ikV_{11,\eta}^*D_{11}Y_{11} + 2ikV_{11,\eta}D_{11}Y_{11}^*$$

$$- \frac{3ik}{2}\frac{U_{11,\eta}^*V_{11,\eta}}{U_{0,\eta}}Y_{11} - \frac{3ik}{2}\frac{U_{11,\eta}V_{11,\eta}^*}{U_{0,\eta}}Y_{11} + \frac{3ik}{2}\frac{U_{11,\eta}V_{11,\eta}}{U_{0,\eta}}Y_{11}^*$$

$$- \frac{ik}{2}\frac{D_{11}^*U_{11,\eta}V_{11}}{U_{0,\eta}} - \frac{ik}{2}\frac{D_{11}U_{11,\eta}^*V_{11}}{U_{0,\eta}} + \frac{ik}{2}\frac{D_{11}U_{11,\eta}V_{11}^*}{U_{0,\eta}}$$

$$\left(P_{31}^{(3)}\right)_2 = -ikY_{11}^*U_{22,\eta} + 2ikU_{11,\eta}^*Y_{22} - 2ikU_{0,\eta}D_{11}^*Y_{22} + ikU_{0,\eta}Y_{11}^*D_{22} \quad (17)$$

$$+ D_{11}^*V_{22,\eta} + V_{11,\eta}^*D_{22} + ikY_{11}U_{20,\eta} + D_{11}V_{20,\eta}$$

$$- ikD_{11}^*Y_{11}U_{11,\eta} + ikD_{11}Y_{11}^*U_{11,\eta} - ikD_{11}Y_{11}U_{11,\eta}^* + 2ikU_{0,\eta}D_{11}^*D_{11}Y_{11}$$

$$- ikU_{0,\eta}D_{11}^2Y_{11}^* - 2D_{11}D_{11}^*V_{11,\eta} - D_{11}^2V_{11,\eta}^*$$

$$\begin{aligned}
\left(P_{31}^{(3)}\right)_3 &= -4k^2 T_{t11}^* Y_{22} + 2k^2 Y_{11}^* T_{t22} + 4ik(T_{n11}^* + P_{11}^*) D_{22} + 2ik T_{n11,\eta}^* Y_{22} \\
&+ 4k^2(1-\eta) D_{11}^* Y_{22} + 4k^2(1-\eta) Y_{11}^* D_{22} - ik Y_{11}^* T_{n22,\eta} - 2ik D_{11}^* (T_{n22} + P_{22}) \\
&+ ik U_{11}^* U_{22} - U_0^\Omega Y_{11}^* U_{22,\eta} + 2U_0^\Omega U_{11,\eta}^* Y_{22} + U_0^\Omega U_{0,\eta} Y_{11}^* D_{22} - 2ik U_{0,\eta} U_{11}^* Y_{22} \\
&+ ik U_{0,\eta} Y_{11}^* U_{22} - U_{0,\eta} V_{11}^* D_{22} + U_{11,\eta}^* V_{22} - U_{0,\eta} D_{11}^* V_{22} + D_{11}^* T_{t22,\eta} \\
&+ T_{t11,\eta}^* D_{22} + 2D_{11}^* D_{22} - 2U_0^\Omega U_{0,\eta} D_{11}^* Y_{22} + V_{11}^* U_{22,\eta} \\
&- 2k^2 Y_{11} T_{t20} + 2ik(T_{n20} + P_{20}) D_{11} + ik Y_{11} T_{n20,\eta} + ik U_{11} U_{20} + U_0^\Omega Y_{11} U_{20,\eta} \\
&+ ik U_{0,\eta} Y_{11} U_{20} + V_{11} U_{20,\eta} + U_{11,\eta} V_{20} - U_{0,\eta} D_{11} V_{20} + D_{11} T_{t20,\eta} \\
&+ 6k^4(1-\eta) Y_{11}^* Y_{11}^2 - 4ik^3 Y_{11}^* Y_{11} (T_{n11} + P_{11}) + 2ik^3 (T_{n11}^* + P_{11}^*) Y_{11}^2 \\
&- 2k^2 T_{t11}^* Y_{11} D_{11} + 2k^2 T_{t11} Y_{11}^* D_{11} + 2k^2 T_{t11} Y_{11} D_{11}^* - U_0^\Omega Y_{11} D_{11}^* U_{11,\eta} \\
&+ U_0^\Omega Y_{11}^* D_{11} U_{11,\eta} - U_0^\Omega Y_{11} D_{11} U_{11,\eta}^* + 2U_0^\Omega U_{0,\eta} Y_{11} D_{11} D_{11}^* - U_0^\Omega U_{0,\eta} Y_{11}^* D_{11}^2 \\
&- ik U_{11}^* U_{11,\eta} Y_{11} - ik U_{11} U_{11,\eta}^* Y_{11} - ik U_{11} U_{11,\eta} Y_{11}^* \\
&+ ik U_{11}^* D_{11} Y_{11} + ik U_{11} D_{11}^* Y_{11} - ik U_{11} D_{11} Y_{11}^* - D_{11}^* V_{11} U_{11,\eta} - D_{11} V_{11}^* U_{11,\eta} \\
&- D_{11} V_{11} U_{11,\eta}^* + U_{0,\eta} V_{11}^* D_{11}^2 + 2U_{0,\eta} V_{11} D_{11} D_{11}^* - 2k^2(1-\eta) D_{11}^2 Y_{11}^* \\
&- ik D_{11}^* T_{n11,\eta} Y_{11} - ik D_{11} T_{n11,\eta}^* Y_{11} + ik D_{11} T_{n11,\eta} Y_{11}^* - 2ik D_{11}^2 (T_{n11}^* + P_{11}^*) \\
&- 2D_{11}^* D_{11} T_{t11,\eta} - D_{11}^2 T_{t11,\eta}^* - 3D_{11}^* D_{11}^2 + 2ik(P_{31}^{(3)})_5 \\
\left(P_{31}^{(3)}\right)_4 &= -4k^2 (T_{n11}^* + P_{11}^*) Y_{22} + 8ik^3(1-\eta) Y_{11}^* Y_{22} + 2k^2 Y_{11}^* (T_{n22} + P_{22}) \\
&- 2ik T_{t11,\eta}^* Y_{22} - 4ik T_{t11}^* D_{22} + ik Y_{11}^* T_{t22,\eta} + 2ik D_{11}^* T_{t22} \\
&+ 2ik U_{11}^* V_{22} - ik V_{11}^* U_{22} - U_0^\Omega Y_{11}^* V_{22,\eta} + 2U_0^\Omega V_{11,\eta}^* Y_{22} + V_{11}^* V_{22,\eta} + V_{11,\eta}^* V_{22} \\
&+ D_{11}^* T_{n22,\eta} - 2ik D_{11}^* Y_{22} + 2ik(1-\eta) D_{11}^* D_{22} + T_{n11,\eta}^* D_{22} + ik Y_{11}^* D_{22} \\
&- 2k^2 Y_{11} (T_{n20} + P_{20}) - ik Y_{11} T_{t20,\eta} - 2ik D_{11} T_{t20} + ik V_{11} U_{20} + U_0^\Omega Y_{11} V_{20,\eta} \\
&+ V_{11} V_{20,\eta} + V_{11,\eta} V_{20} + D_{11} T_{n20,\eta} \\
&+ 4ik^3 T_{t11} Y_{11} Y_{11}^* - 2ik^3 T_{t11}^* Y_{11}^2 + 4ik^3(1-\eta) D_{11} Y_{11} Y_{11}^* + 2ik^3(1-\eta) D_{11}^* Y_{11}^2 \\
&- 2k^2 (T_{n11}^* + P_{11}^*) D_{11} Y_{11} + 2k^2 (T_{n11} + P_{11}) D_{11}^* Y_{11} + 2k^2 (T_{n11} + P_{11}) D_{11} Y_{11}^* \\
&- U_0^\Omega Y_{11} D_{11}^* V_{11,\eta} + U_0^\Omega Y_{11}^* D_{11} V_{11,\eta} - U_0^\Omega Y_{11} D_{11} V_{11,\eta}^* - ik U_{11}^* Y_{11} V_{11,\eta} \\
&+ ik U_{11} Y_{11}^* V_{11,\eta} - ik U_{11} Y_{11} V_{11,\eta}^* - D_{11}^* V_{11} V_{11,\eta} - D_{11} V_{11}^* V_{11,\eta} - D_{11} V_{11} V_{11,\eta}^* \\
&+ 2ik D_{11}^* D_{11} Y_{11} - ik D_{11}^2 Y_{11}^* + ik T_{t11,\eta}^* D_{11} Y_{11} + ik T_{t11,\eta} D_{11}^* Y_{11} - ik T_{t11,\eta} D_{11} Y_{11}^* \\
&+ 2ik T_{t11}^* D_{11}^2 - 2T_{n11,\eta} D_{11}^* D_{11} - T_{n11,\eta}^* D_{11}^2 - 2ik(1-\eta) D_{11}^2 D_{11}^* \\
\left(P_{31}^{(3)}\right)_5 &= -4ik T_{t11}^* Y_{22} + 2ik Y_{11}^* T_{t22} - 2ik \nu_{t0} Y_{11}^* U_{22,\eta} + 4ik \nu_{t0} U_{11,\eta}^* Y_{22}
\end{aligned}
\tag{18}$$

$$\begin{aligned}
\left(P_{31}^{(3)}\right)_4 &= -4k^2 (T_{n11}^* + P_{11}^*) Y_{22} + 8ik^3(1-\eta) Y_{11}^* Y_{22} + 2k^2 Y_{11}^* (T_{n22} + P_{22}) \\
&- 2ik T_{t11,\eta}^* Y_{22} - 4ik T_{t11}^* D_{22} + ik Y_{11}^* T_{t22,\eta} + 2ik D_{11}^* T_{t22} \\
&+ 2ik U_{11}^* V_{22} - ik V_{11}^* U_{22} - U_0^\Omega Y_{11}^* V_{22,\eta} + 2U_0^\Omega V_{11,\eta}^* Y_{22} + V_{11}^* V_{22,\eta} + V_{11,\eta}^* V_{22} \\
&+ D_{11}^* T_{n22,\eta} - 2ik D_{11}^* Y_{22} + 2ik(1-\eta) D_{11}^* D_{22} + T_{n11,\eta}^* D_{22} + ik Y_{11}^* D_{22} \\
&- 2k^2 Y_{11} (T_{n20} + P_{20}) - ik Y_{11} T_{t20,\eta} - 2ik D_{11} T_{t20} + ik V_{11} U_{20} + U_0^\Omega Y_{11} V_{20,\eta} \\
&+ V_{11} V_{20,\eta} + V_{11,\eta} V_{20} + D_{11} T_{n20,\eta} \\
&+ 4ik^3 T_{t11} Y_{11} Y_{11}^* - 2ik^3 T_{t11}^* Y_{11}^2 + 4ik^3(1-\eta) D_{11} Y_{11} Y_{11}^* + 2ik^3(1-\eta) D_{11}^* Y_{11}^2 \\
&- 2k^2 (T_{n11}^* + P_{11}^*) D_{11} Y_{11} + 2k^2 (T_{n11} + P_{11}) D_{11}^* Y_{11} + 2k^2 (T_{n11} + P_{11}) D_{11} Y_{11}^* \\
&- U_0^\Omega Y_{11} D_{11}^* V_{11,\eta} + U_0^\Omega Y_{11}^* D_{11} V_{11,\eta} - U_0^\Omega Y_{11} D_{11} V_{11,\eta}^* - ik U_{11}^* Y_{11} V_{11,\eta} \\
&+ ik U_{11} Y_{11}^* V_{11,\eta} - ik U_{11} Y_{11} V_{11,\eta}^* - D_{11}^* V_{11} V_{11,\eta} - D_{11} V_{11}^* V_{11,\eta} - D_{11} V_{11} V_{11,\eta}^* \\
&+ 2ik D_{11}^* D_{11} Y_{11} - ik D_{11}^2 Y_{11}^* + ik T_{t11,\eta}^* D_{11} Y_{11} + ik T_{t11,\eta} D_{11}^* Y_{11} - ik T_{t11,\eta} D_{11} Y_{11}^* \\
&+ 2ik T_{t11}^* D_{11}^2 - 2T_{n11,\eta} D_{11}^* D_{11} - T_{n11,\eta}^* D_{11}^2 - 2ik(1-\eta) D_{11}^2 D_{11}^* \\
\left(P_{31}^{(3)}\right)_5 &= -4ik T_{t11}^* Y_{22} + 2ik Y_{11}^* T_{t22} - 2ik \nu_{t0} Y_{11}^* U_{22,\eta} + 4ik \nu_{t0} U_{11,\eta}^* Y_{22}
\end{aligned}
\tag{19}$$

$$\begin{aligned}
\left(P_{31}^{(3)}\right)_5 &= -4ik T_{t11}^* Y_{22} + 2ik Y_{11}^* T_{t22} - 2ik \nu_{t0} Y_{11}^* U_{22,\eta} + 4ik \nu_{t0} U_{11,\eta}^* Y_{22}
\end{aligned}
\tag{20}$$

$$\begin{aligned}
& -4ik(1-\eta)D_{11}^*Y_{22} + 2ik(1-\eta)Y_{11}^*D_{22} + 2\nu_{t0}D_{11}^*V_{22,\eta} + 2\nu_{t0}V_{11,\eta}^*D_{22} \\
& + \frac{2\nu_{t0}U_{11,\eta}^*V_{22,\eta}}{U_{0,\eta}} + \frac{2\nu_{t0}V_{11,\eta}^*U_{22,\eta}}{U_{0,\eta}} \\
& -2ikY_{11}T_{t20} + 2ik\nu_{t0}Y_{11}U_{20,\eta} + 2\nu_{t0}D_{11}V_{20,\eta} \\
& + \frac{2\nu_{t0}U_{20,\eta}V_{11,\eta}}{U_{0,\eta}} + \frac{2\nu_{t0}U_{11,\eta}V_{20,\eta}}{U_{0,\eta}} \\
& + 6ik^3(1-\eta)Y_{11}^*Y_{11}^2 - 2k^2(T_{n11}^* + P_{11}^*)Y_{11}^2 + 4k^2(T_{n11} + P_{11})Y_{11}^*Y_{11} - 2ik\nu_{t0}D_{11}^*Y_{11}U_{11,\eta} \\
& + 2ik\nu_{t0}D_{11}Y_{11}^*U_{11,\eta} - 2ik\nu_{t0}D_{11}Y_{11}U_{11,\eta}^* + 4ik(1-\eta)D_{11}D_{11}^*Y_{11} \\
& - 2ik(1-\eta)D_{11}^2Y_{11}^* - 4\nu_{t0}D_{11}D_{11}^*V_{11,\eta} - 2\nu_{t0}D_{11}^2V_{11,\eta}^*
\end{aligned}$$

where:

$$Y_{22} = R_{22} + \eta D_{22} \quad (21)$$

$$T_{n11} + P_{11} = -2ik\nu_{t0}U_{11} \quad (22)$$

$$\begin{aligned}
T_{n22} + P_{22} &= -4ik\nu_{t0}U_{22} - 2ikT_{t11}Y_{11} + 2\nu_{t0}D_{11}V_{11,\eta} \\
&+ \frac{2\nu_{t0}U_{11,\eta}V_{11,\eta}}{U_{0,\eta}} + 2ik\nu_{t0}U_{11,\eta}Y_{11}
\end{aligned} \quad (23)$$

The vector  $\mathbf{P}_{33}$  associated to the differential system (5.19) at the order  $O(\epsilon^3)$  is given by:

$$(P_{33})_1 = 4k^2U_{11}Y_{22} - 4k^2U_{0,\eta}Y_{11}Y_{22} + 4k^2Y_{11}U_{22} - \frac{U_{11,\eta}U_{22,\eta}}{U_{0,\eta}} - ikD_{11}V_{22} - ik \quad (24)$$

$$\begin{aligned}
& + \frac{U_{11,\eta}V_{22}}{U_{0,\eta}} + \frac{ik}{2}V_{11}D_{22} + \frac{ik}{2}\frac{V_{11}U_{22,\eta}}{U_{0,\eta}} - \frac{ik}{2}Y_{11}V_{22,\eta} + ikV_{11,\eta}Y_{22} \\
& + \frac{k^2}{\nu_{t0}}T_{t11}Y_{11}^2 + 2k^2U_{0,\eta}D_{11}Y_{11}^2 - 2k^2U_{11,\eta}Y_{11}^2 - 2ikV_{11,\eta}D_{11}Y_{11} \\
& - \frac{3ik}{2}\frac{U_{11,\eta}V_{11,\eta}}{U_{0,\eta}}Y_{11} - \frac{ik}{2}\frac{D_{11}U_{11,\eta}V_{11}}{U_{0,\eta}}
\end{aligned}$$

$$(P_{33})_2 = -ikY_{11}U_{22,\eta} + 2ikU_{11,\eta}Y_{22} - 2ikU_{0,\eta}D_{11}Y_{22} + ikU_{0,\eta}Y_{11}D_{22} \quad (25)$$

$$+ D_{11}V_{22,\eta} + V_{11,\eta}D_{22} - ikD_{11}Y_{11}U_{11,\eta} + ikU_{0,\eta}D_{11}^2Y_{11} - 3D_{11}^2V_{11,\eta}$$

$$(P_{33})_3 = -4k^2T_{t11}Y_{22} + 2k^2Y_{11}T_{t22} + 4ik(T_{n11} + P_{11})D_{22} + 2ikT_{n11,\eta}Y_{22} \quad (26)$$

$$\begin{aligned}
& + 4k^2(1-\eta)D_{11}Y_{22} + 4k^2(1-\eta)Y_{11}D_{22} - ikY_{11}T_{n22,\eta} - 2ikD_{11}(T_{n22} + P_{22}) \\
& + ikU_{11}U_{22} - U_0^\Omega Y_{11}U_{22,\eta} + 2U_0^\Omega U_{11,\eta}Y_{22} + U_0^\Omega U_{0,\eta}Y_{11}D_{22} - 2ikU_{0,\eta}U_{11}Y_{22} \\
& + ikU_{0,\eta}Y_{11}U_{22} - U_{0,\eta}V_{11}D_{22} + U_{11,\eta}V_{22} - U_{0,\eta}D_{11}V_{22} + D_{11}T_{t22,\eta} \\
& + T_{t11,\eta}D_{22} + 2D_{11}D_{22} - 2U_0^\Omega U_{0,\eta}D_{11}Y_{22} + V_{11}U_{22,\eta} \\
& + 6k^4(1-\eta)Y_{11}^3 - 2ik^3Y_{11}^2(T_{n11} + P_{11}) + 2k^2T_{t11}Y_{11}D_{11} - U_0^\Omega Y_{11}D_{11}U_{11,\eta}
\end{aligned}$$

$$\begin{aligned}
& +U_0^\Omega U_{0,\eta} Y_{11} D_{11}^2 - 3ikU_{11}U_{11,\eta}Y_{11} + ikU_{11}D_{11}Y_{11} - 3D_{11}V_{11}U_{11,\eta} \\
& +3U_{0,\eta}V_{11}D_{11}^2 - 2k^2(1-\eta)D_{11}^2Y_{11} - ikD_{11}T_{n11,\eta}Y_{11} - 2ikD_{11}^2(T_{n11} + P_{11}) \\
& -3D_{11}^2T_{t11,\eta} - 3D_{11}^3 + 2ik(P_{33})_5 \\
(P_{33})_4 = & -4k^2(T_{n11} + P_{11})Y_{22} + 8ik^3(1-\eta)Y_{11}Y_{22} + 2k^2Y_{11}(T_{n22} + P_{22}) \tag{27}
\end{aligned}$$

$$\begin{aligned}
& -2ikT_{t11,\eta}Y_{22} - 4ikT_{t11}D_{22} + ikY_{11}T_{t22,\eta} + 2ikD_{11}T_{t22} + 2ikU_{11}V_{22} \\
& -ikV_{11}U_{22} - U_0^\Omega Y_{11}V_{22,\eta} + 2U_0^\Omega V_{11,\eta}Y_{22} + V_{11}V_{22,\eta} + V_{11,\eta}V_{22} + D_{11}T_{n22,\eta} \\
& -2ikD_{11}Y_{22} + 2ik(1-\eta)D_{11}D_{22} + T_{n11,\eta}D_{22} + ikY_{11}D_{22} \\
& +2ik^3T_{t11}Y_{11}^2 + 6ik^3(1-\eta)D_{11}Y_{11}^2 + 2k^2(T_{n11} + P_{11})D_{11}Y_{11} - U_0^\Omega Y_{11}D_{11}V_{11,\eta} \\
& -ikU_{11}Y_{11}V_{11,\eta} - 3D_{11}V_{11}V_{11,\eta} + ikD_{11}^2Y_{11} + ikT_{t11,\eta}D_{11}Y_{11} + 2ikT_{t11}D_{11}^2 \\
& -3T_{n11,\eta}D_{11}^2 - 2ik(1-\eta)D_{11}^3
\end{aligned}$$

$$\begin{aligned}
(P_{33})_5 = & -4ikT_{t11}Y_{22} + 2ikY_{11}T_{t22} - 2ik\nu_{t0}Y_{11}U_{22,\eta} + 4ik\nu_{t0}U_{11,\eta}Y_{22} \tag{28} \\
& -4ik(1-\eta)D_{11}Y_{22} + 2ik(1-\eta)Y_{11}D_{22} + 2\nu_{t0}D_{11}V_{22,\eta} + 2\nu_{t0}V_{11,\eta}D_{22} \\
& + \frac{2\nu_{t0}U_{11,\eta}V_{22,\eta}}{U_{0,\eta}} + \frac{2\nu_{t0}V_{11,\eta}U_{22,\eta}}{U_{0,\eta}} \\
& +6ik^3(1-\eta)Y_{11}^3 + 2k^2(T_{n11} + P_{11})Y_{11}^2 - 2ik\nu_{t0}D_{11}Y_{11}U_{11,\eta} + 2ik(1-\eta)D_{11}^2Y_{11} \\
& -6\nu_{t0}D_{11}^2V_{11,\eta}
\end{aligned}$$

### Appendix C

In this Appendix the array  $\mathbf{U}_{20}$ , which appears in (5.15) and (5.16), and the inhomogeneous terms of the algebraic systems (5.9), (5.15), (5.16), (5.21) and (5.23) are specified.

$$\mathbf{U}_{20} = \begin{pmatrix} 1 & 0 & S_0^{-1} \\ 0 & 1 & 0 \\ \overline{U}_{20}^{(1)} & 0 & S_0^{-1}\overline{U}_{20}^{(I)} \end{pmatrix} \tag{29}$$

Vector  $\mathbf{U}_{22}^{(P)}$ :

$$\left(\mathbf{U}_{22}^{(P)}\right)_1 = \left[V_{22}^{(P)} - ikU_{11}Y_{11}\right]_1 \tag{30}$$

$$\left(\mathbf{U}_{22}^{(P)}\right)_2 = \left[T_{t22}^{(P)}\right]_1 \tag{31}$$

$$\left(\mathbf{U}_{22}^{(P)}\right)_3 = \left[T_{n22}^{(P)}\right]_1 \tag{32}$$

$$\left(\mathbf{U}_{22}^{(P)}\right)_4 = \left[T_{t22}^{(P)}\right]_{\eta_b} + \frac{1}{4} \frac{(\theta_{b11} - \theta_{c11})^2}{\theta_{r0}(\theta_{b0} - \theta_{c0})} \tag{33}$$

Vector  $\mathbf{U}_{20}^{(P)}$ :

$$\mathbf{U}_{20}^{(P)} = \left( \left[ T_{t20}^{(P)} \right]_1, \left[ T_{n20}^{(P)} \right]_1, \overline{U}_{20}^{(P)} \right)^T \quad (34)$$

Vector  $\mathbf{U}_{20F}^{(I)}$ :

$$\mathbf{U}_{20F}^{(I)} = (-2, -2S_0^{-1}, -2\overline{U}_{20}^{(I)})^T \quad (35)$$

Vector  $\mathbf{U}_{33}^{(P)}$ :

$$\left( \mathbf{U}_{33}^{(P)} \right)_1 = \left[ V_{33}^{(P)} - 2ikU_{11}Y_{22} + ikU_{22}Y_{11} \right]_1 \quad (36)$$

$$\left( \mathbf{U}_{33}^{(P)} \right)_2 = \left[ T_{t33}^{(P)} \right]_1 \quad (37)$$

$$\left( \mathbf{U}_{33}^{(P)} \right)_3 = \left[ T_{n33}^{(P)} \right]_1 \quad (38)$$

$$\left( \mathbf{U}_{33}^{(P)} \right)_4 = \left[ T_{t33}^{(P)} \right]_{\eta_b} + \frac{1}{2} \frac{(\theta_{b11} - \theta_{c11})(\theta_{b22} - \theta_{c22})}{\theta_{r0}(\theta_{b0} - \theta_{c0})} - \frac{1}{8} \frac{(\theta_{b11} - \theta_{c11})^3}{\theta_{r0}(\theta_{b0} - \theta_{c0})^2} \quad (39)$$

Vector  $\mathbf{U}_{31}^{(1)}$ :

$$\mathbf{U}_{31}^{(1)} = \left( \left[ V_{31}^{(P1)} - Y_{11} \right]_1, \left[ T_{t31}^{(P1)} \right]_1, \left[ T_{n31}^{(P1)} \right]_1, \left[ T_{t31}^{(P1)} \right]_{\eta_b} - \frac{F_0}{\theta_{r0}Q} \right)^T \quad (40)$$

Vector  $\mathbf{U}_{31}^{(2F)}$ :

$$\left( \mathbf{U}_{31}^{(2F)} \right)_1 = 0 \quad (41)$$

$$\left( \mathbf{U}_{31}^{(2F)} \right)_2 = 0 \quad (42)$$

$$\left( \mathbf{U}_{31}^{(2F)} \right)_3 = -2S_0^{-1} [Y_{11}]_1 \quad (43)$$

$$\begin{aligned} \left( \mathbf{U}_{31}^{(2F)} \right)_4 &= \left[ T_{t31}^{(2F)} + T_{t11,\eta} \eta_{b20F} + 2T_{t11} \right]_{\eta_b} - \frac{F_0}{\theta_{r0}Q} \\ &\quad + \frac{1}{2} \frac{(\theta_{b11} - \theta_{c11})(\theta_{b20F} - \theta_{c20F})}{\theta_{r0}(\theta_{b0} - \theta_{c0})} \end{aligned} \quad (44)$$

Vector  $\mathbf{U}_{31}^{(3)}$ :

$$\left( \mathbf{U}_{31}^{(3)} \right)_1 = \left[ V_{31}^{(P3)} - 2ikU_{11}^* Y_{22} + ikU_{22} Y_{11}^* - ikU_{20} Y_{11} \right]_1 \quad (45)$$

$$\left( \mathbf{U}_{31}^{(3)} \right)_2 = \left[ T_{t31}^{(P3)} \right]_1 \quad (46)$$

$$\left( \mathbf{U}_{31}^{(3)} \right)_3 = \left[ T_{n31}^{(P3)} \right]_1 \quad (47)$$

$$\begin{aligned} \left( \mathbf{U}_{31}^{(3)} \right)_4 &= \left[ T_{t31}^{(P3)} + T_{t11,\eta} \eta_{b20} \right]_{\eta_b} + \frac{1}{2} \frac{(\theta_{b11}^* - \theta_{c11}^*)(\theta_{b22} - \theta_{c22})}{\theta_{r0}(\theta_{b0} - \theta_{c0})} \\ &\quad + \frac{1}{2} \frac{(\theta_{b11} - \theta_{c11})(\theta_{b20} - \theta_{c20})}{\theta_{r0}(\theta_{b0} - \theta_{c0})} - \frac{1}{8} \frac{(\theta_{b11}^* - \theta_{c11}^*)(\theta_{b11} - \theta_{c11})^2}{\theta_{r0}(\theta_{b0} - \theta_{c0})^2} \end{aligned} \quad (48)$$

where  $\eta_{b20F}$  and  $\eta_{b20}$  are the corrections of the level  $\eta_b$  due to the nonlinear distortions of the basic uniform flow.