Appendix to "Optimal growth and transition to turbulence in channel flow with spanwise magnetic field."

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1. Navier-Stokes Equations with Lorentz force term

In the assumption of low magnetic Reynolds number, the governing equations reduce to the Navier-Stokes system for the velocity $\boldsymbol{v}$ and pressure p with the additional Lorentz force:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v} + \frac{1}{\rho}\left(\boldsymbol{j}\times\boldsymbol{B}_0\right), \tag{1.1}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{1.2}$$

where ν and ρ stand for kinematic viscosity and density of the fluid. The induced electric current density is given by Ohm's law

$$\boldsymbol{j} = \sigma\left(-\nabla\phi + \boldsymbol{v}\times\boldsymbol{B}_0\right). \tag{1.3}$$

Neglecting displacement currents and assuming that the fluid is electrically neutral we require that $\nabla\cdot\boldsymbol{j} = 0$. This leads to an equation for the electric potential ϕ:

$$\nabla^2\phi = \nabla\cdot\left(\boldsymbol{v}\times\boldsymbol{B}_0\right). \tag{1.4}$$

The problem is solved in a rectangular domain with periodicity conditions used in the x- and y-directions following the assumption of flow homogeneity. The no-slip conditions are imposed at the walls:

$$v_x = v_y = v_z = 0 \quad \text{at } z = \pm d/2. \tag{1.5}$$

The electric potential ϕ is also periodic in the x- and y-directions. Since no current flows through the electrically insulating walls and the velocity $\boldsymbol{v}$ is zero at these walls, (1.3) leads to

$$\frac{\partial\phi}{\partial z} = 0 \quad \text{at } z = \pm d/2. \tag{1.6}$$

In the case of spanwise magnetic field, the basic velocity has the classical parabolic profile

$$U_H(z) = -\frac{d^2}{8\nu\rho}\frac{\partial P_0}{\partial x}\left(1 - \frac{4z^2}{d^2}\right) \tag{1.7}$$

with the basic pressure field $P_H(x) = (\partial P_0/\partial x)x$. Finally the basic potential field reads

$$\phi_H(z) = -\frac{d^2 B_0}{8\nu\rho}\frac{\partial P_0}{\partial x}\left(z - \frac{4z^3}{3d^2}\right). \tag{1.8}$$

For the non-dimensionalization, the centerline velocity of the Poiseuille flow is used as the velocity scale U:

$$U \equiv -\frac{d^2}{8\nu\rho}\frac{\partial P_0}{\partial x}. \tag{1.9}$$

The characteristic length is taken to be the channel half width $L \equiv d/2$. The imposed magnetic field and the electric potential scale with B_0 and LUB_0, respectively. Finally the units of time and pressure are taken as L/U and ρU^2. The non-dimensional basic velocity profile is

$$U_H(z) = 1 - z^2, \tag{1.10}$$

and the non-dimensional governing equations and boundary conditions become

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \frac{1}{Re}\nabla^2\boldsymbol{v} + N\left(-\nabla\phi\times\boldsymbol{e} + (\boldsymbol{v}\times\boldsymbol{e})\times\boldsymbol{e}\right), \tag{1.11}$$

$$\nabla\cdot\boldsymbol{v} = 0, \tag{1.12}$$

$$\nabla^2\phi = \nabla\cdot(\boldsymbol{v}\times\boldsymbol{e}), \tag{1.13}$$

$$v_x = v_y = v_z = \frac{\partial\phi}{\partial z} = 0 \quad \text{at } z = \pm 1. \tag{1.14}$$

Two non-dimensional independent parameters can be defined, namely the Reynolds number

$$Re = \frac{UL}{\nu} = -\frac{d^3}{16\nu^2\rho}\frac{\partial P_0}{\partial x}, \tag{1.15}$$

and either the Hartmann number

$$Ha = \frac{d}{2\delta}, \quad \text{where} \quad \delta = \frac{1}{B_0}\sqrt{\frac{\rho\nu}{\sigma}} \tag{1.16}$$

or the magnetic interaction parameter

$$N \equiv \frac{Ha^2}{Re}. \tag{1.17}$$

2. Governing equations for the case of arbitrary orientation of the magnetic field

Let us consider a channel flow subjected to a constant magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{e}$ of arbitrary orientation defined by the unit vector $\boldsymbol{e} = (e_x, e_y, e_z)$. The quasi-static approximation is applied following the assumption of low magnetic Reynolds numbers. Equations (1.1) to (1.6) are the governing equations of the flow.

First, let us consider the basic flow, whose velocity field has only a streamwise component $U_H(z)$. One can easily find the profile of a generalized Hartmann flow

$$U_H(z) = -\frac{1}{\sigma B_0^2 e_z^2}\frac{\partial P_0}{\partial x}\left(1 - \frac{\cosh(2e_z Ha\ z/d)}{\cosh(e_z Ha)}\right) \tag{2.1}$$

with the Hartmann number defined as in (1.16). When the magnetic field is orthogonal to the walls ($\boldsymbol{e} = (0,0,1)$), one recovers the standard Hartmann flow (Hartmann & Lazarus 1937; Hartmann 1937). In that case, the Hartmann number quantifies the ratio between the half channel width $d/2$ and the Hartmann layer thickness δ at which viscous and electromagnetic forces are of the same order. When magnetic field is parallel to the walls ($e_z = 0$), one recovers the parabolic Poiseuille profile as was done in this paper for the case of purely spanwise field $\boldsymbol{e} = (0,1,0)$.

The basic pressure field satisfies

$$P_H(y,z) = -\frac{e_x}{e_z}\frac{\partial P_0}{\partial x}\left(z - \frac{d\sinh(2e_z Ha\ z/d)}{2e_z Ha\cosh(e_z Ha)}\right). \tag{2.2}$$

The basic electric potential depends on z and linearly on y as

$$\phi_H(y,z) = cy - \frac{e_y}{\sigma B_0 e_z^2}\,\frac{\partial P_0}{\partial x}\left(z - \frac{d\sinh\left(2e_z Ha\; z/d\right)}{2e_z Ha\cosh(e_z Ha)}\right) \tag{2.3}$$

where c is a constant ensuring that the net induced current in the spanwise direction is zero. Here we implicitly assume the presence of electrically insulating walls at $y \to \pm\infty$.

Non-dimensionalization is performed using the same characteristic scales as in section 1. The laminar profile now reads

$$U_H(z) = \frac{2}{Ha^2 e_z^2}\left(1 - \frac{\cosh\left(e_z Ha\; z\right)}{\cosh(e_z Ha)}\right). \tag{2.4}$$

The equations for the velocity $\boldsymbol{v}$, electric potential ϕ and pressure p are identical to equations (1.10)–(1.14) with the proviso that $\boldsymbol{e} = (e_x, e_y, e_z)$. The two integral conditions (fixed streamwise volume flux Q per span width and zero total electric current in the y-direction) have to be specified for the nonlinear equations.

For the linear problem we split the fields into the basic flow contribution, which is now the generalized Hartmann flow (2.4), and three-dimensional perturbations in the form of a monochromatic Fourier mode

$$(\boldsymbol{v}_p, \phi_p, p_p) = (\hat{u}(z,t), \hat{v}(z,t), \hat{w}(z,t), \hat{\phi}(z,t), \hat{p}(z,t))\exp(i\alpha x + i\beta y), \tag{2.5}$$

where α and β are the wavenumbers in the streamwise (x) and spanwise (y) directions.

The evolution of infinitesimal perturbations is governed by the generalized linear system

$$\left[\frac{\partial}{\partial t} + i\alpha U_H(z)\right]\hat{u} + \frac{\partial U_H}{\partial z}\hat{w} + i\alpha\hat{p} - \frac{1}{Re}\left[\frac{\partial^2}{\partial z^2} - \alpha^2 - \beta^2\right]\hat{u} +$$

$$+N(1-e_x^2)\hat{u} - Ne_x e_y\hat{v} - Ne_x e_z\hat{w} + i\beta N e_z\hat{\phi} - Ne_y\frac{\partial\hat{\phi}}{\partial z} = 0, \tag{2.6}$$

$$\left[\frac{\partial}{\partial t} + i\alpha U_H(z)\right]\hat{v} + i\beta\hat{p} - \frac{1}{Re}\left[\frac{\partial^2}{\partial z^2} - \alpha^2 - \beta^2\right]\hat{v} +$$

$$+N(1-e_y^2)\hat{v} - Ne_y e_x\hat{u} - Ne_y e_z\hat{w} - i\alpha N e_z\hat{\phi} + Ne_x\frac{\partial\hat{\phi}}{\partial z} = 0, \tag{2.7}$$

$$\left[\frac{\partial}{\partial t} + i\alpha U_H(z)\right]\hat{w} + \frac{\partial\hat{p}}{\partial z} - \frac{1}{Re}\left[\frac{\partial^2}{\partial z^2} - \alpha^2 - \beta^2\right]\hat{w} +$$

$$+N(1-e_z^2)\hat{w} - Ne_z e_y\hat{v} - Ne_z e_x\hat{u} - i\beta N e_x\hat{\phi} + i\alpha N e_y\hat{\phi} = 0, \tag{2.8}$$

$$i\alpha\hat{u} + i\beta\hat{v} + \frac{\partial\hat{w}}{\partial z} = 0, \tag{2.9}$$

$$\left[\frac{\partial^2}{\partial z^2} - \alpha^2 - \beta^2\right]\hat{\phi} - (i\beta e_x - i\alpha e_y)\hat{w} + e_x\frac{\partial\hat{v}}{\partial z} - e_y\frac{\partial\hat{u}}{\partial z} + i\beta e_z\hat{u} - i\alpha e_z\hat{v} = 0, \tag{2.10}$$

with the boundary conditions:

$$\hat{u} = \hat{v} = \hat{w} = 0, \quad \frac{\partial\hat{\phi}}{\partial z} = 0 \quad \text{at } z = \pm 1. \tag{2.11}$$

The adjoint equations are

$$\left[\frac{\partial}{\partial\tau} - i\alpha U_H(z)\right]\tilde{u} - \frac{1}{Re}\left[\frac{\partial^2}{\partial z^2} - \alpha^2 - \beta^2\right]\tilde{u} +$$

$$+N(1-e_x^2)\tilde{u} - Ne_x e_z\tilde{w} - Ne_y e_x\tilde{v} - i\alpha\tilde{p} - e_y\frac{\partial\tilde{\phi}}{\partial z} - i\beta e_z\tilde{\phi} = 0, \tag{2.12}$$

$$\left[\frac{\partial}{\partial\tau} - i\alpha U_H(z)\right]\tilde{v} - \frac{1}{Re}\left[\frac{\partial^2}{\partial z^2} - \alpha^2 - \beta^2\right]\tilde{v} - i\beta\tilde{p} +$$

$$+N(1-e_y^2)\tilde{v} - Ne_xe_y\tilde{u} - Ne_ze_y\tilde{w} + i\alpha e_z\tilde{\phi} - e_x\frac{\partial\tilde{\phi}}{\partial z} = 0, \tag{2.13}$$

$$\left[\frac{\partial}{\partial\tau} - i\alpha U_H(z)\right]\tilde{w} - \frac{1}{Re}\left[\frac{\partial^2}{\partial z^2} - \alpha^2 - \beta^2\right]\tilde{w} - \frac{\partial\tilde{p}}{\partial z} + \frac{\partial U_H}{\partial z}\tilde{u} +$$

$$+N(1-e_z^2)\tilde{w} - Ne_ye_z\tilde{v} - Ne_ze_x\tilde{u} + (i\beta e_x - i\alpha e_y)\tilde{\phi} = 0, \tag{2.14}$$

$$i\alpha\tilde{u} + i\beta\tilde{v} + \frac{\partial\tilde{w}}{\partial z} = 0, \tag{2.15}$$

$$\left[\frac{\partial^2}{\partial z^2} - \alpha^2 - \beta^2\right]\tilde{\phi} + Ne_y\frac{\partial\tilde{u}}{\partial z} - i\beta Ne_z\tilde{u} +$$

$$+(i\beta Ne_x - i\alpha Ne_y)\tilde{w} - Ne_x\frac{\partial\tilde{v}}{\partial z} + i\alpha e_z\tilde{v} = 0, \tag{2.16}$$

with the boundary conditions

$$\tilde{u} = \tilde{v} = \tilde{w} = 0, \quad \frac{\partial\tilde{\phi}}{\partial z} = 0 \quad \text{at } z = \pm 1 \tag{2.17}$$

and $\tau \equiv -t$.

3. Optimal perturbation method

Let us denote by $q_j(z,t), j = 1\ldots 5$ the vector fields

$$(\hat{u}(z,t), \hat{v}(z,t), \hat{w}(z,t), \hat{p}(z,t), \hat{\phi}(z,t)).$$

These quantities evolve according to equations (2.6)–(2.10) which will be formally written here as $F_j(\boldsymbol{q}) = 0, \quad j = 1\ldots 5$.

To get the maximum amplification at a given time T, we maximize the perturbation kinetic energy $E(\boldsymbol{q}(T))$

$$E(\boldsymbol{q}(T)) \equiv \sum_{j=1}^{3}\int q_j(z,T) * q_j^+(z,T)\mathrm{d}z \tag{3.1}$$

at time T with respect to the set of all possible initial perturbations $\boldsymbol{q}(0)$ such that $E(\boldsymbol{q}(0)) = 1$. Recall that the superscript + denotes complex conjugation and spatial integration is performed over the entire channel width.

To solve this problem, the variation $\delta E(\boldsymbol{q}(T))$ with respect to a variation $\delta\boldsymbol{q}(0)$ of the initial perturbation, should be evaluated. Unfortunately, this computation cannot be performed in a straightforward manner. Indeed, the energy $E(\boldsymbol{q}(T))$ is explicitly known in terms of $\boldsymbol{q}(T)$ but only implicitely in terms of $\boldsymbol{q}(0)$ *via* several constraints: normalization of $\boldsymbol{q}(0)$ and more importantly, the integration over time interval $[0,T]$ of the equations $F_j(\boldsymbol{q}(t)) = 0$, which relate $\boldsymbol{q}(0)$ to $\boldsymbol{q}(T)$.

This optimization with constraints necessitates the introduction of Lagrangian multipliers. In the present case, these multipliers are the so-called adjoint fields

$$\tilde{\boldsymbol{q}}(t) \equiv (\tilde{u}(z,t), \tilde{v}(z,t), \tilde{w}(z,t), \tilde{p}(z,t), \tilde{\phi}(z,t)) \tag{3.2}$$

and a normalization scalar γ.

A Lagrangian function L is first defined that depends on direct $\boldsymbol{q}(t)$ and adjoint $\tilde{\boldsymbol{q}}(t)$

variables for $t \in [0,T]$, and γ:

$$L(\boldsymbol{q},\tilde{\boldsymbol{q}},\gamma,T) = E(\boldsymbol{q}(T)) - \gamma(E(\boldsymbol{q}(0)) - 1) \tag{3.3}$$
$$-\sum_{j=1}^{5}\int_0^T dt\,(\langle F_j(\boldsymbol{q}(t)),\tilde{q}_j(t)\rangle + \langle \tilde{q}_j(t), F_j(\boldsymbol{q}(t))\rangle)\,.$$

Here $\langle\cdot,\cdot\rangle$ stands for the scalar product

$$\langle a_1, a_2\rangle \equiv \int \hat{a}_1(z) * \hat{a}_2^{+}(z)\mathrm{d}z. \tag{3.4}$$

Note that, when $\boldsymbol{q}(t)$ satisfies the constraints (direct problem plus normalization at $t=0$), all terms but the first one on the r.h.s. of equation (3.3) are zero and, by consequence, $L = E$ and $\delta L = \delta E$. In addition, at this stage, the adjoint variables and the quantity γ are left unspecified.

Let us now write down formally the variation δL as

$$\delta L = \sum_{j=1}^{3}\left(\int q_j^{+}(z,T)\delta q_j(z,T)\mathrm{d}z - \gamma\int q_j^{+}(z,0)\delta q_j(z,0)\mathrm{d}z\right) \tag{3.5}$$
$$-\sum_{j=1}^{5}\int_0^T dt[\langle \delta F_j(\boldsymbol{q}(t)),\tilde{q}_j(t)\rangle + \langle F_j(\boldsymbol{q}(t)),\delta\tilde{q}_j(t)\rangle] + c.c.,$$

where *c.c.* means comlex conjugate. The expression $\langle F_j(\boldsymbol{q}(t)),\delta\tilde{q}_j(t)\rangle$ in equation (3.5) is zero if the governing equations $F_j(\boldsymbol{q}) = 0$ are satisfied on the time interval $[0,T]$. The main idea is to rewrite quantity $\langle \delta F_j(\boldsymbol{q}(t)),\tilde{q}_j(t)\rangle$ in terms of $\delta q_k(t)$. This can be done by using integrations by parts in space or time. After some tedious algebra, the following identity

$$\sum_{j=1}^{5}\int_0^T dt\langle \delta F_j(\boldsymbol{q}(t)),\tilde{q}_j(t)\rangle = \sum_{j=1}^{5}[\int_0^T dt\langle \tilde{F}_j(\tilde{q}(t),q),\delta q(t)\rangle] \tag{3.6}$$
$$+\sum_{j=1}^{3}[\int \tilde{q}_j^{+}(z,T)\delta q_j(z,T)\mathrm{d}z - \int \tilde{q}_j^{+}(z,0)\delta q_j(z,0)\mathrm{d}z] + B(\boldsymbol{q},\tilde{\boldsymbol{q}})$$

can be established. $\tilde{F}_j$ is an expression containing spatial or time derivatives of $\tilde{\boldsymbol{q}}$ and depends on $\boldsymbol{q}$. Note that the second r.h.s term originates from integration by parts of time derivatives in equations (2.6)–(2.10) and terms $B(\boldsymbol{q},\tilde{\boldsymbol{q}})$ are generated from the boundary terms resulting from integration by parts of spatial derivatives. They, thus, involve only quantities $\boldsymbol{q}$ and $\tilde{\boldsymbol{q}}$ evaluated at the boundaries $z = \pm 1$.

At this stage, it is relevant to use the freedom of the Lagrangian multipliers to enforce several constraints: (i) equations $\tilde{F}_j(\tilde{\boldsymbol{q}}(t),\boldsymbol{q}) = 0, j = 1\ldots 5$ are to be satisfied, which defines the evolution equations (2.12 – 2.16) for $\tilde{\boldsymbol{q}}$ similar to $F_j(\boldsymbol{q}(t))$ for $\boldsymbol{q}$, and (ii) $B(\boldsymbol{q},\tilde{\boldsymbol{q}}) = 0$ are to be satisfied, which can be ensured by imposing boundary conditions (2.17) on $\tilde{\boldsymbol{q}}$, again the counterpart of boundary conditions (2.11) on $\boldsymbol{q}$. This defines a new system which should be simulated as the direct problem. When this system is satisfied, the variation δL reads

$$\delta L = \sum_{j=1}^{3}\left(\int (q_j^{+}(z,T) - \tilde{q}_j^{+}(z,T))\delta q_j(z,T)\mathrm{d}z\right. \tag{3.7}$$

$$- \int (\gamma q_j^+(z,0) - \tilde{q}_j^+(z,0))\delta q_j(z,0)\mathrm{d}z \Big) + c.c.$$

It is easily seen from equations (2.12 – 2.16) that the adjoint system must be integrated backwards in time. Let us thus choose as initial condition for the adjoint variables at time $t = T$

$$\tilde{q}_j(z,T) = q_j(z,T), \quad j = 1\ldots 3 \tag{3.8}$$

and at time $t = 0$

$$\gamma q_j(z,0) = \tilde{q}_j(z,0), \quad j = 1\ldots 3 \tag{3.9}$$

so that the normalization is satisfied. When all these constraints are satisfied, $\delta L = 0$, which means that an optimal perturbation is attained. One may use an iteration procedure which is schematically illustrated by a diagram

$$\begin{array}{ccc} \boldsymbol{q}(z,0) & \xrightarrow{F_j(\boldsymbol{q})=0} & \boldsymbol{q}(z,T) \\ \uparrow & & \downarrow \\ \tilde{\boldsymbol{q}}(z,0) & \xleftarrow{\tilde{F}_j(\tilde{\boldsymbol{q}})=0} & \tilde{\boldsymbol{q}}(z,T) \end{array} \tag{3.10}$$

When the iterative process has converged, an initial optimal perturbation for time T is found.

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