Appendices to "Parametric instability in a rotating cylinder of gas subject to sinusoidal axial compression. Part 2. Weakly nonlinear theory."

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Appendix D: Some properties of the nonlinear coefficients $\Lambda_{\mu\nu_1\nu_2}$

Writing the gradient tensor $\dfrac{\partial u_i^{(\mu)}}{\partial X_j}$ in cylindrical coordinates yields the components

$$\begin{bmatrix} \dfrac{\partial u_r^{(\mu)}}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_r^{(\mu)}}{\partial \theta} - \dfrac{u_\theta^{(\mu)}}{r} & \dfrac{\partial u_r^{(\mu)}}{\partial Z} \\ \dfrac{\partial u_\theta^{(\mu)}}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\theta^{(\mu)}}{\partial \theta} + \dfrac{u_r^{(\mu)}}{r} & \dfrac{\partial u_\theta^{(\mu)}}{\partial Z} \\ \dfrac{\partial u_Z^{(\mu)}}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_Z^{(\mu)}}{\partial \theta} & \dfrac{\partial u_Z^{(\mu)}}{\partial Z} \end{bmatrix} \tag{D.1}$$

which is used in (2.6), together with (I.3.4) and integration over θ and Z, to obtain

$$\Lambda_{\mu\nu_1\nu_2} = \frac{1}{4}\pi h_0 \int_0^1 \left(f_{\mu\nu_1\nu_2}(r) + f_{\mu\nu_2\nu_1}(r) \right) dr \tag{D.2}$$

in which $f_{\mu\nu_1\nu_2} = 0$ unless (2.7) is satisfied, and otherwise

$$\begin{aligned} f_{\mu\nu_1\nu_2} = &\left(\delta_{m_\mu+m_{\nu_1}+m_{\nu_2}} + \delta_{m_\mu+m_{\nu_1}-m_{\nu_2}} + \delta_{m_\mu-m_{\nu_1}+m_{\nu_2}} + \delta_{m_\mu-m_{\nu_1}-m_{\nu_2}}\right) \left[\chi_r^{(\nu_1)}\left(r\chi_r^{(\nu_2)} \frac{d\chi_r^{(\mu)}}{dr} - \chi_\theta^{(\nu_2)}\left(\chi_\theta^{(\mu)} + n_\mu \chi_r^{(\mu)}\right)\right) + \right. \\ &\left. + \chi_\theta^{(\nu_1)}\left(r\chi_r^{(\nu_2)} \frac{d\chi_\theta^{(\mu)}}{dr} - \chi_\theta^{(\nu_2)}\left(\chi_r^{(\mu)} + n_\mu \chi_\theta^{(\mu)}\right)\right)\right] + \\ &+\left(\delta_{m_\mu+m_{\nu_1}+m_{\nu_2}} + \delta_{m_\mu-m_{\nu_1}+m_{\nu_2}} - \delta_{m_\mu+m_{\nu_1}-m_{\nu_2}} - \delta_{m_\mu-m_{\nu_1}-m_{\nu_2}}\right) \frac{m_\mu \pi r}{h_0} \chi_Z^{(\nu_2)}\left(\chi_r^{(\nu_1)}\chi_r^{(\mu)} + \chi_\theta^{(\nu_1)}\chi_\theta^{(\mu)}\right) + \\ &+\left(\delta_{m_\mu-m_{\nu_1}-m_{\nu_2}} + \delta_{m_\mu-m_{\nu_1}+m_{\nu_2}} - \delta_{m_\mu+m_{\nu_1}+m_{\nu_2}} - \delta_{m_\mu+m_{\nu_1}-m_{\nu_2}}\right) \chi_Z^{(\nu_1)}\left(\chi_r^{(\nu_2)} r \frac{d\chi_Z^{(\mu)}}{dr} - \chi_\theta^{(\nu_2)} n_\mu \chi_Z^{(\mu)}\right) + \\ &+\left(\delta_{m_\mu+m_{\nu_1}-m_{\nu_2}} + \delta_{m_\mu-m_{\nu_1}+m_{\nu_2}} - \delta_{m_\mu+m_{\nu_1}+m_{\nu_2}} - \delta_{m_\mu-m_{\nu_1}-m_{\nu_2}}\right) \frac{m_\mu \pi r}{h_0} \chi_Z^{(\mu)}\chi_Z^{(\nu_1)}\chi_Z^{(\nu_2)} \end{aligned} \tag{D.3}$$

where the notation $\delta_0 = 1$ and $\delta_l = 0$ if $l \neq 0$ is used. It is apparent that $\Lambda_{\mu\nu_1\nu_2}$ is real, as stated in the main text. Using (I.A.1), (I.A.5) for the modal eigenfunctions, and the Bessel differential equation to rewrite the second derivative of J_n in terms of J_n and J_n', (D.2) and (D.3) can be used to express $\Lambda_{\mu\nu_1\nu_2}$ as a sum of integrals of triple products of the Bessel function and its derivative (including powers of r in the case of non-axisymmetric, geostrophic modes).

Taking the complex conjugate of (2.6) and using the fact that $\Lambda_{\mu\nu_1\nu_2}$ is real gives

$$\Lambda_{\mu^*\nu_1^*\nu_2^*} = -\Lambda_{\mu\nu_1\nu_2}. \tag{D.4}$$

Applying the divergence theorem, (I.3.2), and the boundary conditions on $\mathbf{u}^{(\mu)}$,

$$\int u_i^{(\nu_1)} u_j^{(\nu_2)} \frac{\partial u_i^{(\mu)^*}}{\partial X_j} d^3\mathbf{X} = -\int u_i^{(\mu)^*} u_j^{(\nu_2)} \frac{\partial u_i^{(\nu_1)}}{\partial X_j} d^3\mathbf{X} = -\int u_i^{(\mu^*)} u_j^{(\nu_2)} \frac{\partial u_i^{(\nu_1^*)^*}}{\partial X_j} d^3\mathbf{X}. \tag{D.5}$$

In the two cases $\nu_1 = \mu^*,\ \nu_2 = \nu$ and $\nu_1 = \nu,\ \nu_2 = \mu^*$, (D.5) gives

$$\int u_i^{(\mu^*)} u_j^{(\nu)} \frac{\partial u_i^{(\mu)^*}}{\partial X_j} d^3\mathbf{X} = 0, \qquad \int u_i^{(\nu)} u_j^{(\mu^*)} \frac{\partial u_i^{(\mu)^*}}{\partial X_j} d^3\mathbf{X} = -\int u_i^{(\mu^*)} u_j^{(\mu^*)} \frac{\partial u_i^{(\nu^*)^*}}{\partial X_j} d^3\mathbf{X} \tag{D.6}$$

and hence

$$2\Lambda_{\mu\mu^*\nu} = -\Lambda_{\nu^*\mu^*\mu^*} = \Lambda_{\nu\mu\mu} \tag{D.7}$$

a result which will be used in appendix E and in which the second equality follows from (D.4). Finally, (2.8) follows from (2.7) and $n_{\nu^*} = -n_\nu$ when $n_\mu \neq 0$, and otherwise from (D.4) (with $\nu_1 = \nu_2^* = \nu$), the fact that modes with $n = m = 0$ are real ($\mu^* = \mu$) and symmetry of $\Lambda_{\mu\nu_1\nu_2}$ with respect to its last two indices.

Appendix E: Calculation of $\Lambda_{\sigma\mu_+\mu_+}$, $\Lambda_{\mu_+\mu_+\sigma}$ and G

Throughout this appendix, μ_+ is axisymmetric and σ geostrophic. If σ is non-axisymmetric, (2.7) implies that $\Lambda_{\sigma\mu_+\mu_+} = \Lambda_{\mu_+\mu_+\sigma} = 0$ so we take $n_\sigma = 0$ in the remainder of this appendix. The modes encountered are then all axisymmetric. For convenience sake we reproduce some properties of axisymmetric modes from [I]. (I.A.1) or (I.A.5) gives

$$\chi_r^{(\mu)} = N^{(\mu)} \omega^{(\mu)} k^{(\mu)} J_0'\left(k^{(\mu)} r\right) \tag{E.1}$$

$$\chi_\theta^{(\mu)} = N^{(\mu)} k^{(\mu)} J_0'\left(k^{(\mu)} r\right) \tag{E.2}$$

and either

$$\chi_Z^{(\mu)} = N^{(\mu)} \frac{\omega^{(\mu)} k^{(\mu)2} h_0}{m\pi} J_0\left(k^{(\mu)} r\right) \tag{E.3}$$

if $m_\mu \neq 0$, or

$$\chi_Z^{(\mu)} = 0 \tag{E.4}$$

if $m_\mu = 0$, where the normalisation constant $N^{(\mu)}$ is

$$N^{(\mu)} = \left(\pi h_0\right)^{-1/2} \left(k^{(\mu)} J_0\left(k^{(\mu)}\right)\right)^{-1}. \tag{E.5}$$

From (I.A.2), the transverse wavenumber $k^{(\mu)}$ in the above expressions is a positive root of $J_0'(k) = 0$, while the modal frequency $\omega^{(\mu)}$ is zero for geostrophic modes ($m_\mu = 0$) and given by (I.A.3) as

$$\omega^{(\mu)} = \pm\left(1 + \left(\frac{k^{(\mu)} h_0}{m_\mu \pi}\right)^2\right)^{-1/2} \tag{E.6}$$

if $m_\mu \neq 0$, where the choice of signs leads to two conjugate modes having the same $k^{(\mu)}$.

A variety of integrals of products of Bessel functions arise in the course of the analysis and we give their values here. Let k_1 and k_2 be any two positive zeros of the Bessel function J_0'. The simplest type of integral we will encounter is given by

$$\int_0^1 r J_0'(k_1 r) J_0'(k_2 r) dr = \begin{cases} 0 & ; \ k_1 \neq k_2 \\ \dfrac{1}{2} J_0^2(k_1) & ; \ k_1 = k_2 \end{cases} \tag{E.7}$$

while the other integrals have integrands which are triple products of Bessel functions and can be expressed in terms of

$$I = \int_0^1 J_0'^2(k_1 r) J_0'(k_2 r) dr \tag{E.8}$$

as

$$\int_0^1 rJ_0^2(k_1r)J_0(k_2r)dr = \frac{4k_1^2}{k_2\left(k_2^2-4k_1^2\right)}\ I \tag{E.9}$$

$$\int_0^1 rJ_0'^2(k_1r)J_0(k_2r)dr = 2\frac{2k_1^2-k_2^2}{k_2\left(k_2^2-4k_1^2\right)}\ I \tag{E.10}$$

$$\int_0^1 rJ_0(k_1r)J_0'(k_1r)J_0'(k_2r)dr = \frac{2k_1}{k_2^2-4k_1^2}\ I \tag{E.11}$$

E.1 Calculation of $\Lambda_{\sigma\mu_+\mu_+}$ and $\Lambda_{\mu_+\mu_+\sigma}$

The integrand in (D.2) is expressed using (D.3), (E.1)-(E.5) and the Bessel differential equation and the resulting integrals of triple products of Bessel functions rewritten in terms of

$$I_\sigma = \int_0^1 J_0'^2\left(k^{(\mu_+)}r\right)J_0'\left(k^{(\sigma)}r\right)dr \tag{E.12}$$

using (E.8)-(E.11) with $k_1 = k^{(\mu_+)}$ and $k_2 = k^{(\sigma)}$. Thus, we obtain

$$\Lambda_{\sigma\mu_+\mu_+} = \frac{4\omega_+ k^{(\mu_+)2} I_\sigma}{(\pi h_0)^{1/2}\left(k^{(\sigma)2}-4k^{(\mu_+)2}\right)J_0^2\left(k^{(\mu_+)}\right)J_0\left(k^{(\sigma)}\right)} \tag{E.13}$$

and (A.3), while (E.11) with $k_1 = k^{(\mu_+)}$ and $k_2 = k^{(\sigma)}$ can be used to derive an alternative form of (E.13), namely (A.2).

E.2 Calculation of G

The quantity G is given by (3.6), which can be rewritten using (3.5) as

$$G = \sum_\lambda \left\{ 2\frac{\Lambda_{\lambda\mu_+\mu_+}\Lambda_{\mu_+\mu_+^*\lambda}}{\omega^{(\lambda)}-2\omega_+} + 4\frac{\Lambda_{\lambda\mu_+\mu_+^*}\Lambda_{\mu_+\mu_+\lambda}}{\omega^{(\lambda)}} \right\} \tag{E.14}$$

The first term in brackets can be reexpressed using $2\Lambda_{\mu_+\mu_+^*\lambda} = \Lambda_{\lambda\mu_+\mu_+}$, which follows from (D.7). The condition (2.7) implies that only the families $n_\lambda = m_\lambda = 0$ and $n_\lambda = 0$, $m_\lambda = 2m_+$ contribute to the sum. If $\lambda \in M$, the second term in brackets has an apparent division by zero, indicating that it should be dropped according to the discussion following equation (3.2). Thus, we obtain

$$G = -\frac{1}{2\omega_+}\sum_{\sigma\in M}\Lambda^2_{\sigma\mu_+\mu_+} + \sum_{\lambda\in M_2}\left\{\frac{\Lambda^2_{\lambda\mu_+\mu_+}}{\omega^{(\lambda)}-2\omega_+} + 4\frac{\Lambda_{\lambda\mu_+\mu_+^*}\Lambda_{\mu_+\mu_+\lambda}}{\omega^{(\lambda)}}\right\} \tag{E.15}$$

where M_2 denotes the modal family $n=0$, $m=2m_+$. To determine the contribution to G from the first sum in (E.15), we first note that (E.7) implies that the set of functions $J_0'\left(k^{(\sigma)}r\right)$, $\sigma\in M$ are orthogonal on $0<r<1$ with weighting function r. This set of functions is also complete and thus forms an orthogonal basis. The function $J_0\left(k^{(\mu_+)}r\right)J_0'\left(k^{(\mu_+)}r\right)$ may be expanded as

$$J_0\left(k^{(\mu_+)}r\right)J_0'\left(k^{(\mu_+)}r\right) = \sum_{\sigma\in M} c_\sigma J_0'\left(k^{(\sigma)}r\right) \tag{E.16}$$

in $0<r<1$. This expansion is introduced into the integral of (A.2) and (E.7) employed to obtain

$$c_\sigma = \frac{\left(\pi h_0\right)^{1/2} J_0^2\left(k^{(\mu_+)}\right)}{\omega_+ k^{(\mu_+)} J_0\left(k^{(\sigma)}\right)}\Lambda_{\sigma\mu_+\mu_+} \tag{E.17}$$

Squaring (E.16), multiplying by r, integrating over the interval $0<r<1$ and using (E.7) and (E.17), the result gives the first sum in (E.15) and hence the corresponding contribution to G is

$$-\frac{\omega_+ k^{(\mu_+)2}}{\pi h_0 J_0^4\left(k^{(\mu_+)}\right)}\int_0^1 r J_0^2\left(k^{(\mu_+)}r\right)J_0'^2\left(k^{(\mu_+)}r\right)dr. \tag{E.18}$$

Turning attention to the remainder of G, namely the second sum in (E.15), we evaluate the Λ's using (D.2), (D.3), (E.1)-(E.6), the Bessel differential equation and (E.8)-(E.11) with $k_1 = k^{(\mu_+)}$ and $k_2 = k^{(\lambda)}$. Thus, we find that

$$\Lambda_{\lambda\mu_+\mu_+} = 0 \tag{E.19}$$

$$\Lambda_{\lambda\mu_+\mu_+^*} = -\frac{\omega^{(\lambda)}}{\left(\pi h_0\right)^{1/2} J_0^2\left(k^{(\mu_+)}\right)J_0\left(k^{(\lambda)}\right)}\ I_\lambda \tag{E.20}$$

$$\Lambda_{\mu_+\mu_+\lambda} = -\frac{2\omega_+ + \omega^{(\lambda)}}{2\left(\pi h_0\right)^{1/2} J_0^2\left(k^{(\mu_+)}\right)J_0\left(k^{(\lambda)}\right)}\ I_\lambda \tag{E.21}$$

where

$$I_\lambda = \int_0^1 J_0'^2\left(k^{(\mu_+)} r\right) J_0'\left(k^{(\lambda)} r\right) dr \tag{E.22}$$

These results are used in the second sum in (E.15). Recognising that modes $\lambda \in M_2$ divide into positive and negative frequencies and that both classes have wavenumbers $k^{(\lambda)}$ which coincide with the $k^{(\sigma)}$ from before, the second contribution to G is found to be

$$\frac{8\omega_+}{\pi h_0 J_0^4\left(k^{(\mu_+)}\right)} \sum_{\sigma \in M} \frac{I_\sigma^2}{J_0^2\left(k^{(\sigma)}\right)}. \tag{E.23}$$

This sum of squares may be evaluated in much the same way as was the first sum in (E.15). The function $r^{-1} J_0'^2\left(k^{(\mu_+)} r\right)$ is expanded using the same basis set as before and the coefficients determined using (E.7) and (E.12). In this way, (E.23) is reexpressed as

$$\frac{4\omega_+}{\pi h_0 J_0^4\left(k^{(\mu_+)}\right)} \int_0^1 r^{-1} J_0'^4\left(k^{(\mu_+)} r\right) dr. \tag{E.24}$$

Combining the two components, (E.18) and (E.24),

$$G = \frac{\omega_+}{\pi h_0 J_0^4\left(k^{(\mu_+)}\right)} \int_0^{k^{(\mu_+)}} \left(4 J_0'^2(\xi) - \xi^2 J_0^{\,2}(\xi)\right) \frac{J_0'^2(\xi)}{\xi} d\xi \tag{E.25}$$

which leads to (A.1) when the Bessel-function identity

$$\int_0^{k^{(\mu_+)}} \xi\, J_0^{\,2}(\xi) J_0'^2(\xi) d\xi = \frac{1}{3} \int_0^{k^{(\mu_+)}} \left(\xi + \frac{2}{\xi}\right) J_0'^4(\xi) d\xi \tag{E.26}$$

is used. Numerical evaluation of the integral in (A.1) for successive positive zeroes, $k^{(\mu_+)}$, of J_0' shows that the first few values of $h_0 G / \omega_+$ are 2.686, 7.932, 15.221 and 24.254, staying positive for the first 132 zeroes of J_0', but becoming negative thereafter.

Appendix F: Derivation of equation (B.2) and calculation of $\Gamma_\sigma^{(1)}$ and Γ_σ

Throughout this appendix, μ_+ is axisymmetric and σ geostrophic.

F.1 Derivation of (B.2)

Evaluation of the surface integral in (B.2) involves boundary-layer analysis to obtain the normal derivative of $\langle \mathbf{u} \rangle$ at the end walls. The flow outside the boundary layers is given correct to $O(\varepsilon)$ by the first two terms in (2.3). Taking the fast-time average of (2.4) and (3.2), nonzero contributions to the average arise from terms with zero frequency. Thus, $\left\langle B_\mu^{[1]} \right\rangle = 0$ from (2.4), while averaging (3.2) gives

$$\langle \mathbf{u} \rangle = \varepsilon \sum_\mu c_\mu \, \mathbf{u}^{(\mu)} \tag{F.1}$$

at leading order, outside the boundary layers, where

$$c_\mu = \mathcal{A}_\mu \tag{F.2}$$

if $m_\mu = 0$,

$$c_\mu = \frac{2}{\omega^{(\mu)}} \Lambda_{\mu\mu_+\mu_+^*} \left| A_+ \right|^2 \tag{F.3}$$

if $n_\mu = 0$ and $m_\mu = 2m_+$ ((F.3) comes from the $\nu_1 = \nu_2^*$ terms in the sum of (3.2)), and $c_\mu = 0$ for all other modes. (F.1) will provide matching conditions for the boundary-layer analysis to come and indicates that the mean flow is $O(\varepsilon)$. It is interesting to note that, although $u_r^{(\mu)}$ and $u_Z^{(\mu)}$ are nonzero for the family $n_\mu = 0$, $m_\mu = 2m_+$, they cancel out in conjugate pairs when the sum in (F.1) is taken. In consequence, the leading-order mean flow is purely azimuthal. Whereas the geostrophic contribution to the mean flow, represented by (F.2), is independent of Z, the other has $\cos(2m_+\pi Z / h_0)$ dependence.

Turning attention to the boundary layers on $Z = 0,\ h_0$, we first derive a leading-order, boundary-layer equation for $\langle \mathbf{u} \rangle$ from (I.2.17) and (I.2.18). Equation (I.2.17) can be rewritten as

$$\frac{\partial u_\xi}{\partial \xi} = -\overline{Re}^{-1/2} \nabla_\perp . \mathbf{u}_\perp \tag{F.4}$$

where $\overline{Re}^{-1/2}\xi$ is distance from the end wall, scaled appropriately for the boundary layer, u_ξ is the ξ-component of $\mathbf{u}$, and $\nabla_\perp$ is the projection of the gradient operator perpendicular to the cylinder axis (i.e. parallel to the wall). Since $u_\xi = 0$ at the wall, integration of (F.4) with respect to ξ shows that u_ξ is $O\left(\overline{Re}^{-1/2}\right) = O(\varepsilon)$ smaller than $\mathbf{u}_\perp$ within the layer, i.e. $\mathbf{u}$ is dominantly parallel to the wall, as usual in boundary-layer theory. More precisely, since $\mathbf{u} = O\left(\varepsilon^{1/2}\right)$, $\mathbf{u}_\perp = O\left(\varepsilon^{1/2}\right)$ and $u_\xi = O\left(\varepsilon^{3/2}\right)$. Similar reasoning based on $\mathbf{U} = o(\varepsilon)$, $\nabla.\mathbf{U} = 0$ and $U_\xi = 0$ at the wall shows that $\mathbf{U}_\perp = o(\varepsilon)$ and $U_\xi = o\left(\varepsilon^2\right)$ within the boundary layer, while applying the same argument to the fast-time averaged version of (F.4) and using $\langle \mathbf{u} \rangle = O(\varepsilon)$ gives $\langle \mathbf{u}_\perp \rangle = O(\varepsilon)$ and $\langle u_\xi \rangle = O\left(\varepsilon^2\right)$.

Rewriting (I.2.18) using $\mathbf{V} = \mathbf{U} + \mathbf{u}$ as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{e}_Z \times \mathbf{u} + \nabla\lambda = \mathbf{e}_Z \frac{\partial}{\partial t}\left(\left(1 - \left(\frac{h}{h_0}\right)^2\right) u_Z\right) + Re^{-1}\mathcal{D}\left(\mathbf{u}_\perp + \mathbf{e}_Z \left(\frac{h}{h_0}\right)^2 u_Z\right) -$$
$$-\mathbf{u}.\nabla\left(\mathbf{U}_\perp + \mathbf{e}_Z\left(\frac{h}{h_0}\right)^2 U_Z\right) - \mathbf{U}.\nabla\left(\mathbf{u}_\perp + \mathbf{e}_Z\left(\frac{h}{h_0}\right)^2 u_Z\right) - \mathbf{u}.\nabla\left(\mathbf{u}_\perp + \mathbf{e}_Z\left(\frac{h}{h_0}\right)^2 u_Z\right) \quad \text{(F.5)}$$

and examining the right-hand side with the above orders of magnitude in mind and the aim of dropping terms which are smaller than $O(\varepsilon)$, the first term has the negligible magnitude $O(\varepsilon^{5/2})$ due to the $O(\varepsilon)$ piston amplitude and $u_\xi = O(\varepsilon^{3/2})$. The second (viscous) term can be approximated by replacing Re^{-1} by $\overline{Re}^{-1}$, neglecting u_Z compared with $\mathbf{u}_\perp$ and dropping the X and Y-derivatives compared with the Z-derivatives in the operator $\mathcal{D}$ (given by (I.2.12)) on the grounds that the boundary layer is thin. Thus, the viscous term becomes $\frac{\partial^2 \mathbf{u}_\perp}{\partial \xi^2}$. The third and fourth terms are negligible because $\mathbf{u} = O(\varepsilon^{1/2})$, $\mathbf{u}.\nabla = O(\varepsilon^{1/2})$, $\mathbf{U} = o(\varepsilon)$ and $\mathbf{U}.\nabla = o(\varepsilon)$, while we neglect u_Z compared with $\mathbf{u}_\perp$ in the fifth term. Finally, $\mathbf{u}$ is replaced by its dominant component, $\mathbf{u}_\perp$, on the left-hand side to obtain

$$\frac{\partial \mathbf{u}_\perp}{\partial t} + \mathbf{e}_Z \times \mathbf{u}_\perp + \nabla\lambda = \frac{\partial^2 \mathbf{u}_\perp}{\partial \xi^2} - \mathbf{u}.\nabla\mathbf{u}_\perp \quad \text{(F.6)}$$

correct to $O(\varepsilon)$. The component of (F.6) normal to the wall implies that λ is independent of ξ and hence imposed from outside the layer, as usual for the pressure in boundary-layer theory. The other components yield

$$\frac{\partial \mathbf{u}_\perp}{\partial t} + \mathbf{e}_Z \times \mathbf{u}_\perp - \frac{\partial^2 \mathbf{u}_\perp}{\partial \xi^2} = -\mathbf{u}.\nabla\mathbf{u}_\perp - \nabla_\perp\lambda \quad \text{(F.7)}$$

correct to $O(\varepsilon)$.

Taking the fast-time average of (F.7), the time derivative can be expressed as

$$\left\langle \frac{\partial \mathbf{u}_\perp}{\partial t} \right\rangle = \varepsilon \, \frac{\partial \langle \mathbf{u}_\perp \rangle}{\partial T} \quad \text{(F.8)}$$

using reasoning similar to that employed in deriving (3.9). Since $\langle \mathbf{u}_\perp \rangle = O(\varepsilon)$, (F.8) is $O(\varepsilon^2)$ and is therefore dropped, leading to

$$\mathbf{e}_Z \times \langle \mathbf{u}_\perp \rangle - \frac{\partial^2 \langle \mathbf{u}_\perp \rangle}{\partial \xi^2} = -\langle \mathbf{u}.\nabla\mathbf{u}_\perp \rangle - \nabla_\perp \langle \lambda \rangle \quad \text{(F.9)}$$

Writing

$$\mathbf{u}.\nabla\mathbf{u}_\perp = \mathbf{u}_\perp.\nabla_\perp\mathbf{u}_\perp + \overline{Re}^{1/2} u_\xi \frac{\partial \mathbf{u}_\perp}{\partial \xi} \tag{F.10}$$

$\mathbf{u}_\perp$ is evaluated using the leading-order, boundary-layer expression

$$\mathbf{u}_\perp = \varepsilon^{1/2} \sum_{\nu=\mu_+,\,\mu_+^*} A_\nu e^{-i\omega^{(\nu)}t} \mathbf{w}_\perp^{(\nu)} \tag{F.11}$$

where

$$\mathbf{w}_\perp^{(\mu)} = \mathbf{u}^{(\mu)} - \frac{1}{2}\left(\left(\mathbf{u}^{(\mu)} + i\mathbf{e}_Z \times \mathbf{u}^{(\mu)}\right)e^{-\gamma_+^{(\mu)}\xi} + \left(\mathbf{u}^{(\mu)} - i\mathbf{e}_Z \times \mathbf{u}^{(\mu)}\right)e^{-\gamma_-^{(\mu)}\xi}\right) \tag{F.12}$$

is the Ekman-layer profile of mode μ (recall equation (I.B.3)), $\mathbf{u}^{(\mu)}$ is evaluated at the wall, and $\gamma_\pm^{(\mu)}$ are given by (I.B.4). The normal velocity u_ξ in (F.10) is expressed using (F.4), (F.11) and the boundary condition $u_\xi = 0$ at $\xi = 0$ as

$$u_\xi = \varepsilon^{1/2}\,\overline{Re}^{-1/2} \sum_{\nu=\mu_+,\,\mu_+^*} A_\nu e^{-i\omega^{(\nu)}t} w^{(\nu)} \tag{F.13}$$

where $w^{(\mu)}$ is defined by

$$\frac{\partial w^{(\mu)}}{\partial \xi} = -\nabla_\perp . \mathbf{w}_\perp^{(\mu)} \tag{F.14}$$

and $w^{(\mu)} = 0$ at $\xi = 0$. The fast-time average of (F.10) can now be calculated, leading to

$$\left\langle \mathbf{u}.\nabla\mathbf{u}_\perp \right\rangle = \varepsilon\, |A_+|^2 \mathbf{f}(\xi) \tag{F.15}$$

where

$$\mathbf{f} = 2\,\mathrm{Re}\left\{ \mathbf{w}_\perp^{(\mu_+)^*} . \nabla_\perp \mathbf{w}_\perp^{(\mu_+)} + w^{(\mu_+)^*} \frac{\partial \mathbf{w}_\perp^{(\mu_+)}}{\partial \xi} \right\} \tag{F.16}$$

and Re denotes the real part. Thus, (F.9) yields the leading-order, mean-flow, boundary-layer equation

$$\mathbf{e}_Z \times \langle \mathbf{u}_\perp \rangle - \frac{\partial^2 \langle \mathbf{u}_\perp \rangle}{\partial \xi^2} = -\varepsilon \, |A_+|^2 \mathbf{f}(\xi) - \nabla_\perp \langle \lambda \rangle \tag{F.17}$$

in which $\langle \lambda \rangle$ is independent of ξ and imposed from outside the layer, while the other term on the right represents mean-flow forcing by nonlinear interactions of the primary modes within the boundary-layer.

Equation (F.17) is to be solved for $\langle \mathbf{u}_\perp \rangle$ subject to the wall boundary condition $\langle \mathbf{u}_\perp \rangle = 0$ at $\xi = 0$ and the matching condition that

$$\langle \mathbf{u}_\perp \rangle \to \varepsilon \sum_\mu c_\mu \, \mathbf{u}^{(\mu)} \tag{F.18}$$

as $\xi \to \infty$, where $\mathbf{u}^{(\mu)}$ is to be evaluated at the wall. The viscous term in (F.17) is negligible outside the layer, allowing the determination of $\nabla_\perp \langle \lambda \rangle$, independent of ξ, by taking the $\xi \to \infty$ limit and using (F.18). Substituting the result into (F.17), we have

$$\mathbf{e}_Z \times \langle \mathbf{u}_\perp \rangle - \frac{\partial^2 \langle \mathbf{u}_\perp \rangle}{\partial \xi^2} = \varepsilon \left\{ \sum_\mu c_\mu \mathbf{e}_Z \times \mathbf{u}^{(\mu)} + |A_+|^2 \big(\mathbf{f}(\infty) - \mathbf{f}(\xi) \big) \right\} \tag{F.19}$$

with the boundary conditions on $\langle \mathbf{u}_\perp \rangle$ given above. Recalling that the non-zero c_μ are given by either (F.2) or (F.3), the solution of the above problem can be shown to be

$$\langle \mathbf{u}_\perp \rangle = \varepsilon \sum_{\mu \in M_g} \mathcal{A}_\mu \, \mathbf{w}_\perp^{(\mu)} + \varepsilon \, |A_+|^2 \left\{ 2 \sum_{\mu \in M_2} \frac{\Lambda_{\mu\mu_+\mu_+^*}}{\omega^{(\mu)}} \, \mathbf{u}^{(\mu)} - \mathbf{W} \right\} \tag{F.20}$$

where, as before, M_g denotes all geostrophic modes and M_2 the modal family $n = 0$, $m = 2m_+$ and $\mathbf{W}$ is determined from

$$\mathbf{e}_Z \times \mathbf{W} - \frac{\partial^2 \mathbf{W}}{\partial \xi^2} = \mathbf{f}(\xi) - \mathbf{f}(\infty) \tag{F.21}$$

with the boundary conditions $\mathbf{W} \to 0$ as $\xi \to \infty$ and

$$\mathbf{W} = 2 \sum_{\mu \in M_2} \frac{\Lambda_{\mu\mu_+\mu_+^*}}{\omega^{(\mu)}} \, \mathbf{u}^{(\mu)} \tag{F.22}$$

at $\xi = 0$.

The surface integral in (B.2) is evaluated using

$$\mathbf{u}^{(\sigma)^*}.\left(\mathbf{n}.\nabla\langle\mathbf{u}\rangle\right) = -\overline{Re}^{1/2}\mathbf{u}^{(\sigma)^*}.\left.\frac{\partial\langle\mathbf{u}_\perp\rangle}{\partial\xi}\right|_{\xi=0} \tag{F.23}$$

and (F.20). Each term of the geostrophic sum in (F.20) contributes $-\varepsilon^{-1}\overline{Re}^{-1/2}D_{\sigma\mu}\mathcal{A}_\mu$ to (B.2), where $D_{\sigma\mu}$ is given by equation (I.B.5). However, as noted in appendix I.B, $D_{\sigma\mu} = 0$ for geostrophic modes $\sigma \neq \mu$, so the only term in the sum which is nonzero is $\mu = \sigma$. The first sum inside the brackets of (F.20) is independent of ξ and so does not contribute to (F.23). Thus, we obtain (B.2) with

$$\Gamma_\sigma^{(1)} = \int_{Z=0,h_0} \mathbf{u}^{(\sigma)^*}.\left.\frac{\partial\mathbf{W}}{\partial\xi}\right|_{\xi=0} d^2\mathbf{X} \tag{F.24}$$

Note that $\mathbf{W}$ is a sum of axisymmetric modes according to (F.22), hence the r, θ and Z components of $\frac{\partial\mathbf{W}}{\partial\xi}$ are independent of θ, whereas those of $\mathbf{u}^{(\sigma)^*}$ have $e^{-in_\sigma\theta}$ dependence. Integrating over θ, $\Gamma_\sigma^{(1)} = 0$ unless $n_\sigma = 0$. For this reason, we restrict attention to $\sigma \in M$ in the remainder of this appendix (this, among other things, makes $\mathbf{u}^{(\sigma)}$ real).

It is perhaps worth noting that (F.20)-(F.22) describe the response of the mean-flow boundary layer to three forcing mechanisms: a) the geostrophic component, (F.1) with (F.2), of the mean flow outside the layer, represented by the first term in (F.20), b) the forced component, (F.1) with (F.3), of the mean flow outside the layer, corresponding to the first term in brackets in (F.20) and the right-hand side of (F.22), and c) mean nonlinear forcing within the layer, expressed by the right-hand side of (F.21). Of these, as we have seen, the first leads to the damping term in (B.2), while (b) and (c) both contribute to the nonlinear term in (B.2) via (F.24). Whereas (b) expresses effects of nonlinearity outside the layer, effects which are responsible for generating the forced component of the mean flow there, (c) arises from nonlinearity within the boundary layer. Both types force the geostrophic flow via the mean viscous stress on the end walls.

F.2 Calculation of $\Gamma_\sigma^{(1)}$ and Γ_σ

The quantity

$$\mathbf{v} = e^{-2^{-1/2}\xi}\left\{\mathbf{u}^{(\sigma)}\cos\left(2^{-1/2}\xi\right) + \mathbf{e}_Z \times \mathbf{u}^{(\sigma)}\sin\left(2^{-1/2}\xi\right)\right\} \tag{F.25}$$

can be shown to satisfy

$$\mathbf{e}_Z \times \mathbf{v} + \frac{\partial^2\mathbf{v}}{\partial\xi^2} = 0 \tag{F.26}$$

We scalar multiply (F.21) by $\mathbf{v}$ and integrate from $\xi = 0$ to $\xi = \infty$ by parts using (F.26). The values of $\mathbf{W}$, $\mathbf{v}$ and $\dfrac{\partial \mathbf{v}}{\partial \xi}$ at $\xi = 0$ which arise are evaluated using (F.22) and (F.25), leading to

$$\mathbf{u}^{(\sigma)} \boldsymbol{.} \frac{\partial \mathbf{W}}{\partial \xi}\bigg|_{\xi=0} = 2^{1/2} \sum_{\lambda \in M_2} \frac{\Lambda_{\lambda\mu_+\mu_+^*}}{\omega^{(\lambda)}} \mathbf{u}^{(\lambda)} \boldsymbol{.} \left(\mathbf{e}_Z \times \mathbf{u}^{(\sigma)} - \mathbf{u}^{(\sigma)}\right) + \int_0^\infty \left(\mathbf{f}(\xi) - \mathbf{f}(\infty)\right) \boldsymbol{.} \mathbf{v} \; d\xi \tag{F.27}$$

for the integrand in (F.24), where $\mathbf{u}^{(\lambda)}$ and $\mathbf{u}^{(\sigma)}$ are to be evaluated at the wall. The two terms on the right of (F.27) imply two components of $\Gamma_\sigma^{(1)}$, corresponding to the two forcing mechanisms (b) and (c) described above and which we examine separately in the remainder of this appendix.

The terms in the sum of (F.27) can be expressed using (I.3.4) for $\mathbf{u}^{(\lambda)}$ and $\mathbf{u}^{(\sigma)}$ and (E.1)-(E.6), the resulting contributions to the integral in (F.24) being evaluated using (E.7). Only the two terms with $k^{(\lambda)} = k^{(\sigma)}$ yield nonzero contributions, leading to

$$\Gamma_\sigma^{(1,1)} = -\frac{2^{5/2}}{h_0 \omega^{(\lambda)}} \Lambda_{\lambda\mu_+\mu_+^*} \tag{F.28}$$

where $\lambda \in M_2$ is such that $k^{(\lambda)} = k^{(\sigma)}$. Using (E.13), (E.20) and $I_\lambda = I_\sigma$, this result can be rewritten as

$$\Gamma_\sigma^{(1,1)} = \frac{2^{1/2}\left(k^{(\sigma)2} - 4k^{(\mu_+)2}\right)}{h_0 \omega_+ k^{(\mu_+)2}} \Lambda_{\sigma\mu_+\mu_+} \tag{F.29}$$

for the contribution to $\Gamma_\sigma^{(1)}$ arising from the sum in (F.27).

The integral in (F.27) is more complicated to calculate. Using (I.3.4) with (E.1)-(E.6) in (F.12), we find

$$\mathbf{w}_\perp^{(\mu_+)} = k^{(\mu_+)} N^{(\mu_+)} \cos\left(\frac{m_+ \pi Z}{h_0}\right) \mathbf{F}(\xi) \; J_0'\left(k^{(\mu_+)} r\right) \tag{F.30}$$

where $Z = 0$ or $Z = h_0$, depending on which end wall is considered, and the components of $\mathbf{F}$ are given by (A.7), (A.8) and $F_Z = 0$. Using (F.30) on the right-hand side of (F.14), the Bessel differential equation and integration with respect to ξ yield

$$w^{(\mu_+)} = k^{(\mu_+)2} N^{(\mu_+)} \cos\left(\frac{m_+ \pi Z}{h_0}\right) \mathcal{F}(\xi)\; J_0\left(k^{(\mu_+)} r\right) \tag{F.31}$$

where $\mathcal{F}(\xi)$ is given by (A.9). Employing (F.30) and (F.31) in (F.16) and evaluating the second derivatives of Bessel functions which arise using the Bessel differential equation, we find

$$f_r = 2k^{(\mu_+)2} N^{(\mu_+)2} \,\mathrm{Re}\left\{ k^{(\mu_+)} \left(\mathcal{F}^* \frac{dF_r}{d\xi} - |F_r|^2 \right) J_0\left(k^{(\mu_+)} r\right) J_0'\left(k^{(\mu_+)} r\right) - \left(|F_r|^2 + |F_\theta|^2\right) \frac{J_0'^2\left(k^{(\mu_+)} r\right)}{r} \right\} \tag{F.32}$$

$$f_\theta = 2k^{(\mu_+)3} N^{(\mu_+)2} \,\mathrm{Re}\left\{ \mathcal{F}^* \frac{dF_\theta}{d\xi} - F_r^* F_\theta \right\} \; J_0\left(k^{(\mu_+)} r\right) J_0'\left(k^{(\mu_+)} r\right), \tag{F.33}$$

where Re denotes the real part. (F.32) and (F.33) are used in the integral of (F.27), with $\mathbf{v}$ expressed via (F.25). The result is, in turn, introduced into (F.24) and the integrals over r evaluated in terms of $\Lambda_{\sigma\mu_+\mu_+}$ using (A.2), (E.12) and (E.13) to obtain the second component of $\Gamma_\sigma^{(1)}$, which is added to (F.29), (G.16) and (H.7) to derive the final expression, (A.4), for Γ_σ. Since, as shown in the corresponding appendices, each of the components $\Gamma_\sigma^{(1)}$, $\Gamma_\sigma^{(2)}$ and $\Gamma_\sigma^{(3)}$ are zero if $n_\sigma \neq 0$, the same is true of Γ_σ, as stated in appendix B.

Appendix G: Derivation of (B.7), (B.10) and calculation of $\Gamma_\sigma^{(2)}$

Throughout this appendix, μ_+ is axisymmetric and σ geostrophic.

G.1 Derivation of (B.7)

The sum in (B.6) has three components which are expressed using (2.4) and (3.2):

$$\sum_{\lambda_1,\lambda_2} \Lambda_{\sigma\lambda_1\lambda_2} \left\langle B_{\lambda_1}^{[1]} B_{\lambda_2}^{[1]} \right\rangle = \sum_{\nu_1,\nu_2=\mu_+,\mu_+^*} \Lambda_{\sigma\nu_1\nu_2} \left\langle A_{\nu_1} A_{\nu_2} e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}\right)t} \right\rangle \tag{G.1}$$

$$\sum_{\lambda_1,\lambda_2} \Lambda_{\sigma\lambda_1\lambda_2} \left\langle B_{\lambda_1}^{[1]} B_{\lambda_2}^{[2]} \right\rangle = \sum_{\substack{\lambda \\ \nu=\mu_+,\mu_+^*}} \Lambda_{\sigma\nu\lambda} A_\nu \mathcal{A}_\lambda \left\langle e^{-i\left(\omega^{(\nu)}+\omega^{(\lambda)}\right)t} \right\rangle + \sum_{\substack{\lambda \\ \nu_1,\nu_2,\nu_3= \\ \mu_+,\mu_+^*}} \frac{\Lambda_{\sigma\nu_1\lambda}\Lambda_{\lambda\nu_2\nu_3} A_{\nu_1} A_{\nu_2} A_{\nu_3}}{\omega^{(\lambda)} - \omega^{(\nu_2)} - \omega^{(\nu_3)}} \left\langle e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}+\omega^{(\nu_3)}\right)t} \right\rangle \tag{G.2}$$

$$\sum_{\lambda_1,\lambda_2}\Lambda_{\sigma\lambda_1\lambda_2}\left\langle B_{\lambda_1}^{[2]}B_{\lambda_2}^{[2]}\right\rangle = \sum_{\lambda_1,\lambda_2}\Lambda_{\sigma\lambda_1\lambda_2}\mathcal{A}_{\lambda_1}\mathcal{A}_{\lambda_2}\left\langle e^{-i\left(\omega^{(\lambda_1)}+\omega^{(\lambda_2)}\right)t}\right\rangle$$
$$+2\sum_{\substack{\lambda_1,\lambda_2\\ \nu_1,\nu_2=\mu_+,\mu_+^*}}\frac{\Lambda_{\sigma\lambda_1\lambda_2}\Lambda_{\lambda_1\nu_1\nu_2}A_{\nu_1}A_{\nu_2}\mathcal{A}_{\lambda_2}}{\omega^{(\lambda_1)}-\omega^{(\nu_1)}-\omega^{(\nu_2)}}\left\langle e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}+\omega^{(\lambda_2)}\right)t}\right\rangle$$
$$+\sum_{\substack{\lambda_1,\lambda_2\\ \nu_1,\nu_2,\nu_3,\nu_4=\\ \mu_+,\mu_+^*}}\frac{\Lambda_{\sigma\lambda_1\lambda_2}\Lambda_{\lambda_1\nu_1\nu_2}\Lambda_{\lambda_2\nu_3\nu_4}A_{\nu_1}A_{\nu_2}A_{\nu_3}A_{\nu_4}}{\left(\omega^{(\lambda_1)}-\omega^{(\nu_1)}-\omega^{(\nu_2)}\right)\left(\omega^{(\lambda_2)}-\omega^{(\nu_3)}-\omega^{(\nu_4)}\right)}\left\langle e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}+\omega^{(\nu_3)}+\omega^{(\nu_4)}\right)t}\right\rangle \quad \text{(G.3)}$$

The time average in the sum of (G.1) is zero unless $\nu_1 = \nu_2^*$, but in that case $\Lambda_{\sigma\nu_1\nu_2} = 0$ from (2.8), making (G.1) zero. Likewise, the time average in the first sum of (G.2) is nonzero when $\lambda = \nu^*$, but then $\Lambda_{\sigma\nu\lambda} = 0$, while the frequency of the exponential in the second sum of (G.2) takes one of the non-zero values $\pm\omega_+$ or $\pm 3\omega_+$, so its time average is zero. We conclude that neither (G.1) nor (G.2) contributes to (B.6).

Turning attention to (G.3), the time average in the first sum is nonzero if both λ_1 and λ_2 are geostrophic, yielding the right-hand side of (B.7). It is also nonzero if $\lambda_1 = \lambda_2^*$, but then $\Lambda_{\sigma\lambda_1\lambda_2} = 0$. The second sum has a nonzero average when $\nu_1 = \nu_2^*$ and $m_{\lambda_2} = 0$. In that case, the condition (2.7) for nonzero $\Lambda_{\sigma\lambda_1\lambda_2}$ implies $m_{\lambda_1} = 0$, so $\Lambda_{\lambda_1\nu_1\nu_2} = 0$ from (2.8). Applying condition (2.7) to the third sum, it is zero unless $n_\sigma = 0$, implying $\sigma^* = \sigma$. The average in the third sum is nonzero if ν_1, ν_2, ν_3 and ν_4 consist of some permutation of μ_+, μ_+, μ_+^* and μ_+^*. Using (D.4) and $\sigma^* = \sigma$, such terms are antisymmetric under conjugation of λ_1, λ_2, ν_1, ν_2, ν_3 and ν_4, so they sum to zero.

G.2 Derivation of (B.10)

In what follows, we will need order of magnitude estimates for both $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$ both inside and outside the boundary layers. From its definition, (B.4), $\tilde{\mathbf{u}}$ is $O\left(\varepsilon^{1/2}\right)$ everywhere. On the other hand, $\hat{\mathbf{u}}$ is $O\left(\varepsilon^{3/2}\right)$ outside the layers because we have already subtracted out the first two orders in the definition $\hat{\mathbf{u}} = \mathbf{u} - \tilde{\mathbf{u}}$. However, within the layers the sum of a large number of high-order modes acting in concert causes $\hat{\mathbf{u}}$ to rise to $O\left(\varepsilon^{1/2}\right)$. This is so because $\hat{\mathbf{u}}$, rather than $\tilde{\mathbf{u}}$, contains the layers, and hence must increase from $O\left(\varepsilon^{3/2}\right)$ outside the layers to express the velocity variations across the layer.

Examining the second integral on the right of (B.5), the contribution from the side-wall boundary layer can be shown to be negligible as follows. As usual in boundary layers, the component of velocity normal to the wall is asymptotically smaller than the tangential component, $O\left(\varepsilon^{1/2}\right)$. In particular, both $\tilde{u}_r$ and $\hat{u}_r$ are $O\left(\varepsilon^{3/2}\right)$ in the side-wall layer. Writing the integrand using cylindrical coordinates and recalling that $\mathbf{u}^{(\sigma)} = 0$ at the side wall (so $\mathbf{u}^{(\sigma)} = O(\varepsilon)$ within the layer), the integrand is $O\left(\varepsilon^2\right)$ and thus the contribution of the side-wall layer to the integral is only $O(\varepsilon^3)$, negligible to the order we are working. In consequence, modes with large $k^{(\lambda)} = O(\varepsilon^{-1})$, needed to represent the side-wall layer, are unimportant here.

Given the orders of magnitude of $\hat{\mathbf{u}}$ inside and outside the boundary layers, the leading-order expression $\tilde{\mathbf{u}} \sim \varepsilon^{1/2} \sum_{\nu=\mu_+,\mu_+^*} A_\nu e^{-i\omega^{(\nu)}t}\mathbf{u}^{(\nu)}$ is sufficiently accurate for the evaluation of the second integral on the right of (B.5), leading to (B.8). As we saw above, the contribution of high-order λ, making up the side-wall boundary layers, is negligible. Furthermore, according to (2.7), only modes with $m_\lambda = m_+$ contribute to the sum in (B.8), excluding high-order ones with $m_\lambda \pi / h_0 = O(\varepsilon^{-1})$ which form the end-wall boundary layers. We conclude that high-order modes are negligible and hence $\hat{B}_\lambda$ can be replaced by $\varepsilon^{3/2} B_\lambda^{[3]}$ in (B.8), leading to (B.9), with $B_\lambda^{[3]}$ from (B.1).

Using $\omega_0 = 2\omega_+ + \varepsilon\Delta$ and equation (B.1) in the sum on the right-hand side of (B.9),

$$
\begin{aligned}
\sum_{\substack{\lambda \\ \nu=\mu_+,\mu_+^*}} \Lambda_{\sigma\nu\lambda} A_\nu \left\langle B_\lambda^{[3]} e^{-i\omega^{(\nu)}t} \right\rangle = & \sum_{\substack{\lambda \\ \nu=\mu_+,\mu_+^*}} \Lambda_{\sigma\nu\lambda} A_\nu \mathcal{B}_\lambda \left\langle e^{-i\left(\omega^{(\nu)}+\omega^{(\lambda)}\right)t} \right\rangle \\
- \sum_{\substack{\lambda \\ \nu_1,\nu_2=\mu_+,\mu_+^*}} & \Lambda_{\sigma\nu_2\lambda} C_{\lambda\nu_1} A_{\nu_1} A_{\nu_2} \left\{ \frac{\omega^{(\nu_1)} - 2\omega_+}{\omega^{(\nu_1)} - \omega^{(\lambda)} - 2\omega_+} e^{i\Delta T} \left\langle e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}-2\omega_+\right)t} \right\rangle \right. \\
& \left. + \frac{\omega^{(\nu_1)} + 2\omega_+}{\omega^{(\nu_1)} - \omega^{(\lambda)} + 2\omega_+} e^{-i\Delta T} \left\langle e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}+2\omega_+\right)t} \right\rangle \right\} \\
& -i\varepsilon^{-1}\overline{Re}^{-1/2} \sum_{\substack{\lambda \\ \nu_1,\nu_2=\mu_+,\mu_+^*}} \frac{\Lambda_{\sigma\nu_2\lambda} D_{\lambda\nu_1}}{\omega^{(\nu_1)} - \omega^{(\lambda)}} A_{\nu_1} A_{\nu_2} \left\langle e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}\right)t} \right\rangle \\
& -2 \sum_{\substack{\lambda_1,\lambda_2 \\ \nu_1,\nu_2=\mu_+,\mu_+^*}} \frac{\Lambda_{\sigma\nu_2\lambda_2} \Lambda_{\lambda_2\nu_1\lambda_1}}{\omega^{(\nu_1)} + \omega^{(\lambda_1)} - \omega^{(\lambda_2)}} A_{\nu_1} A_{\nu_2} \mathcal{A}_{\lambda_1} \left\langle e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}+\omega^{(\lambda_1)}\right)t} \right\rangle \\
-\frac{1}{2} \sum_{\substack{\lambda \\ \nu_1,\nu_2,\nu_3,\nu_4= \\ \mu_+,\mu_+^*}} & \frac{\Lambda_{\sigma\nu_4\lambda} F_{\lambda\nu_1\nu_2\nu_3}}{\omega^{(\nu_1)} + \omega^{(\nu_2)} + \omega^{(\nu_3)} - \omega^{(\lambda)}} A_{\nu_1} A_{\nu_2} A_{\nu_3} A_{\nu_4} \left\langle e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}+\omega^{(\nu_3)}+\omega^{(\nu_4)}\right)t} \right\rangle. \qquad \text{(G.4)}
\end{aligned}
$$

The sums on the right of (G.4) are treated as follows:

i) The time average in the first sum is nonzero when $\lambda = \nu^*$, but then $\Lambda_{\sigma\nu\lambda} = 0$ from (2.8).

ii) In the second sum, the time averages are both zero unless $\nu_1 = \nu_2$, while $C_{\lambda\nu_1} \neq 0$ requires $\lambda = \nu_1$ or $\lambda = \nu_1^*$ (recall properties i and iii following (I.3.10)). The case $\lambda = \nu_1^* = \nu_2^*$ gives zero $\Lambda_{\sigma\nu_2\lambda}$ by (2.8), so we focus on $\lambda = \nu_1 = \nu_2$. Of the two time averages, the first is nonzero if $\lambda = \nu_1 = \nu_2 = \mu_+$ and the second if $\lambda = \nu_1 = \nu_2 = \mu_+^*$. Using $C_{\mu_+\mu_+} = C_{\mu_+^*\mu_+^*} = -C$ (which follows from (I.A.10) and the definition, $C = C_{\mu_+\mu_+^*} = C_{\mu_+^*\mu_+}$, of the coefficient C), the corresponding terms in the second sum of (G.4) are evaluated to obtain

$$\frac{1}{2}C\left(\Lambda_{\sigma\mu_+\mu_+}A_+^2 e^{i\Delta T} + \Lambda_{\sigma\mu_+^*\mu_+^*}A_+^{*2}e^{-i\Delta T}\right) \tag{G.5}$$

as the piston-motion contribution to (G.4).

iii) The time average in the third sum of (G.4) is nonzero if either $\nu_1 = \nu_2^* = \mu_+$ or $\nu_1 = \nu_2^* = \mu_+^*$, leading to

$$-\frac{1}{2}i\varepsilon^{-1}\overline{Re}^{-1/2}\Gamma_\sigma^{(2)}\left|A_+\right|^2 \tag{G.6}$$

for the viscous contribution to (G.4), where

$$\Gamma_\sigma^{(2)} = 2\left\{\sum_{\lambda\neq\mu_+}\frac{\Lambda_{\sigma\mu_+^*\lambda}D_{\lambda\mu_+}}{\omega_+ - \omega^{(\lambda)}} - \sum_{\lambda\neq\mu_+^*}\frac{\Lambda_{\sigma\mu_+\lambda}D_{\lambda\mu_+^*}}{\omega_+ + \omega^{(\lambda)}}\right\} \tag{G.7}$$

in which terms with a division by zero have been excluded from the sums, in keeping with the remark following equation (B.1).

iv) The time average in the fourth sum of (G.4) is nonzero if $\nu_1 = \nu_2^*$ and λ_1 is geostrophic. Thus, the contribution to (G.4) is

$$\frac{1}{2}\left|A_+\right|^2\sum_{\lambda\in M_g}\Xi_{\sigma\lambda}\mathcal{A}_\lambda \tag{G.8}$$

where

$$\Xi_{\sigma\lambda} = 4\left\{\sum_{\mu\neq\mu_+^*}\frac{\Lambda_{\sigma\mu_+\mu}\Lambda_{\mu\mu_+^*\lambda}}{\omega_+ + \omega^{(\mu)}} - \sum_{\mu\neq\mu_+}\frac{\Lambda_{\sigma\mu_+^*\mu}\Lambda_{\mu\mu_+\lambda}}{\omega_+ - \omega^{(\mu)}}\right\} \tag{G.9}$$

and, once again, terms with a division by zero have been dropped. Note that $\Xi_{\sigma\lambda}$ is only defined for geostrophic σ and λ.

v) Using (2.7) and (3.5) for the fifth sum, $F_{\lambda\nu_1\nu_2\nu_3} = 0$ unless $n_\lambda = 0$, but then $\Lambda_{\sigma\nu_4\lambda} \neq 0$ requires $n_\sigma = 0$, implying $\sigma^* = \sigma$. The time average in the fifth sum is nonzero if ν_1, ν_2, ν_3 and ν_4 consist of some permutation of μ_+, μ_+, μ_+^* and μ_+^*. Using (D.4), (3.5) and $\sigma^* = \sigma$, such terms are antisymmetric under conjugation of λ, ν_1, ν_2, ν_3 and ν_4, so they sum to zero.

Finally, the sum of the contributions (G.5), (G.6) and (G.8) gives (B.10).

G.3 Some properties of $\Xi_{\sigma\lambda}$

Changing summation index from μ to μ^* in the second sum of (G.9),

$$\Xi_{\sigma\lambda} = 4\sum_{\mu\neq\mu_+^*} \frac{\Lambda_{\sigma\mu_+\mu}\Lambda_{\mu\mu_+^*\lambda} - \Lambda_{\sigma\mu_+^*\mu^*}\Lambda_{\mu^*\mu_+\lambda}}{\omega_+ + \omega^{(\mu)}}. \tag{G.10}$$

From (2.7), it follows that all terms in (G.10) are zero unless $n_\sigma = n_\lambda$. Thus, $\Xi_{\sigma\lambda} = 0$ unless $n_\sigma = n_\lambda$, i.e. $\Xi_{\sigma\lambda}$ only couples modes of the same n. If $n_\sigma = 0$ or $n_\lambda = 0$, we deduce that $n_\sigma = n_\lambda = 0$, otherwise $\Xi_{\sigma\lambda} = 0$. Given $n_\sigma = n_\lambda = 0$, both modes are real and (D.4) implies $\Lambda_{\sigma\mu_+\mu}\Lambda_{\mu\mu_+^*\lambda} = \Lambda_{\sigma\mu_+^*\mu^*}\Lambda_{\mu^*\mu_+\lambda}$, hence $\Xi_{\sigma\lambda} = 0$ from (G.10). We conclude that $\Xi_{\sigma\lambda} = 0$ if either $n_\sigma = 0$ or $n_\lambda = 0$, as stated in appendix B.

G.4 Calculation of $\Gamma_\sigma^{(2)}$

Since $D_{\lambda\mu} = 0$ if $n_\lambda \neq n_\mu$, only terms with $n_\lambda = 0$ contribute to (G.7). From (2.7) it follows that $\Gamma_\sigma^{(2)} = 0$ if $n_\sigma \neq 0$, hence we specialise to axisymmetric σ in the remainder of this appendix.

Changing the summation index from λ to λ^* in the first sum of (G.7) and using $\sigma^* = \sigma$,

$$\Gamma_\sigma^{(2)} = 2\sum_{\lambda\neq\mu_+^*} \frac{\Lambda_{\sigma^*\mu_+^*\lambda^*}D_{\lambda^*\mu_+} - \Lambda_{\sigma\mu_+\lambda}D_{\lambda\mu_+^*}}{\omega_+ + \omega^{(\lambda)}} \tag{G.11}$$

Employing (D.4) and the identity $D_{\mu\nu}^* = D_{\mu^*\nu^*}$ (which follows from the complex conjugates of (I.B.4), (I.B.5)),

$$\Gamma_\sigma^{[2]} = -4\sum_{\lambda\neq\mu_+^*} \frac{\Lambda_{\sigma\mu_+\lambda}D_{\lambda\mu_+^*}^r}{\omega_+ + \omega^{(\lambda)}} \tag{G.12}$$

where $D_{\lambda\mu_+^*}^r$ denotes the real part of $D_{\lambda\mu_+^*}$. According to (2.7), we may restrict the sum in (G.12) to λ in the modal family $n_\lambda = 0$, $m_\lambda = m_+$. The quantity $D_{\lambda\mu_+^*}^r$ is determined for that modal family from (I.B.5) using (I.3.4) and (E.1)-(E.7). We find that

$$D_{\lambda\mu_+^*}^r = -\operatorname{sgn}\left(\omega^{(\lambda)}\right)\left(\frac{1}{2}\omega_+\left(1-\omega_+^2\right)\left(1-\omega^{(\lambda)2}\right)\right)^{1/2} \tag{G.13}$$

for $\lambda \neq \mu_+$ and

$$D^r_{\mu_+\mu_+^*} = 2^{-1/2}\left(1-\omega_+^2\right)\left\{\frac{1}{h_0}\left(\left(1+\omega_+\right)^{1/2}+\left(1-\omega_+\right)^{1/2}\right)-\omega_+^{1/2}\right\} \tag{G.14}$$

when $\lambda = \mu_+$. The coefficients $\Lambda_{\sigma\mu_+\lambda}$ in (G.12) are calculated from (D.2), (D.3) and (E.1)-(E.6) as

$$\Lambda_{\sigma\mu_+\lambda} = -\frac{\omega_+ + \omega^{(\lambda)}}{\left(\pi h_0\right)^{1/2} J_0\left(k^{(\mu_+)}\right) J_0\left(k^{(\lambda)}\right) J_0\left(k^{(\sigma)}\right)} \int_0^1 J_0'\left(k^{(\mu_+)}r\right) J_0'\left(k^{(\lambda)}r\right)\left(J_0'\left(k^{(\sigma)}r\right)+\frac{1}{2}k^{(\sigma)}rJ_0\left(k^{(\sigma)}r\right)\right)dr \tag{G.15}$$

Using (G.13) and (G.15) in (G.12), it is apparent that the terms in the sum arising from mode λ and its conjugate (which have the same $k^{(\lambda)}$, but opposite signs for $\omega^{(\lambda)}$) cancel unless $\lambda = \mu_+$ or μ_+^*. The integral in (G.15) is evaluated in terms of $\Lambda_{\sigma\mu_+\mu_+}$ for $\lambda = \mu_+$ and μ_+^* using (E.8) and (E.10) with $k_1 = k^{(\mu_+)}$, $k_2 = k^{(\sigma)}$, together with (E.12) and (E.13). Employing the results in (G.12), as well as (G.13) and (G.14), we obtain

$$\Gamma_\sigma^{(2)} = 2^{1/2}\,\omega_+^{-1}\left(1-\omega_+^2\right)\Lambda_{\sigma\mu_+\mu_+}\left\{\omega_+^{1/2}-\frac{1}{h_0}\left(\left(1+\omega_+\right)^{1/2}+\left(1-\omega_+\right)^{1/2}\right)\right\} \tag{G.16}$$

Appendix H: Derivation of (B.11) and calculation of $\Gamma_\sigma^{(3)}$

Throughout this appendix, μ_+ is axisymmetric and σ geostrophic.

H.1 Derivation of (B.11)

Since, as discussed at the beginning of appendix G, $\hat{\mathbf{u}} = O\left(\varepsilon^{3/2}\right)$ outside the boundary layers, the contribution to the third integral on the right of (B.5) from that region is negligible, as is the side-wall layer contribution following reasoning similar to that used in the second paragraph of section F.2. Thus, the integral is dominated by the end-wall boundary layers.

At leading order, (F.11) and (F.12) (with $\mathbf{u}^{(\mu)}$ evaluated at the wall, as it is throughout this appendix) give the velocity in the boundary layer, whereas

$$\tilde{\mathbf{u}} = \varepsilon^{1/2} \sum_{\nu=\mu_+,\,\mu_+^*} A_\nu e^{-i\omega^{(\nu)}t}\mathbf{u}^{(\nu)} \tag{H.1}$$

(also at leading order). Subtraction of (H.1) from (F.11) gives

$$\hat{\mathbf{u}} = \mathbf{u} - \tilde{\mathbf{u}} \sim \varepsilon^{1/2} \sum_{\nu=\mu_+,\,\mu_+^*} A_\nu e^{-i\omega^{(\nu)} t}\left(\mathbf{w}_\perp^{(\nu)} - \mathbf{u}^{(\nu)}\right). \tag{H.2}$$

Since $\mathbf{u}^{(\sigma)}$ is geostrophic, it is independent of Z, leading to

$$\begin{gathered}\hat{\mathbf{u}}.\left(\hat{\mathbf{u}}.\nabla\right)\mathbf{u}^{(\sigma)^*} = \hat{\mathbf{u}}.\left(\hat{\mathbf{u}}.\nabla_\perp\right)\mathbf{u}^{(\sigma)^*} \sim \\ \varepsilon \sum_{\substack{\nu_1,\nu_2 \\ =\mu_+,\,\mu_+^*}} A_{\nu_1} A_{\nu_2} e^{-i\left(\omega^{(\nu_1)}+\omega^{(\nu_2)}\right)t}\left(\mathbf{w}_\perp^{(\nu_1)} - \mathbf{u}^{(\nu_1)}\right).\left(\left(\mathbf{w}_\perp^{(\nu_2)} - \mathbf{u}^{(\nu_2)}\right).\nabla_\perp\right)\mathbf{u}^{(\sigma)^*}.\end{gathered} \tag{H.3}$$

Taking the time average, only the terms with $\nu_1 = \nu_2^*$ contribute, hence

$$\begin{aligned}\left\langle \hat{\mathbf{u}}.\left(\hat{\mathbf{u}}.\nabla\right)\mathbf{u}^{(\sigma)^*}\right\rangle \sim \varepsilon\left|A_+\right|^2 \Big\{&\left(\mathbf{w}_\perp^{(\mu_+)} - \mathbf{u}^{(\mu_+)}\right).\left(\left(\mathbf{w}_\perp^{(\bar{\mu}_+)} - \mathbf{u}^{(\bar{\mu}_+)}\right).\nabla_\perp\right)\mathbf{u}^{(\sigma)^*} \\ &+\left(\mathbf{w}_\perp^{(\bar{\mu}_+)} - \mathbf{u}^{(\bar{\mu}_+)}\right).\left(\left(\mathbf{w}_\perp^{(\mu_+)} - \mathbf{u}^{(\mu_+)}\right).\nabla_\perp\right)\mathbf{u}^{(\sigma)^*}\Big\}\end{aligned} \tag{H.4}$$

whose volume integral yields (B.11) with

$$\begin{aligned}\Gamma_\sigma^{(3)} = \int_{Z=0,h_0} \int_0^\infty \Big\{&\left(\mathbf{w}_\perp^{(\mu_+)} - \mathbf{u}^{(\mu_+)}\right).\left(\left(\mathbf{w}_\perp^{(\bar{\mu}_+)} - \mathbf{u}^{(\bar{\mu}_+)}\right).\nabla_\perp\right)\mathbf{u}^{(\sigma)^*} \\ &+\left(\mathbf{w}_\perp^{(\bar{\mu}_+)} - \mathbf{u}^{(\bar{\mu}_+)}\right).\left(\left(\mathbf{w}_\perp^{(\mu_+)} - \mathbf{u}^{(\mu_+)}\right).\nabla_\perp\right)\mathbf{u}^{(\sigma)^*}\Big\} d\xi\, d^2\mathbf{X}\end{aligned} \tag{H.5}$$

H.2 Calculation of $\Gamma_\sigma^{(3)}$

Expressing (H.5) in cylindrical coordinates, axisymmetry of mode μ_+ and $e^{-in_\sigma\theta}$ dependence of the components of $\mathbf{u}^{(\sigma)^*}$ imply $e^{-in_\sigma\theta}$ dependence of the integrand. Thus, taking the integral over θ, $\Gamma_\sigma^{(3)} = 0$ unless $n_\sigma = 0$, hence we specialise to axisymmetric σ in the remainder of this appendix.

Since $\mathbf{u}^{(\sigma)}$ is then real, (H.5) yields

$$\Gamma_\sigma^{(3)} = 2\ \mathrm{Re}\left\{ \int_{Z=0,h_0} \int_0^\infty \left(\mathbf{w}_\perp^{(\mu_+)} - \mathbf{u}^{(\mu_+)}\right).\left(\left(\mathbf{w}_\perp^{(\mu_+)} - \mathbf{u}^{(\mu_+)}\right)^*.\nabla\mathbf{u}^{(\sigma)}\right) d\xi\, d^2\mathbf{X} \right\} \tag{H.6}$$

where we have used the identity $\mathbf{w}_\perp^{(\mu^*)} = \mathbf{w}_\perp^{(\mu)^*}$ (which can be derived from (F.12)). Expressing $\mathbf{w}_\perp^{(\mu_+)}$ in (H.6) using (F.12), the integral over ξ is carried out and the resulting surface integrals evaluated using (I.3.4) for $\mathbf{u}^{(\sigma)}$ and $\mathbf{u}^{(\mu_+)}$ (evaluated at the end-walls) and (E.1)-(E.6). The integrals over r which arise are determined from (E.8) and (E.10) with $k_1 = k^{(\mu_+)}$ and $k_2 = k^{(\sigma)}$ and written in terms of $\Lambda_{\sigma\mu_+\mu_+}$ using (E.13). Thus, we obtain

$$\Gamma_{\sigma}^{(3)} = \frac{2^{3/2}\left(1-\omega_{+}^{2}\right)\Lambda_{\sigma\mu_{+}\mu_{+}}}{h_{0}\omega_{+}\left(\left(1+\omega_{+}\right)^{1/2}+\left(1-\omega_{+}\right)^{1/2}\right)} \qquad \text{(H.7)}$$