Appendices to "Two-dimensional resonant piston-like sloshing in a moonpool"

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Appendix A. Realisation of the Galerkin scheme based on the functional basis (3.31)

The use of Poisson integrals makes it possible to get explicit analytical expressions for elements of the matrix problem (3.25)-(3.26). The non-zero elements of the sub-matrices D and p are as follows

$$D_{k,i} = \frac{\pi\Gamma(m_k+1)\Gamma(m_i+1)}{r_k^{(1)} r_i^{(1)}}\Bigg\{\frac{b(h-d)}{4\Gamma(m_k+\frac{3}{2})\Gamma(m_i+\frac{3}{2})}+ + 2^{m_i+m_k}\pi^{-2-m_i-m_k}(h-d)^2\sum_{j=1}^{\infty}\frac{J_{m_k+\frac{1}{2}}(\pi j)J_{m_i+\frac{1}{2}}(\pi j)}{j^{2+m_i+m_k}\tanh(\pi j(b-\frac{1}{2})/(h-d))}+ + 2^{m_i+m_k-1}(h-d)^{1-m_i-m_k}\boxed{\sum_{j=1}^{\infty}\left[\frac{J_{m_k+\frac{1}{2}}(\kappa_j^{(1)}(h-d))J_{m_i+\frac{1}{2}}(\kappa_j^{(1)}(h-d))}{N_j^{(1)}(\kappa_j^{(1)})^{2+m_i+m_k}}\right]}\Bigg\}, \quad i=1,\ldots,N_1; \qquad (\mathrm{A}\,1a)$$

$$D_{k,i+N_1} = -\frac{\pi\Gamma(m_k+1)\Gamma(m_i+1)}{r_k^{(1)} r_i^{(1)}}\times \times\left\{2^{m_i+m_k}\pi^{-2-m_i-m_k}(h-d)^2\sum_{j=1}^{\infty}\frac{J_{m_k+\frac{1}{2}}(\pi j)J_{m_i+\frac{1}{2}}(\pi j)}{j^{2+m_i+m_k}\sinh(\pi j(b-\frac{1}{2})/(h-d))}\right\}, \quad i=1,\ldots,N_2; \qquad (\mathrm{A}\,1b)$$

$$D_{k,N_1+N_2+N_3+2} = -\frac{\sqrt{\pi}\Gamma(m_k+1)(h-d)}{2r_k^{(1)}\Gamma(m_k+\frac{3}{2})} \qquad (\mathrm{A}\,1c)$$

for $k=1,\ldots,N_1$;

$$D_{k+N_1,i} = \frac{\pi\Gamma(m_k+1)\Gamma(m_i+1)}{r_k^{(1)} r_i^{(1)}}\Bigg\{-\frac{(h-d)}{8\Gamma(m_k+\frac{3}{2})\Gamma(m_i+\frac{3}{2})}- - 2^{m_i+m_k}\pi^{-2-m_i-m_k}(h-d)^2\sum_{j=1}^{\infty}\frac{J_{m_k+\frac{1}{2}}(\pi j)J_{m_i+\frac{1}{2}}(\pi j)}{j^{2+m_i+m_k}\sinh(\pi j(b-\frac{1}{2})/(h-d))}\Bigg\}, \quad i=1,\ldots,N_1; \qquad (\mathrm{A}\,2a)$$

$$D_{k+N_1,N_1+i} = \frac{\pi\Gamma(m_k+1)\Gamma(m_i+1)}{r_k^{(1)} r_i^{(1)}}\times$$
$$\times\left\{-\frac{(h-d)^3}{8\Gamma(m_k+\frac{5}{2})\Gamma(m_i+\frac{3}{2})}+\frac{h-d}{16\Gamma(m_k+\frac{3}{2})\Gamma(m_i+\frac{3}{2})}+\right.$$
$$+2^{m_i+m_k}\pi^{-2-m_i-m_k}(h-d)^2\sum_{j=1}^{\infty}\frac{J_{m_k+\frac{1}{2}}(\pi j)J_{m_i+\frac{1}{2}}(\pi j)}{j^{2+m_i+m_k}}\times$$
$$\left.\times\left[\coth\left(\frac{\pi j(b-\frac{1}{2})}{h-d}\right)+\coth\left(\frac{\pi j}{2(h-d)}\right)\right]\right\},\quad i=1,\dots,N_2; \quad (\mathrm{A}\,2b)$$

$$D_{k+N_1,N_1+N_2+i} = -\frac{\Gamma(m_k+1)\Gamma(m_i+1)\ 2^{m_i-\frac{1}{2}}\pi^{-1-m_i-m_k}(h-d)^{\frac{1}{2}-m_k}}{r_k^{(1)} r_i^{(2)}}\times$$
$$\times\sum_{j=1}^{\infty}\frac{(-1)^j\ J_{m_i+\frac{1}{2}}(\pi j)\ \tilde{I}_{m_k+\frac{1}{2}}(2\pi j(h-d))}{j^{2+m_i+m_k}\ (1-\exp(-4\pi j(h-d)))},\quad i=1,\dots,N_3; \quad (\mathrm{A}\,2c)$$

$$D_{k+N_1,N_1+N_2+N_3+1} = -D_{k+N_1,N_1+N_2+N_3+2} = -\frac{\sqrt{\pi}\Gamma(m_k+1)(h-d)}{2r_k^{(1)}\Gamma(m_k+\frac{3}{2})}; \quad (\mathrm{A}\,2d)$$

for $k=1,\dots,N_2$;

$$D_{k+N_1+N_2,N_1+i} = \frac{\pi\Gamma(m_k+1)\Gamma(m_i+1)}{r_k^{(2)} r_i^{(1)}}\left\{\frac{(h-d)^2}{8\Gamma(m_k+\frac{3}{2})\Gamma(m_i+\frac{3}{2})}-\right.$$
$$-\frac{1}{64\Gamma(m_k+\frac{5}{2})\Gamma(m_i+\frac{3}{2})}-2^{2m_k+m_i+\frac{1}{2}}\ \pi^{-2-m_i-m_k}\ (h-d)^{m_k+\frac{3}{2}}\times$$
$$\left.\times\sum_{j=1}^{\infty}\frac{(-1)^j J_{m_i+\frac{1}{2}}(\pi j)\tilde{I}_{m_k+\frac{1}{2}}\left(\frac{\pi j}{2(h-d)}\right)}{j^{2+m_i+m_k}\ (1-\exp(-\frac{\pi j}{h-d}))}\right\},\quad i=1,\dots,N_2; \quad (\mathrm{A}\,3a)$$

$$D_{k+N_1+N_2,N_1+N_2+i} = \frac{\pi\Gamma(m_i+1)\Gamma(m_k+1)}{r_k^{(2)} r_i^{(2)}}\left\{\boxed{\frac{1}{8\Gamma(m_i+\frac{3}{2})\Gamma(m_k+\frac{3}{2})}\left(d-\frac{1}{\Lambda}\right)}+\right.$$
$$+2^{m_i+m_k-2}\pi^{-2-m_i-m_k}\times$$
$$\left.\times\boxed{\sum_{j=1}^{\infty}\frac{J_{m_i+\frac{1}{2}}(\pi j)J_{m_k+\frac{1}{2}}(\pi j)}{j^{2+m_i+m_k}}\times\left(\coth(2\pi j(h-d))-\frac{2\pi j-\Lambda\tanh(2\pi jd)}{\Lambda-2\pi j\tanh(2\pi jd)}\right)}\right\},$$
$$i=1,\dots,N_3; \quad (\mathrm{A}\,3b)$$

$$D_{k+N_1+N_2,N_1+N_2+N_3+1} = \frac{\sqrt{\pi}\Gamma(m_k+1)}{4r_k^{(2)}\Gamma(m_k+\frac{3}{2})}; \quad (\mathrm{A}\,3c)$$

for $k=1,\dots,N_3$,

$$D_{N_1+N_2+N_3+1,i} = \frac{\sqrt{\pi}(h-d)\Gamma(m_i+1)}{2r_i^{(1)}\Gamma(m_i+\frac{3}{2})}, \quad i = 1,\ldots,N_1; \tag{A 4a}$$

$$D_{N_1+N_2+N_3+1,N_1+i} = -\frac{\sqrt{\pi}(h-d)\Gamma(m_i+1)}{2r_i^{(1)}\Gamma(m_i+\frac{3}{2})}, \quad i = 1,\ldots,N_2; \tag{A 4b}$$

$$D_{N_1+N_2+N_3+2,N_1+i} = \frac{\sqrt{\pi}\Gamma(m_i+1)(h-d)}{2r_i^{(1)}\Gamma(m_i+\frac{3}{2})}, \quad i = 1,\ldots,N_2; \tag{A 4c}$$

$$D_{N_1+N_2+N_3+2,N_1+N_2+i} = -\frac{\sqrt{\pi}\Gamma(m_i+1)}{4r_i^{(2)}\Gamma(m_i+\frac{3}{2})}, \quad i = 1,\ldots,N_3 \tag{A 4d}$$

and

$$p_{k,i} = \pi\Gamma(m_k+1)\Gamma(m_i+1)2^{m_k+m_i-1}(h-d)^{1-m_k-m_i}\times$$
$$\times \boxed{\frac{I_{m_k+\frac{1}{2}}(\mathcal{K}(h-d))I_{m_i+\frac{1}{2}}(\mathcal{K}(h-d))}{\mathcal{K}^{m_k+m_i+2}\ N_0\ r_k^{(1)}r_i^{(1)}}}, \quad i,k = 1,\ldots,N_1. \tag{A 5}$$

Here, $\Gamma(\cdot)$ is the gamma-function, $J_\nu(\cdot)$, $\nu \geqslant 0$ is the Bessel function of the second kind, $\tilde{I}_\nu(\cdot) = I_\nu(\cdot)/\exp(\cdot)$, $\nu \geqslant 0$, where $I_\nu(\cdot)$ is the Bessel function of the first kind, and $r_k^{(1)}$, $r_k^{(2)}$ are defined by (3.32).

The vector $\boldsymbol{b}$ in the right-hand side has the following non-zero elements

$$b_{N_1+N_2+N_3+2+i} = \frac{\sqrt{\pi}\Gamma(m_k+1)(h-d)^2}{8\Gamma(m_k+\frac{5}{2})r_k^{(1)}}, \quad i = 1,\ldots,N_1;$$
$$b_{2N_1+N_2+N_3+2+i} = \frac{\sqrt{\pi}\Gamma(m_i+1)}{4r_k^{(1)}}\left(\frac{(b-\frac{1}{2})^2}{\Gamma(m_i+\frac{3}{2})} - \frac{(h-d)^2}{2\Gamma(m_i+\frac{5}{2})}\right), \ i = 1,\ldots,N_2; \tag{A 6}$$
$$b_{2N_1+2N_2+2N_3+3} = \left(b-\frac{1}{2}\right).$$

Appendix B. Tables demonstrating the convergence

q	$d=0.2$	$d=0.5$	$d=1.0$	$d=1.5$	$d=3.0$	$d=4.0$
1	0.54388112	0.47925707	0.39825051	0.33968429	0.23094939	0.18393324
2	0.90821713	0.70529128	0.51430407	0.40477625	0.24354644	0.18698152
3	0.92943583	0.71627085	0.51868087	0.40667755	0.24365607	0.18698395
4	0.93063259	0.71666837	0.51877725	0.40670197	0.24365611	0.18698413
5	0.93064711	0.71666839	0.51877801	0.40670281	0.24365627	0.18698414
6	0.93065773	0.71667048	0.51877880	0.40670310	0.24365627	0.18698414
7	0.93065626	0.71667095	0.51877862	0.40670310	—	—
8	0.93065118	0.71667070	0.51877920	—	—	—
9	0.93065117	0.71667070	0.51877920	—	—	—

TABLE 1. Roots of (3.39) versus $q = N1 = N2 = N3$ and various $0 < d < h$ for the experimental case associated with dimensionless $h = 5.72222$ and $B = 2$.

q	$d = 0.15$	$d = 0.25$	$d = 0.5$	$d = 0.75$	$d = 1.5$	$d = 2.0$
1	0.9060715	0.8392199	0.70082570	0.59796359	0.40003308	0.30846586
2	1.1723834	1.0390523	0.80455059	0.65468361	0.40857502	0.30974655
3	1.1798184	1.0434947	0.80612905	0.65531651	0.40860977	0.30975024
4	1.1802283	1.0436399	0.80615378	0.65532377	0.40861018	0.30975027
5	1.1802403	1.0436425	0.80615464	0.65532410	0.40861018	0.30975027
6	1.1802412	1.0436426	0.80615464	0.65532410	—	—
7	1.1802413	1.0436427	—	—	—	—
8	1.1802411	1.0436429	—	—	—	—
9	1.1802411	1.0436429	—	—	—	—

TABLE 2. The same as in Table 1, but for the experimental case $h = 2.86111$ and $B = 1$.

q	$\tilde{\mathcal{A}}_1^{(1)}$	$\tilde{\mathcal{A}}_1^{(2)}$	$\tilde{\mathcal{A}}_0^{(1)}$	$\tilde{\mathcal{A}}_0^{(2)}$
1	-5.442221	-4.461939	-0.7768481	-0.0659394
2	-1.788784	1.657756	-0.2187737	0.9765799
3	-1.727617	1.684100	-0.2060396	0.9795395
4	-1.726415	1.684566	-0.2058046	0.9796068
5	-1.726404	1.684570	-0.2058026	0.9796074
6	-1.726395	1.684572	-0.2058008	0.9796076
7	-1.726397	1.684572	-0.2058012	0.9796076
8	-1.726390	1.684571	-0.2058001	0.9796074
9	-1.726390	1.684571	-0.2058001	0.9796074

TABLE 3. Convergence to the scaled amplitudes $\tilde{\mathcal{A}}_j^{(i)} = \mathcal{A}_j^{(i)}/\epsilon,\ j = 0, 1; i = 1, 2$ defined by (4.2) and (4.5) versus $q = N1 = N2 = N3$ for $h = 5.72222, d = 1, B = 2$ and $\Lambda = 0.4$. The values of d, h and B are associated with one of our model test cases.

q	$\tilde{\mathcal{A}}_1^{(1)}$	$\tilde{\mathcal{A}}_1^{(2)}$	$\tilde{\mathcal{A}}_0^{(1)}$	$\tilde{\mathcal{A}}_0^{(2)}$
1	0.1764988	-1.153833	0.02494924	0.40232393
2	-0.0227025	-4.600383	-0.00501437	-0.01379929
3	-0.1032949	-4.965190	-0.02252254	-0.06006656
4	-0.1052375	-4.973530	-0.02293699	-0.06113310
5	-0.1052550	-4.973597	-0.02294075	-0.06114273
6	-0.1052727	-4.973662	-0.02294452	-0.06115236
7	-0.1052683	-4.973648	-0.02294358	-0.06114996
8	-0.1052822	-4.973691	-0.02294657	-0.06115752
9	-0.1052822	-4.973691	-0.02294657	-0.06115751

TABLE 4. The same as in Table 3, but for $\Lambda = 0.6$.

q	$\tilde{\mathcal{A}}_1^{(1)}$	$\tilde{\mathcal{A}}_1^{(2)}$	$\tilde{\mathcal{A}}_0^{(1)}$	$\tilde{\mathcal{A}}_0^{(2)}$
1	0.313615	-1.555687	0.045382	0.353708063
2	-10.27102	-6.889712	-1.793831	-0.280838555
3	-12.01568	-5.527816	-2.050071	-0.008747921
4	-12.04921	-5.490236	-2.054792	-0.001675700
5	-12.04948	-5.489935	-2.054830	-0.001619321
6	-12.04972	-5.489616	-2.054865	-0.001561095
7	-12.04967	-5.489693	-2.054857	-0.001574827
8	-12.04983	-5.489450	-2.054880	-0.001531820
9	-12.04983	-5.489451	-2.054880	-0.001531900

TABLE 5. The same as in Table 3, but for the resonant frequency $\Lambda = \Lambda_* \approx 0.5188$.

Appendix C. Auxiliary formulae

In order to get convenient approximate formulae for various amplitude parameters, we note that for dimensionless statement many of the velocity amplitudes coincide with wave amplitudes of the free surface.

C.1. *Piston-like amplitude and wave elevations at w1-w3*

In terms of our Galerkin scheme, calculations of sin/cosine components in (4.1) are as follows

$$\mathcal{A}_1^{(i)} = 2\int_{-\frac{1}{2}}^{0} \frac{\partial \varphi^{(i)}}{\partial z}\Big|_{z=0} dx = 2\sum_{j=1}^{N_3} D_{N_1+N_2+N_3+2,N_1+N_2+j}\ \alpha_j^{(3,i)}, \quad i=1,2. \qquad \text{(C 1)}$$

The cosine- and sinusoidal components of the elevation at w1-w5 (at $x = x_{w_m}$, see definition (4.10)) can be computed as

$$\mathcal{A}_3^{(i)}(x_{w_m}) = \mathcal{A}_1^{(i)} + \Lambda \sum_{k=1}^{N_3} \alpha_k^{(3,i)}\ \gamma_k(x_{w_m}), \qquad \text{(C 2)}$$

where

$$\gamma_k(x_{w_m}) = \frac{2^{m_k-\frac{1}{2}}\Gamma(m_k+1)}{\pi^{m_k+1} r_k^{(2)}}\ \sum_{j=1}^{\infty} \frac{\cos(2\pi j x_{w_m})\ J_{m_k+\frac{1}{2}}(\pi j)}{j^{m_k+\frac{3}{2}}(\Lambda/(2\pi j) - \tanh(2\pi j d))}.$$

C.2. *Amplitude of the outgoing wave and wave elevations away from the moonpool*

Our Galerkin method gives the explicit formulae for (4.5) as follows

$$\mathcal{A}_0^{(i)} = \frac{\sinh(\mathcal{K}h)}{N_0}\ \sum_{j=1}^{N_1} \beta_j\ \alpha_j^{(1,i)}, \quad i=1,2, \qquad \text{(C 3)}$$

where

$$\beta_j = \sqrt{\pi}\Gamma(m_j+1)2^{m_j-\frac{1}{2}}(h-d)^{\frac{1}{2}-m_j} \frac{I_{m_j+\frac{1}{2}}(\mathcal{K}(h-d))}{r_j^{(1)}\mathcal{K}^{m_j+\frac{1}{2}}}, \quad j \geqslant 1. \qquad \text{(C 4)}$$

Accounting for radiation component at $x_{w_p} = -w_p - b$ needs computations of $\mathcal{A}_2^{(i)}$

from (4.8). These are

$$\mathcal{A}_2^{(i)} = \sum_{k=1}^{N_1} d_k^{(2)} \alpha_k^{(1,i)}, \ i = 1, 2, \tag{C 5}$$

where

$$d_k^{(2)} = \Lambda\sqrt{\pi}\frac{2^{m_k-\frac{1}{2}}(h-d)^{\frac{1}{2}-m_k}\Gamma(m_k+1)}{r_k^{(1)}}\sum_{j=1}^{\infty}\frac{(-1)^j\exp(-\kappa_j^{(1)})J_{m_k+\frac{1}{2}}(\kappa_j^{(1)}(h-d))}{(\kappa_j^{(1)})^{m_k+\frac{3}{2}}N_j^{(1)}}.$$

C.3. *Computational expression for $\mathcal{I}_i$ and added mass*

In context of our numerical method, the first integral of (4.14) can be computed as

$$\mathcal{I}_i = -\frac{1}{2\epsilon}\left\{\left(b-\frac{1}{2}\right)A_{-1}^{(1)} + \sum_{k=1}^{N_1} C_k^{(1)}\alpha_k^{(1,1)} + \sum_{k=1}^{N_2} C_k^{(2)}\alpha_k^{(2,1)}\right\}, \tag{C 6}$$

where

$$C_k^{(1)} = \frac{(\frac{1}{4}-b^2)\sqrt{\pi}\Gamma(m_k+1)}{4r_k^{(1)}\Gamma(m_k+\frac{3}{2})} - \frac{2^{m_k+\frac{1}{2}}(h-d)^2\Gamma(m_k+1)}{\pi^{m_k+2}r_k^{(1)}}\ \sum_{j=1}^{\infty}\frac{(-1)^j J_{m_k+\frac{1}{2}}(\pi j)}{j^{m_k+\frac{5}{2}}},$$

$$C_k^{(2)} = \frac{2^{m_k+\frac{1}{2}}(h-d)^2\Gamma(m_k+1)}{\pi^{m_k+2}r_k^{(1)}}\ \sum_{j=1}^{\infty}\frac{(-1)^j J_{m_k+\frac{1}{2}}(\pi j)}{j^{m_k+\frac{5}{2}}}.$$

Analogously, A_{33} can be written in the form

$$A_{33} = \frac{1}{2}\left(b-\frac{1}{2}\right)\left[(h-d) - \frac{(b-\frac{1}{2})^2}{3(h-d)}\right] + \\ + \left\{\left(b-\tfrac{1}{2}\right)\mathcal{A}_{-1}^{(2)} + \sum_{k=1}^{N_1} C_k^{(1)}\alpha_k^{(1,2)} + \sum_{k=1}^{N_2} C_k^{(2)}\alpha_k^{(2,2)}\right\}. \tag{C 7}$$