

**Appendix to “Low-dimensional models for turbulent plane  
Couette flow in a minimal flow unit”**

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**Equations for the 9-uncoupled-mode model**

$$\begin{aligned}
 \dot{a}_{0,0}^{(1)[1]} &= A_{00} a_{0,0}^{(1)[1]} / Re \\
 &+ 2B_{01} \Re(a_{0,1}^{(1)[1]} a_{0,1}^{(1)[2]*}) - 2B_{02} \Re(a_{0,2}^{(1)[1]} a_{0,2}^{(1)[2]*}) - 2B_{03} \Re(a_{0,3}^{(1)[1]} a_{0,3}^{(1)[2]*}) \\
 &+ 2B_{11}^{[1]} \Re(a_{1,1}^{(1)[1]} a_{1,1}^{(1)[2]*} + a_{1,-1}^{(1)[1]} a_{1,-1}^{(1)[2]*}) - B_{11}^{[2]} (|a_{1,1}^{(1)[2]}|^2 + |a_{1,-1}^{(1)[2]}|^2) \\
 &+ 2B_{12}^{[1]} \Re(a_{1,2}^{(1)[1]} a_{1,2}^{(1)[2]*} + a_{1,-2}^{(1)[1]} a_{1,-2}^{(1)[2]*}) + B_{12}^{[2]} (|a_{1,2}^{(1)[2]}|^2 + |a_{1,-2}^{(1)[2]}|^2) \\
 \\
 e_{01}^{[1]} \dot{a}_{0,1}^{(1)[1]} &= A'_{01} a_{0,1}^{(1)[1]} / Re + A''_{01} a_{0,1}^{(1)[2]} - B_{01} a_{0,0}^{(1)[1]} a_{0,1}^{(1)[2]} \\
 &+ 0,1[1] C_{1,-1[1]}^{1,0[1]} (a_{1,0}^{(1)[1]} a_{1,-1}^{(1)[1]*} - a_{1,1}^{(1)[1]} a_{1,0}^{(1)[1]*}) - 0,1[1] C_{0,1[2]}^{0,2[1]} a_{0,2}^{(1)[1]} a_{0,1}^{(1)[2]*} \\
 &+ 0,1[1] C_{1,-1[2]}^{1,0[1]} (a_{1,0}^{(1)[1]} a_{1,-1}^{(1)[2]*} - a_{1,1}^{(1)[2]} a_{1,0}^{(1)[1]*}) \\
 &- 0,1[1] C_{1,-2[1]}^{1,-1[1]} (a_{1,-1}^{(1)[1]} a_{1,-2}^{(1)[1]*} - a_{1,2}^{(1)[1]} a_{1,1}^{(1)[1]*}) - 0,1[1] C_{0,2[2]}^{0,3[1]} a_{0,3}^{(1)[1]} a_{0,2}^{(1)[2]*} \\
 &+ 0,1[1] C_{1,-2[2]}^{1,-1[1]} (a_{1,-1}^{(1)[1]} a_{1,-2}^{(1)[2]*} - a_{1,2}^{(1)[2]} a_{1,1}^{(1)[1]*}) \\
 &- 0,1[1] C_{1,-2[1]}^{1,-1[2]} (a_{1,-1}^{(1)[2]} a_{1,-2}^{(1)[1]*} - a_{1,2}^{(1)[1]} a_{1,1}^{(1)[2]*}) - 0,1[1] C_{0,2[1]}^{0,3[2]} a_{0,3}^{(1)[2]} a_{0,2}^{(1)[1]*} \\
 &+ 0,1[1] C_{1,-2[2]}^{1,-1[2]} (a_{1,-1}^{(1)[2]} a_{1,-2}^{(1)[2]*} - a_{1,2}^{(1)[2]} a_{1,1}^{(1)[2]*}) \\
 \\
 e_{01}^{[2]} \dot{a}_{0,1}^{(1)[2]} &= A'''_{01} a_{0,1}^{(1)[2]} / Re - 0,1[2] C_{0,1[2]}^{0,2[2]} a_{0,2}^{(1)[2]} a_{0,1}^{(1)[2]*} + 0,1[2] C_{0,2[2]}^{0,3[2]} a_{0,3}^{(1)[2]} a_{0,2}^{(1)[2]*} \\
 &- 0,1[2] C_{1,-1[1]}^{1,0[1]} (a_{1,0}^{(1)[1]} a_{1,-1}^{(1)[1]*} - a_{1,1}^{(1)[1]} a_{1,0}^{(1)[1]*}) \\
 &- 0,1[2] C_{1,-1[2]}^{1,0[1]} (a_{1,0}^{(1)[1]} a_{1,-1}^{(1)[2]*} - a_{1,1}^{(1)[2]} a_{1,0}^{(1)[1]*}) \\
 &+ 0,1[2] C_{1,-2[1]}^{1,-1[1]} (a_{1,-1}^{(1)[1]} a_{1,-2}^{(1)[1]*} - a_{1,2}^{(1)[1]} a_{1,1}^{(1)[1]*})
 \end{aligned}$$

$$\begin{aligned}
& - 0,1[2]C_{1,-2[2]}^{1,-1[1]}(a_{1,-1}^{(1)[1]}a_{1,-2}^{(1)[2]*} - a_{1,2}^{(1)[2]}a_{1,1}^{(1)[1]*}) \\
& - 0,1[2]C_{1,-2[1]}^{1,-1[2]}(a_{1,-1}^{(1)[2]}a_{1,-2}^{(1)[1]*} - a_{1,2}^{(1)[1]}a_{1,1}^{(1)[2]*}) \\
& + 0,1[2]C_{1,-2[2]}^{1,-1[2]}(a_{1,-1}^{(1)[2]}a_{1,-2}^{(1)[2]*} - a_{1,2}^{(1)[2]}a_{1,1}^{(1)[2]*})
\end{aligned}$$

$$\begin{aligned}
e_{02}^{[1]} \dot{a}_{0,2}^{(1)[1]} &= A'_{02}a_{0,2}^{(1)[1]}/Re + A''_{02}a_{0,2}^{(1)[2]} + B_{02}a_{0,0}^{(1)[1]}a_{0,2}^{(1)[2]} + 0,1[1]C_{0,1[2]}^{0,2[1]}a_{0,1}^{(1)[1]}a_{0,1}^{(1)[2]} \\
& - 0,2[1]C_{1,-2[1]}^{1,0[1]}(a_{1,0}^{(1)[1]}a_{1,-2}^{(1)[1]*} + a_{1,2}^{(1)[1]}a_{1,0}^{(1)[1]*}) + 0,1[1]C_{0,2[1]}^{0,3[2]}a_{0,3}^{(1)[2]}a_{0,1}^{(1)[1]*} \\
& - 0,2[1]C_{1,-2[2]}^{1,0[1]}(a_{1,0}^{(1)[1]}a_{1,-2}^{(1)[2]*} + a_{1,2}^{(1)[2]}a_{1,0}^{(1)[1]*}) - 0,2[1]C_{1,-1[2]}^{1,1[2]}a_{1,1}^{(1)[2]}a_{1,-1}^{(1)[2]*} \\
& - 0,2[1]C_{1,-1[2]}^{1,1[1]}(a_{1,1}^{(1)[1]}a_{1,-1}^{(1)[2]*} + a_{1,1}^{(1)[2]}a_{1,-1}^{(1)[1]*}) - 0,2[1]C_{0,1[2]}^{0,3[1]}a_{0,3}^{(1)[1]}a_{0,1}^{(1)[2]*}
\end{aligned}$$

$$\begin{aligned}
e_{02}^{[2]} \dot{a}_{0,2}^{(1)[2]} &= A'''_{02}a_{0,2}^{(1)[2]}/Re + 0,1[2]C_{0,1[2]}^{0,2[2]}(a_{0,1}^{(1)[2]})^2 + 0,2[2]C_{1,-1[1]}^{1,1[1]}a_{1,1}^{(1)[1]}a_{1,-1}^{(1)[1]*} \\
& - 0,2[2]C_{1,-2[1]}^{1,0[1]}(a_{1,0}^{(1)[1]}a_{1,-2}^{(1)[1]*} + a_{1,2}^{(1)[1]}a_{1,0}^{(1)[1]*}) - 0,2[2]C_{1,-1[2]}^{1,1[2]}a_{1,1}^{(1)[2]}a_{1,-1}^{(1)[2]*} \\
& + 0,2[2]C_{1,-2[2]}^{1,0[1]}(a_{1,0}^{(1)[1]}a_{1,-2}^{(1)[2]*} + a_{1,2}^{(1)[2]}a_{1,0}^{(1)[1]*}) - 0,2[1]C_{0,1[2]}^{0,3[2]}a_{0,3}^{(1)[2]}a_{0,1}^{(1)[2]*} \\
& - 0,2[2]C_{1,-1[2]}^{1,1[1]}(a_{1,1}^{(1)[1]}a_{1,-1}^{(1)[2]*} + a_{1,1}^{(1)[2]}a_{1,-1}^{(1)[1]*})
\end{aligned}$$

$$\begin{aligned}
\dot{a}_{1,0}^{(1)[1]} &= A_{10}a_{1,0}^{(1)[1]}/Re + 0,1[1]C_{1,-1[2]}^{1,0[1]}(a_{0,1}^{(1)[1]}a_{1,-1}^{(1)[2]} - a_{1,1}^{(1)[2]}a_{0,1}^{(1)[1]*}) \\
& - 0,2[1]C_{1,-2[2]}^{1,0[1]}(a_{0,2}^{(1)[1]}a_{1,-2}^{(1)[2]} + a_{1,2}^{(1)[2]}a_{0,2}^{(1)[1]*}) \\
& - 1,0[1]C_{1,-1[1]}^{0,1[1]}(a_{0,1}^{(1)[1]}a_{1,-1}^{(1)[1]} - a_{1,1}^{(1)[1]}a_{0,1}^{(1)[1]*}) \\
& - 1,0[1]C_{1,-1[1]}^{0,1[2]}(a_{0,1}^{(1)[2]}a_{1,-1}^{(1)[1]} - a_{1,1}^{(1)[1]}a_{0,1}^{(1)[2]*}) \\
& + 1,0[1]C_{1,-1[2]}^{0,1[2]}(a_{0,1}^{(1)[2]}a_{1,-1}^{(1)[2]} - a_{1,1}^{(1)[2]}a_{0,1}^{(1)[2]*}) \\
& + 1,0[1]C_{1,-2[1]}^{0,2[1]}(a_{0,2}^{(1)[1]}a_{1,-2}^{(1)[1]} + a_{1,2}^{(1)[1]}a_{0,2}^{(1)[1]*}) \\
& - 1,0[1]C_{1,-2[1]}^{0,2[2]}(a_{0,2}^{(1)[2]}a_{1,-2}^{(1)[1]} + a_{1,2}^{(1)[1]}a_{0,2}^{(1)[2]*}) \\
& + 1,0[1]C_{1,-2[2]}^{0,2[2]}(a_{0,2}^{(1)[2]}a_{1,-2}^{(1)[2]} + a_{1,2}^{(1)[2]}a_{0,2}^{(1)[2]*})
\end{aligned}$$

$$\begin{aligned}
e_{11}^{[1]} \dot{a}_{1,1}^{(1)[1]} &= A'_{11}a_{1,1}^{(1)[1]}/Re + A''_{11}a_{1,1}^{(1)[2]} - B_{11}^{[1]}a_{0,0}^{(1)[1]}a_{1,1}^{(1)[2]} + 1,1[1]C_{1,0[1]}^{0,1[1]}a_{0,1}^{(1)[1]}a_{1,0}^{(1)[1]} \\
& - 1,1[1]C_{1,0[1]}^{0,1[2]}a_{0,1}^{(1)[2]}a_{1,0}^{(1)[1]} + 1,1[1]C_{1,-1[2]}^{0,2[1]}a_{0,2}^{(1)[1]}a_{1,-1}^{(1)[2]} - 1,1[1]C_{1,-1[1]}^{0,2[2]}a_{0,2}^{(1)[2]}a_{1,-1}^{(1)[1]} \\
& + 1,1[1]C_{1,-1[2]}^{0,2[2]}a_{0,2}^{(1)[2]}a_{1,-1}^{(1)[2]} + 1,1[1]C_{1,-2[1]}^{0,3[1]}a_{0,3}^{(1)[1]}a_{1,-2}^{(1)[1]} + 1,1[1]C_{1,-2[2]}^{0,3[1]}a_{0,3}^{(1)[1]}a_{1,-2}^{(1)[2]}
\end{aligned}$$

$$\begin{aligned}
& - {}^{1,1[1]}C_{1,-2[1]}^{0,3[2]} a_{0,3}^{(1)[2]} a_{1,-2}^{(1)[1]} + {}^{1,1[1]}C_{1,-2[2]}^{0,3[2]} a_{0,3}^{(1)[2]} a_{1,-2}^{(1)[2]} - {}^{1,1[1]}C_{0,1[1]}^{1,2[1]} a_{1,2}^{(1)[1]} a_{0,1}^{(1)[1]*} \\
& - {}^{1,1[1]}C_{0,1[2]}^{1,2[1]} a_{1,2}^{(1)[1]} a_{0,1}^{(1)[2]*} - {}^{1,1[1]}C_{0,1[1]}^{1,2[2]} a_{1,2}^{(1)[2]} a_{0,1}^{(1)[1]*} \\
& + {}^{1,1[1]}C_{0,1[2]}^{1,2[2]} a_{1,2}^{(1)[2]} a_{0,1}^{(1)[2]*}
\end{aligned}$$

$$\begin{aligned}
e_{11}^{[2]} \dot{a}_{1,1}^{(1)[2]} &= A_{11}''' a_{1,1}^{(1)[2]} / Re + A_{11}'''' a_{1,1}^{(1)[2]} + 0.5 B_{11}^{[2]} a_{0,0}^{(1)[1]} a_{1,1}^{(1)[2]} + {}^{1,1[2]}C_{1,0[1]}^{0,1[1]} a_{0,1}^{(1)[1]} a_{1,0}^{(1)[1]} \\
& + {}^{1,1[2]}C_{1,0[1]}^{0,1[2]} a_{0,1}^{(1)[2]} a_{1,0}^{(1)[1]} + {}^{1,1[2]}C_{1,-1[1]}^{0,2[1]} a_{0,2}^{(1)[1]} a_{1,-1}^{(1)[1]} + {}^{1,1[2]}C_{1,-1[2]}^{0,2[1]} a_{0,2}^{(1)[1]} a_{1,-1}^{(1)[2]} \\
& + {}^{1,1[2]}C_{1,-1[1]}^{0,2[2]} a_{0,2}^{(1)[2]} a_{1,-1}^{(1)[1]} + {}^{1,1[2]}C_{1,-1[2]}^{0,2[2]} a_{0,2}^{(1)[2]} a_{1,-1}^{(1)[2]} + {}^{1,1[2]}C_{1,-2[1]}^{0,3[1]} a_{0,3}^{(1)[1]} a_{1,-2}^{(1)[1]} \\
& + {}^{1,1[2]}C_{1,-2[2]}^{0,3[1]} a_{0,3}^{(1)[1]} a_{1,-2}^{(1)[2]} - {}^{1,1[2]}C_{1,-2[1]}^{0,3[2]} a_{0,3}^{(1)[2]} a_{1,-2}^{(1)[1]} + {}^{1,1[2]}C_{1,-2[2]}^{0,3[2]} a_{0,3}^{(1)[2]} a_{1,-2}^{(1)[2]} \\
& + {}^{1,1[2]}C_{0,1[1]}^{1,2[1]} a_{1,2}^{(1)[1]} a_{0,1}^{(1)[1]*} - {}^{1,1[2]}C_{0,1[2]}^{1,2[1]} a_{1,2}^{(1)[1]} a_{0,1}^{(1)[2]*} - {}^{1,1[2]}C_{0,1[1]}^{1,2[2]} a_{1,2}^{(1)[2]} a_{0,1}^{(1)[1]*} \\
& + {}^{1,1[2]}C_{1,-2[2]}^{0,1[2]} a_{1,2}^{(1)[2]} a_{0,1}^{(1)[2]*}
\end{aligned}$$

$$\begin{aligned}
e_{11}^{[1]} \dot{a}_{1,-1}^{(1)[1]} &= A_{11}' a_{1,-1}^{(1)[1]} / Re + A_{11}'' a_{1,-1}^{(1)[2]} - B_{11}^{[1]} a_{0,0}^{(1)[1]} a_{1,-1}^{(1)[2]} + {}^{1,1[1]}C_{0,1[1]}^{1,2[1]} a_{0,1}^{(1)[1]} a_{1,-2}^{(1)[1]} \\
& + {}^{1,1[1]}C_{0,1[1]}^{1,2[2]} a_{0,1}^{(1)[1]} a_{1,-2}^{(1)[2]} + {}^{1,1[1]}C_{0,1[2]}^{1,2[1]} a_{0,1}^{(1)[2]} a_{1,-2}^{(1)[1]} - {}^{1,1[1]}C_{0,1[2]}^{1,2[2]} a_{0,1}^{(1)[2]} a_{1,-2}^{(1)[2]} \\
& - {}^{1,1[1]}C_{1,0[1]}^{0,1[1]} a_{1,0}^{(1)[1]} a_{0,1}^{(1)[1]*} + {}^{1,1[1]}C_{1,0[1]}^{0,1[2]} a_{1,0}^{(1)[1]} a_{0,1}^{(1)[2]*} \\
& - {}^{1,1[1]}C_{1,-1[1]}^{0,2[2]} a_{1,1}^{(1)[1]} a_{0,2}^{(1)[2]*} + {}^{1,1[1]}C_{1,-1[2]}^{0,2[1]} a_{1,1}^{(1)[2]} a_{0,2}^{(1)[1]*} \\
& + {}^{1,1[1]}C_{1,-1[2]}^{0,2[2]} a_{1,1}^{(1)[2]} a_{0,2}^{(1)[2]*} - {}^{1,1[1]}C_{1,-2[1]}^{0,3[1]} a_{1,2}^{(1)[1]} a_{0,3}^{(1)[1]*} \\
& + {}^{1,1[1]}C_{1,-2[1]}^{0,3[2]} a_{1,2}^{(1)[1]} a_{0,3}^{(1)[2]*} - {}^{1,1[1]}C_{1,-2[2]}^{0,3[1]} a_{1,2}^{(1)[2]} a_{0,3}^{(1)[1]*} \\
& - {}^{1,1[1]}C_{1,-2[2]}^{0,3[2]} a_{1,2}^{(1)[2]} a_{0,3}^{(1)[2]*}
\end{aligned}$$

$$\begin{aligned}
e_{11}^{[2]} \dot{a}_{1,-1}^{(1)[2]} &= A_{11}''' a_{1,-1}^{(1)[2]} / Re + A_{11}'''' a_{1,-1}^{(1)[2]} + 0.5 B_{11}^{[2]} a_{0,0}^{(1)[1]} a_{1,-1}^{(1)[2]} - {}^{1,1[2]}C_{0,1[1]}^{1,2[1]} a_{0,1}^{(1)[1]} a_{1,-2}^{(1)[1]} \\
& + {}^{1,1[2]}C_{0,1[1]}^{1,2[2]} a_{0,1}^{(1)[1]} a_{1,-2}^{(1)[2]} + {}^{1,1[2]}C_{0,1[2]}^{1,2[1]} a_{0,1}^{(1)[2]} a_{1,-2}^{(1)[1]} - {}^{1,1[2]}C_{1,-2[2]}^{0,1[2]} a_{0,1}^{(1)[2]} a_{1,-2}^{(1)[2]} \\
& - {}^{1,1[2]}C_{1,0[1]}^{0,1[1]} a_{1,0}^{(1)[1]} a_{0,1}^{(1)[1]*} - {}^{1,1[2]}C_{1,0[1]}^{0,1[2]} a_{1,0}^{(1)[1]} a_{0,1}^{(1)[2]*} \\
& + {}^{1,1[2]}C_{1,-1[1]}^{0,2[1]} a_{1,1}^{(1)[1]} a_{0,2}^{(1)[1]*} + {}^{1,1[2]}C_{1,-1[1]}^{0,2[2]} a_{1,1}^{(1)[1]} a_{0,2}^{(1)[2]*} \\
& + {}^{1,1[2]}C_{1,-1[2]}^{0,2[1]} a_{1,1}^{(1)[2]} a_{0,2}^{(1)[1]*} \\
& + {}^{1,1[2]}C_{1,-1[2]}^{0,2[2]} a_{1,1}^{(1)[2]} a_{0,2}^{(1)[2]*} - {}^{1,1[2]}C_{1,-2[1]}^{0,3[1]} a_{1,2}^{(1)[1]} a_{0,3}^{(1)[1]*} \\
& + {}^{1,1[2]}C_{1,-2[1]}^{0,3[2]} a_{1,2}^{(1)[1]} a_{0,3}^{(1)[2]*} - {}^{1,1[2]}C_{1,-2[2]}^{0,3[1]} a_{1,2}^{(1)[2]} a_{0,3}^{(1)[1]*}
\end{aligned}$$

$$\begin{aligned}
& - 1,1[2]C_{1,-2[2]}^{0,3[2]}a_{1,2}^{(1)[2]}a_{0,3}^{(1)[2]*} \\
e_{03}^{[1]} \dot{a}_{0,3}^{(1)[1]} & = A'_{03}a_{0,3}^{(1)[1]}/Re + A''_{03}a_{0,3}^{(1)[2]} + B_{03}a_{0,0}^{(1)[1]}a_{0,3}^{(1)[2]} + 0,1[1]C_{0,2[2]}^{0,3[1]}a_{0,1}^{(1)[1]}a_{0,2}^{(1)[2]} \\
& + 0,2[1]C_{0,1[2]}^{0,3[1]}a_{0,1}^{(1)[2]}a_{0,2}^{(1)[1]} + 0,3[1]C_{1,-2[1]}^{1,1[1]}(a_{1,1}^{(1)[1]}a_{1,-2}^{(1)[1]*} - a_{1,2}^{(1)[1]}a_{1,-1}^{(1)[1]*}) \\
& - 0,3[1]C_{1,-2[2]}^{1,1[1]}(a_{1,1}^{(1)[1]}a_{1,-2}^{(1)[2]*} - a_{1,2}^{(1)[2]}a_{1,-1}^{(1)[1]*}) \\
& - 0,3[1]C_{1,-2[1]}^{1,1[2]}(a_{1,1}^{(1)[2]}a_{1,-2}^{(1)[1]*} - a_{1,2}^{(1)[1]}a_{1,1}^{(1)[2]*}) \\
& - 0,3[1]C_{1,-2[2]}^{1,1[2]}(a_{1,1}^{(1)[2]}a_{1,-2}^{(1)[2]*} - a_{1,2}^{(1)[2]}a_{1,-1}^{(1)[2]*}) \\
e_{03}^{[2]} \dot{a}_{0,3}^{(1)[2]} & = A'''_{03}a_{0,3}^{(1)[2]}/Re + 0,3[2]C_{0,2[2]}^{0,1[2]}a_{0,1}^{(1)[2]}a_{0,2}^{(1)[2]} + 0,3[2]C_{1,-2[1]}^{1,1[1]}a_{1,1}^{(1)[1]}a_{1,-2}^{(1)[1]*} \\
& - 0,3[2]C_{1,-2[2]}^{1,1[1]}(a_{1,1}^{(1)[1]}a_{1,-2}^{(1)[2]*} - a_{1,2}^{(1)[2]}a_{1,-1}^{(1)[1]*}) - 0,3[2]C_{1,-1[1]}^{1,2[1]}a_{1,2}^{(1)[1]}a_{1,-1}^{(1)[1]*} \\
& + 0,3[2]C_{1,-2[1]}^{1,1[2]}(a_{1,1}^{(1)[2]}a_{1,-2}^{(1)[1]*} - a_{1,2}^{(1)[1]}a_{1,-1}^{(1)[2]*}) \\
& - 0,3[2]C_{1,-2[2]}^{1,1[2]}(a_{1,1}^{(1)[2]}a_{1,-2}^{(1)[2]*} - a_{1,2}^{(1)[2]}a_{1,-1}^{(1)[2]*}) \\
e_{12}^{[1]} \dot{a}_{1,2}^{(1)[1]} & = A'_{12}a_{1,2}^{(1)[1]}/Re + A''_{12}a_{1,2}^{(1)[2]} - B_{12}^{[1]}a_{0,0}^{(1)[1]}a_{1,2}^{(1)[2]} + 1,2[1]C_{1,1[1]}^{0,1[1]}a_{0,1}^{(1)[1]}a_{1,1}^{(1)[1]} \\
& - 1,2[1]C_{1,1[2]}^{0,1[1]}a_{0,1}^{(1)[1]}a_{1,1}^{(1)[2]} + 1,2[1]C_{1,1[1]}^{0,1[2]}a_{0,1}^{(1)[2]}a_{1,1}^{(1)[1]} + 1,2[1]C_{1,1[2]}^{0,1[2]}a_{0,1}^{(1)[2]}a_{1,1}^{(1)[2]} \\
& + 1,2[1]C_{1,0[1]}^{0,2[1]}a_{0,2}^{(1)[1]}a_{1,0}^{(1)[1]} + 1,2[1]C_{1,0[1]}^{0,2[2]}a_{0,2}^{(1)[2]}a_{1,0}^{(1)[1]} + 1,2[1]C_{0,3[1]}^{1,-1[1]}a_{1,-1}^{(1)[1]}a_{0,3}^{(1)[1]} \\
& - 1,2[1]C_{0,3[2]}^{1,-1[1]}a_{1,-1}^{(1)[1]}a_{0,3}^{(1)[2]} - 1,2[1]C_{0,3[1]}^{1,-1[2]}a_{1,-1}^{(1)[2]}a_{0,3}^{(1)[1]} \\
& - 1,2[1]C_{0,3[2]}^{1,-1[2]}a_{1,-1}^{(1)[2]}a_{0,3}^{(1)[2]} \\
e_{12}^{[2]} \dot{a}_{1,2}^{(1)[2]} & = A'''_{12}a_{1,2}^{(1)[2]}/Re + A''''_{12}a_{1,2}^{(1)[2]} - 0.5B_{12}^{[2]}a_{0,0}^{(1)[1]}a_{1,2}^{(1)[2]} + 1,2[2]C_{1,1[1]}^{0,1[1]}a_{0,1}^{(1)[1]}a_{1,1}^{(1)[1]} \\
& + 1,2[2]C_{1,1[2]}^{0,1[1]}a_{0,1}^{(1)[1]}a_{1,1}^{(1)[2]} - 1,2[2]C_{1,1[1]}^{0,1[2]}a_{0,1}^{(1)[2]}a_{1,1}^{(1)[1]} - 1,2[2]C_{1,1[2]}^{0,1[2]}a_{0,1}^{(1)[2]}a_{1,1}^{(1)[2]} \\
& + 1,2[2]C_{1,0[1]}^{0,2[1]}a_{0,2}^{(1)[1]}a_{1,0}^{(1)[1]} - 1,2[2]C_{1,0[1]}^{0,2[2]}a_{0,2}^{(1)[2]}a_{1,0}^{(1)[1]} - 1,2[2]C_{0,3[1]}^{1,-1[1]}a_{1,-1}^{(1)[1]}a_{0,3}^{(1)[1]} \\
& - 1,2[2]C_{0,3[2]}^{1,-1[1]}a_{1,-1}^{(1)[1]}a_{0,3}^{(1)[2]} + 1,2[2]C_{0,3[1]}^{1,-1[2]}a_{1,-1}^{(1)[2]}a_{0,3}^{(1)[1]} \\
& + 1,2[2]C_{0,3[2]}^{1,-1[2]}a_{1,-1}^{(1)[2]}a_{0,3}^{(1)[2]} \\
e_{12}^{[1]} \dot{a}_{1,-2}^{(1)[1]} & = A'_{12}a_{1,-2}^{(1)[1]}/Re + A''_{12}a_{1,-2}^{(1)[2]} - B_{12}^{[1]}a_{0,0}^{(1)[1]}a_{1,-2}^{(1)[2]} + 1,2[1]C_{1,0[1]}^{0,2[1]}a_{1,0}^{(1)[1]}a_{0,2}^{(1)[1]*}
\end{aligned}$$

$$\begin{aligned}
& + {}^{1,2[1]}C_{1,0[1]}^{0,2[2]} a_{1,0}^{(1)[1]} a_{0,2}^{(1)[2]*} - {}^{1,2[1]}C_{0,3[1]}^{1,-1[1]} a_{1,1}^{(1)[1]} a_{0,3}^{(1)[1]*} \\
& + {}^{1,2[1]}C_{0,3[2]}^{1,-1[1]} a_{1,1}^{(1)[1]} a_{0,3}^{(1)[2]*} + {}^{1,2[1]}C_{0,3[1]}^{1,-1[2]} a_{1,1}^{(1)[2]} a_{0,3}^{(1)[1]*} \\
& + {}^{1,2[1]}C_{0,3[2]}^{1,-1[2]} a_{1,1}^{(1)[2]} a_{0,3}^{(1)[2]*} - {}^{1,2[1]}C_{1,1[1]}^{0,1[1]} a_{1,-1}^{(1)[1]} a_{0,1}^{(1)[1]*} \\
& - {}^{1,2[1]}C_{1,1[1]}^{0,1[2]} a_{1,-1}^{(1)[1]} a_{0,1}^{(1)[2]*} + {}^{1,2[1]}C_{1,1[2]}^{0,1[1]} a_{1,-1}^{(1)[2]} a_{0,1}^{(1)[1]*} \\
& - {}^{1,2[1]}C_{1,1[2]}^{0,1[2]} a_{1,-1}^{(1)[2]} a_{0,1}^{(1)[2]*}
\end{aligned}$$

$$\begin{aligned}
e_{12}^{[2]} \dot{a}_{1,-2}^{(1)[2]} &= A_{12}''' a_{1,-2}^{(1)[2]} / Re + A_{12}'''' a_{1,-2}^{(1)[2]} - 0.5 B_{12}^{[2]} a_{0,0}^{(1)[1]} a_{1,-2}^{(1)[2]} \\
& + {}^{1,2[2]}C_{1,0[1]}^{0,2[1]} a_{1,0}^{(1)[1]} a_{0,2}^{(1)[1]*} - {}^{1,2[2]}C_{1,0[1]}^{0,2[2]} a_{1,0}^{(1)[1]} a_{0,2}^{(1)[2]*} \\
& + {}^{1,2[2]}C_{0,3[1]}^{1,-1[1]} a_{1,1}^{(1)[1]} a_{0,3}^{(1)[1]*} + {}^{1,2[2]}C_{0,3[2]}^{1,-1[1]} a_{1,1}^{(1)[1]} a_{0,3}^{(1)[2]*} \\
& - {}^{1,2[2]}C_{0,3[1]}^{1,-1[2]} a_{1,1}^{(1)[2]} a_{0,3}^{(1)[1]*} - {}^{1,2[2]}C_{0,3[2]}^{1,-1[2]} a_{1,1}^{(1)[2]} a_{0,3}^{(1)[2]*} \\
& - {}^{1,2[2]}C_{1,1[1]}^{0,1[1]} a_{1,-1}^{(1)[1]} a_{0,1}^{(1)[1]*} + {}^{1,2[2]}C_{1,1[1]}^{0,1[2]} a_{1,-1}^{(1)[1]} a_{0,1}^{(1)[2]*} \\
& - {}^{1,2[2]}C_{1,1[2]}^{0,1[1]} a_{1,-1}^{(1)[2]} a_{0,1}^{(1)[1]*} + {}^{1,2[2]}C_{1,1[2]}^{0,1[2]} a_{1,-1}^{(1)[2]} a_{0,1}^{(1)[2]*}
\end{aligned}$$

$$\begin{aligned}
e_{01}^{[1]} &= 0.9425, & e_{01}^{[2]} &= 0.0575, & e_{02}^{[1]} &= 0.9943, & e_{02}^{[2]} &= 0.0057, & e_{03}^{[1]} &= 0.9963 \\
e_{03}^{[2]} &= 0.0037, & e_{11}^{[1]} &= 0.8744, & e_{11}^{[2]} &= 0.1256, & e_{12}^{[1]} &= 0.8977, & e_{12}^{[2]} &= 0.1020
\end{aligned}$$

$$\begin{aligned}
A_{00} &= -10.760, & A_{01}' &= -9.269, & A_{01}'' &= 0.1281, & A_{01}''' &= -0.5567 \\
A_{02}' &= -21.6678, & A_{02}'' &= -0.0503, & A_{02}''' &= -0.2163, & A_{10} &= 9.9411 \\
A_{11}' &= -8.3273, & A_{11}'' &= 0.0267, & A_{11}''' &= -6.7573, & A_{11}'''' &= -0.0126 \\
A_{03}' &= -27.8530, & A_{03}'' &= -0.0563, & A_{03}''' &= -0.1143 \\
A_{12}' &= -16.2213, & A_{12}'' &= 0.2393, & A_{12}''' &= -2.1371, & A_{12}'''' &= 0.0037
\end{aligned}$$

$$\begin{aligned}
B_{01} &= 0.0386, & B_{02} &= 0.0074, & B_{03} &= 0.0226, \\
B_{11}^{[1]} &= 0.0009, & B_{11}^{[2]} &= 0.0004, & B_{12}^{[1]} &= 0.0856, & B_{12}^{[2]} &= 0.0040 \\
{}^{0,1[1]}C_{0,1[2]}^{0,2[1]} &= 0.0642, & {}^{0,1[1]}C_{1,-1[1]}^{1,0[1]} &= 0.1782, & {}^{0,1[1]}C_{1,-1[2]}^{1,0[1]} &= 0.0145, \\
{}^{0,1[1]}C_{1,-2[1]}^{1,-1[1]} &= 0.1626, & {}^{0,1[1]}C_{1,-2[2]}^{1,-1[1]} &= 0.0001, & {}^{0,1[1]}C_{1,-2[1]}^{1,-1[2]} &= 0.0073, \\
{}^{0,1[1]}C_{1,-2[2]}^{1,-1[2]} &= 0.0058, & {}^{0,1[1]}C_{0,2[2]}^{0,3[1]} &= 0.0008, & {}^{0,1[1]}C_{0,2[1]}^{0,3[2]} &= 0.0001, \\
{}^{0,1[2]}C_{0,1[2]}^{0,2[2]} &= 0.0005, & {}^{0,1[2]}C_{1,-1[1]}^{1,0[1]} &= 0.0078, & {}^{0,1[2]}C_{1,-1[2]}^{1,0[1]} &= 0.0136,
\end{aligned}$$

$$\begin{aligned}
0,1[2]C_{1,-2[1]}^{1,-1[1]} &= 0.0050, & 0,1[2]C_{1,-2[2]}^{1,-1[1]} &= 0.0023, & 0,1[2]C_{1,-2[1]}^{1,-1[2]} &= 0.0031, \\
1,1[1]C_{0,1[1]}^{1,2[2]} &= 0.0044, & 0,1[2]C_{0,2[2]}^{0,3[2]} &= 0.0002, & 0,2[1]C_{1,-2[1]}^{1,0[1]} &= 0.2250, \\
0,2[1]C_{1,-2[2]}^{1,0[1]} &= 0.0125, & 0,2[1]C_{1,-1[2]}^{1,1[1]} &= 0.0526, & 0,2[1]C_{1,-1[2]}^{1,1[2]} &= 0.0069, \\
0,2[1]C_{0,1[2]}^{0,3[1]} &= 0.1511, & 0,2[2]C_{1,-1[2]}^{1,0[1]} &= 0.0095, & 0,2[2]C_{1,-2[2]}^{1,0[1]} &= 0.0021, \\
0,2[2]C_{1,-1[1]}^{1,1[1]} &= 0.0065, & 0,2[2]C_{1,-1[2]}^{1,1[1]} &= 0.0056, & 0,2[2]C_{1,-1[2]}^{1,1[2]} &= 0.0023, \\
0,2[2]C_{0,1[2]}^{0,3[2]} &= 0.0005, & 1,0[1]C_{1,-1[1]}^{0,1[2]} &= 0.0170, & 1,0[1]C_{1,-1[1]}^{0,1[1]} &= 0.0833, \\
1,0[1]C_{1,-1[2]}^{0,1[2]} &= 0.0150, & 1,0[1]C_{1,-2[1]}^{0,2[1]} &= 0.0263, & 1,0[1]C_{1,-2[1]}^{0,2[2]} &= 0.0005, \\
1,0[1]C_{1,-2[2]}^{0,2[2]} &= 0.0024, & 1,1[1]C_{1,0[1]}^{0,1[1]} &= 0.0949, & 1,1[1]C_{1,0[1]}^{0,1[2]} &= 0.0248, \\
1,1[1]C_{1,-1[2]}^{0,2[1]} &= 0.0165, & 1,1[1]C_{1,-1[1]}^{0,2[2]} &= 0.0033, & 1,1[1]C_{1,-1[2]}^{0,2[2]} &= 0.0047, \\
1,1[1]C_{1,-2[1]}^{0,3[1]} &= 0.2969, & 1,1[1]C_{1,-2[2]}^{0,3[1]} &= 0.0259, & 1,1[1]C_{1,-2[1]}^{0,3[2]} &= 0.0040, \\
1,1[1]C_{1,-2[2]}^{0,3[2]} &= 0.0009, & 1,1[1]C_{0,1[1]}^{1,2[1]} &= 0.1880, & 1,1[1]C_{0,1[2]}^{1,2[1]} &= 0.0302, \\
1,1[1]C_{0,1[2]}^{1,2[2]} &= 0.0070, & 1,2[2]C_{0,3[1]}^{1,-1[2]} &= 0.0033, & 1,2[2]C_{0,3[2]}^{1,-1[2]} &= 0.0016, \\
1,1[2]C_{1,0[1]}^{0,1[1]} &= 0.0289, & 1,1[2]C_{1,0[1]}^{0,1[2]} &= 0.0045, & 1,1[2]C_{1,-1[1]}^{0,2[1]} &= 0.0361, \\
1,1[2]C_{1,-1[2]}^{0,2[1]} &= 0.0035, & 1,1[2]C_{1,-1[1]}^{0,2[2]} &= 0.0009, & 1,1[2]C_{1,-1[2]}^{0,2[2]} &= 0.0011, \\
1,1[2]C_{1,-2[1]}^{0,3[1]} &= 0.0307, & 1,1[2]C_{1,-2[2]}^{0,3[1]} &= 0.0076, & 1,1[2]C_{1,-2[1]}^{0,3[2]} &= 0.0012, \\
1,1[2]C_{1,-2[2]}^{0,3[2]} &= 0.0017, & 1,1[2]C_{0,1[1]}^{1,2[1]} &= 0.0015, & 1,1[2]C_{0,1[2]}^{1,2[1]} &= 0.00432, \\
1,1[2]C_{0,1[1]}^{1,2[2]} &= 0.00136, & 1,1[2]C_{1,-2[2]}^{0,1[2]} &= 0.0029, & 0,3[1]C_{1,-2[1]}^{1,1[1]} &= 0.1965, \\
0,3[1]C_{1,-2[2]}^{1,1[1]} &= 0.0279, & 0,3[1]C_{1,-2[1]}^{1,1[2]} &= 0.0532, & 0,3[1]C_{1,-2[2]}^{1,1[2]} &= 0.00434, \\
0,3[2]C_{0,2[2]}^{0,1[2]} &= 0.0003, & 0,3[2]C_{1,-2[1]}^{1,1[1]} &= 0.0018, & 0,3[2]C_{1,-2[2]}^{1,1[1]} &= 0.0074, \\
0,3[2]C_{1,-2[1]}^{1,1[2]} &= 0.0001, & 0,3[2]C_{1,-2[2]}^{1,1[2]} &= 0.000026, & 0,3[2]C_{1,-1[1]}^{1,2[1]} &= 0.0018, \\
1,2[1]C_{1,1[1]}^{0,1[1]} &= 0.0254, & 1,2[1]C_{1,1[2]}^{0,1[1]} &= 0.0088, & 1,2[1]C_{1,1[1]}^{0,1[2]} &= 0.0352, \\
1,2[1]C_{1,1[2]}^{0,1[2]} &= 0.0012, & 1,2[1]C_{1,0[1]}^{0,2[1]} &= 0.1987, & 1,2[1]C_{1,0[1]}^{0,2[2]} &= 0.0099,
\end{aligned}$$

$$\begin{aligned}
{}^{1,2[1]}C_{0,3[1]}^{1,-1[2]} &= 0.4934, & {}^{1,2[1]}C_{0,3[2]}^{1,-1[1]} &= 0.0022, & {}^{1,2[1]}C_{0,3[1]}^{1,-1[2]} &= 0.0225, \\
{}^{1,2[1]}C_{0,3[2]}^{1,-1[2]} &= 0.0012, & {}^{1,2[2]}C_{1,1[1]}^{0,1[1]} &= 0.0044, & {}^{1,2[2]}C_{1,1[2]}^{0,1[1]} &= 0.0072, \\
{}^{1,2[2]}C_{1,1[1]}^{0,1[2]} &= 0.0093, & {}^{1,2[2]}C_{1,1[2]}^{0,1[2]} &= 0.0029, & {}^{1,2[2]}C_{1,0[1]}^{0,2[1]} &= 0.0250, \\
{}^{1,2[2]}C_{1,0[1]}^{0,2[2]} &= 0.0045, & {}^{1,2[2]}C_{0,3[1]}^{1,-1[1]} &= 0.0020, & {}^{1,2[2]}C_{0,3[2]}^{1,-1[1]} &= 0.0066,
\end{aligned}$$

The  $e_{n_x, n_z}^{[m]}$  coefficients on the left hand side of the above ODEs result from the fact that the uncoupled modes are no longer orthonormal, but are merely orthogonal.

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