Appendix to "On nonlinear convection in mushy layers. Part 2. Mixed oscillatory and stationary modes of convection"

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Appendix B

The expressions for the solutions V_{10}, θ_{10} and ϕ_{10} are given below

$$(V_{10}, \theta_{10})=\sum_{l,p=-N}^{N}[(B_{lp}^{(o)}, E_{lp}^{(o)})(A_l^+A_p^+\eta_l^+\eta_p^+ + A_l^-A_p^+\eta_l^-\eta_p^+ + BA_lA_p^+\eta_l\eta_p^+)+(B_{lp}^{(o)*}, E_{lp}^{(o)*})(A_l^+A_p^-\eta_l^+\eta_p^- + A_l^-A_p^-\eta_l^-\eta_p^- + BA_lA_p^-\eta_l\eta_p^-)+B(B_{lp}^{(s)}, E_{lp}^{(s)})(A_l^+A_p\eta_l^+\eta_p + A_l^-A_p\eta_l^-\eta_p + BA_lA_p\eta_l\eta_p)], \qquad \text{(B1a)}$$

$$\phi_{10} = \sum_{lp=-N}^{N}[f_{lp}^{(11)}A_l^+A_p^+\eta_l^+\eta_p^+ + f_{lp}^{(12)}A_l^+A_p^-\eta_l^+\eta_p^- + f_{lp}^{(21)}A_l^-A_p^+\eta_l^-\eta_p^+ + f_{lp}^{(22)}A_l^-A_p^-\eta_l^-\eta_p^- + B(f_{lp}^{(10)}A_l^+A_p\eta_l^+\eta_p + f_{lp}^{(20)}A_l^-A_p\eta_l^-\eta_p + f_{lp}^{(01)}A_lA_p^+\eta_l\eta_p^+ + f_{lp}^{(02)}A_lA_p^-\eta_l\eta_p^-)+Bf_{lp}^{(00)}A_lA_p\eta_l\eta_p]+\sum_{m=-N}^{N}(f_m^+A_m^+\eta_m^+ + f_m^-A_m^-\eta_m^- + g_m^+A_m^+\eta_m^+ + g_m^-A_m^-\eta_m^- + g_mA_m\eta_m), \qquad \text{(B1b)}$$

where the coefficient functions $B_{lp}^{(o)}$, $E_{lp}^{(o)}$, $B_{lp}^{(s)}$ and $E_{lp}^{(s)}$, which are introduced in (B1a), have the following expressions:

$$(B_{lp}^{(o)}, E_{lp}^{(o)})=(B_{lp}^{(0)}, E_{lp}^{(0)})+(B_{lp}^{(1)}, E_{lp}^{(1)})\cos(2\pi z)+(B_{lp}^{(2)}, E_{lp}^{(2)})\sin(2\pi z)+(B_{lp}^{(3)}, E_{lp}^{(3)})\cos(\pi z)\exp[i\omega_{01}S_p(z-1)]+(B_{lp}^{(4)}, E_{lp}^{(4)})\sin(\pi z)\exp[i\omega_{01}S_p(z-1)], \qquad \text{(B2a)}$$

$$(B_{lp}^{(s)}, E_{lp}^{(s)})=(B_{lp}^{(5)}, E_{lp}^{(5)})\sin(2\pi z)+(B_{lp}^{(6)}, E_{lp}^{(6)})\sin(\pi z). \qquad \text{(B2b)}$$

Here

$$(B_{lp}^{(0)}, E_{lp}^{(0)})=(1, a_{lp}^2\sqrt{G})C_1\exp(r_{lp}^{(1)}z)+(1, a_{lp}^2\sqrt{G})C_2\exp(-r_{lp}^{(1)}z)+(1, -a_{lp}^2\sqrt{G})C_3\exp(r_{lp}^{(2)}z)+(1, -a_{lp}^2\sqrt{G})C_4\exp(-r_{lp}^{(2)}z)+(1, -GR_{00})L_{lp}^{(2)} \quad \text{for } \psi_{lp}\neq\pm1, \qquad \text{(B2c)}$$

where

$a_{lp}^2 \equiv 2\pi^2(1+\psi_{lp})$, $r_{lp}^{(1)} \equiv (a_{lp}^2+\sqrt{G}\, R_{00}\, a_{lp})^{0.5}$, $r_{lp}^{(2)} \equiv (a_{lp}^2-\sqrt{G}\, R_{00}\, a_{lp})^{0.5}$, $L_{lp}^{(2)} \equiv i\, \omega_{01}K_1S_p\pi^3/[CG\sqrt{G}(\pi^2-\omega_{01}^2)(a_{lp}^2-GR_{00}^2)]$, $C_1 \equiv [M_{lp}^{(1)}\exp(-r_{lp}^{(1)})-M_{lp}^{(3)}]/[\exp(r_{lp}^{(1)})-\exp(-r_{lp}^{(1)})]$, $C_2 \equiv [M_{lp}^{(1)}\exp(r_{lp}^{(1)})-M_{lp}^{(3)}]/[\exp(r_{lp}^{(1)})-\exp(-r_{lp}^{(1)})]$, $C_3 \equiv [M_{lp}^{(2)}\exp(-r_{lp}^{(2)})-M_{lp}^{(4)}]/[\exp(r_{lp}^{(2)})-\exp(-r_{lp}^{(2)})]$, $C_4 \equiv [M_{lp}^{(2)}\exp(r_{lp}^{(2)})-M_{lp}^{(4)})]/[\exp(r_{lp}^{(2)})-\exp(-r_{lp}^{(2)})]$, $M_{lp}^{(1)} \equiv 0.5[\sqrt{G}\, R_{00}L_{lp}^{(2)}/a_{lp}^2 - L_{lp}^{(2)}+L_{lp}^{(3)}+L_{lp}^{(4)}]$, $L_{lp}^{(3)} \equiv -B_{lp}^{(1)}-B_{lp}^{(3)}\exp(-i\omega_{01}S_p)$, $L_{lp}^{(4)} \equiv [-E_{lp}^{(1)}-E_{lp}^{(3)}\exp(-i\omega_{01}S_p)]/a_{lp}^2$, $M_{lp}^{(2)} \equiv 0.5[-\sqrt{G}\, R_{00}L_{lp}^{(2)}/a_{lp}^2-L_{lp}^{(2)}+L_{lp}^{(3)}-L_{lp}^{(4)}]$, $M_{lp}^{(3)} \equiv 0.5[\sqrt{G}\, R_{00}L_{lp}^{(2)}/a_{lp}^2-L_{lp}^{(2)}+L_{lp}^{(5)} + L_{lp}^{(6)}]$, $L_{lp}^{(5)} \equiv -B_{lp}^{(1)}+B_{lp}^{(3)}$, $L_{lp}^{(6)} \equiv (-E_{lp}^{(1)}+E_{lp}^{(3)})/a_{lp}^2$, $M_{lp}^{(4)} \equiv 0.5[-\sqrt{G}\, R_{00}L_{lp}^{(2)}/a_{lp}^2-L_{lp}^{(2)}+L_{lp}^{(5)}-L_{lp}^{(6)}]$; (B2d)

$B_{lp}^{(0)}=R_{00}[a_{lp}^2(L_{lp}^{(6)}-L_{lp}^{(4)})z^3/6+a_{lp}^2L_{lp}^{(4)}z^2/2]+L_{lp}^{(1)}z^2/2+\{-L_{lp}^{(3)}-0.5L_{lp}^{(1)}-R_{00}[a_{lp}^2(L_{lp}^{(6)}-L_{lp}^{(4)})/6+a_{lp}^2L_{lp}^{(4)}/2]+L_{lp}^{(5)}\}z+L_{lp}^{(3)}$ for $\psi_{lp}= -1$, $E_{lp}^{(0)}=a_{lp}^2(L_{lp}^{(6)}-L_{lp}^{(4)})z+a_{lp}^2L_{lp}^{(4)}$ for $\psi_{lp} = -1$, (B2e)

where

$L_{lp}^{(1)} \equiv -i\omega_{01}\pi^3S_pK_1/[CG\sqrt{G}\,(\pi^2-\omega_{01}^2)]$; (B2f)

$B_{lp}^{(0)}=C_5\exp(r_{lp}^{(1)}z)+C_6\exp(-r_{lp}^{(1)}z)+C_7\, z+C_8+0.25L_{lp}^{(1)}z$ for $\psi_{lp}=1$, $E_{lp}^{(0)}=2\pi\sqrt{G}[C_5 \exp(r_{lp}^{(1)}z)+C_6\exp(-r_{lp}^{(1)}z)-C_7\, z-C_8]-\sqrt{G}\, L_{lp}^{(1)}(0.5\pi z^2+0.25/\pi)$ for $\psi_{lp}=1$, (B2g)

where

$C_5 \equiv [L_{lp}^{(8)}-L_{lp}^{(7)}\exp(-r_{lp}^{(1)})]/[\exp(r_{lp}^{(1)})-\exp(-r_{lp}^{(1)})]$, $L_{lp}^{(7)} \equiv 0.5L_{lp}^{(3)}+L_{lp}^{(4)}a_{lp}^2/(4\pi\sqrt{G})+L_{lp}^{(1)}/(16\pi^2)$, $L_{lp}^{(8)} \equiv 0.5L_{lp}^{(5)}+L_{lp}^{(6)}a_{lp}^2/(4\pi\sqrt{G})-L_{lp}^{(1)}[1/8-\pi\sqrt{G}/4-\sqrt{G}/(8\pi)]$, $C_6 \equiv [L_{lp}^{(7)}\exp(r_{lp}^{(1)}) -L_{lp}^{(8)}]/[\exp(r_{lp}^{(1)})-\exp(-r_{lp}^{(1)})]$, $C_7 \equiv -C_5\exp(r_{lp}^{(1)})-C_6\exp(-r_{lp}^{(1)})-C_8-0.25L_{lp}^{(1)}+L_{lp}^{(5)}$, $C_8 \equiv -L_{lp}^{(7)} + L_{lp}^{(3)}$; (B2h)

$B_{lp}^{(1)}=\{-2i\omega_{01}S_pK_1\ \pi^3/[CG\sqrt{G}(\pi^2-\omega_{01}^2)]\}/\{a_{lp}^2+4\pi^2-GR_{00}a_{lp}^2/(a_{lp}^2+4\pi^2)\}$, $E_{lp}^{(1)}=-GR_{00}a_{lp}^2 B_{lp}^{(1)}/(a_{lp}^2+4\pi^2)$; (B2i)

$B_{lp}^{(2)}=\{\pi^2(1-\psi_{lp})-2K_1\pi^4(a_{lp}^2+4\pi^2)/[R_{00}CG\sqrt{G}(\pi^2-\omega_{01}^2)]\}/\{GR_{00}a_{lp}^2-(a_{lp}^2+4\pi^2)^2/R_{00}\}$,

$E_{lp}^{(2)}=2K_1\pi^4/[R_{00}CG\sqrt{G}(\pi^2-\omega_{01}^2)]-(a_{lp}^2+4\pi^2)B_{lp}^{(2)}/R_{00}$; (B2j)

$E_{lp}^{(3)}=E_{lp}^{(4)}=0$ for $\psi_{lp}=-1$, $B_{lp}^{(3)}=-i\omega_{01}S_pK_1\pi^3(\omega_{01}^2-5\pi^2)/[CG\sqrt{G}(\pi^2-\omega_{01}^2)^3]$ for $\psi_{lp}=-1$,

$B_{lp}^{(4)}=K_1\pi^4(3\pi^2+\omega_{01}^2)/[CG\sqrt{G}(\pi^2-\omega_{01}^2)^3]$ for $\psi_{lp}=-1$; (B2k)

$(E_{lp}^{(3)}, E_{lp}^{(4)})=[(C_9C_{13}-C_{11}C_{15}), (C_{10}C_{15}-C_9C_{12})]/(C_{10}C_{13}-C_{11}C_{12})$ for $\psi_{lp}\neq -1$, $(B_{lp}^{(3)}, B_{lp}^{(4)}) =\{E_{lp}^{(3)}[(-a_{lp}^2-\pi^2-\omega_{01}^2), (-2i\omega_{01}S_p\pi)]+E_{lp}^{(4)}[(2i\omega_{01}S_p\pi), (-a_{lp}^2-\pi^2-\omega_{01}^2)]\}/(GR_{00}a_{lp}^2)$ for $\psi_{lp}\neq -1$, (B2l)

where

$C_9\equiv i\omega_{01}S_pK_1\pi^3a_{lp}^2/[CG\sqrt{G}(\pi^2-\omega_{01}^2)]$, $C_{10}\equiv C_{13}\equiv -a_{lp}^2R_{00}+[(a_{lp}^2+\pi^2+\omega_{01}^2)+4\omega_{01}^2\pi^2]/GR_{00}$, $C_{11}\equiv -4i\omega_{01}S_p\pi(\pi^2+\omega_{01}^2+a_{lp}^2)/GR_{00}\equiv -C_{12}$, $C_{14}\equiv -3K_1\pi^4a_{lp}^2/[CG\sqrt{G}(\pi^2-\omega_{01}^2)]$; (B2m)

$B_{lp}^{(5)}=[\pi^2(1-\psi_{lp})-2K_1\pi^2(a_{lp}^2+4\pi^2)/(CG\sqrt{G}\,R_{00})]/[GR_{00}\,a_{lp}^2-(a_{lp}^2+4\pi^2)^2/R_{00}]$, $E_{lp}^{(5)}=2K_1\pi^2/(CG\sqrt{G}\,R_{00})-(a_{lp}^2+4\pi^2)B_{lp}^{(5)}/R_{00}$; (B2n)

$B_{lp}^{(6)}=3K_1\pi^2(a_{lp}^2+\pi^2)/\{CG\sqrt{G}\,[(a_{lp}^2+\pi^2)^2-GR_{00}^2a_{lp}^2]\}$, $E_{lp}^{(6)}=-GR_{00}\,a_{lp}^2B_{lp}^{(6)}/(a_{lp}^2+\pi^2)$. (B2o)

The coefficient functions f_m^+, f_m^-, g_m^+, g_m^-, g_m and $f_{lp}^{(ij)}$ (i, j =0, 1, 2) , which are given in (B1b), have the following expressions:

$f_m^+=\{-2\pi^3\omega_{11}/[CG(\pi^2-\omega_{01}^2)^2]\}\{iS_m[(\pi^2+\omega_{01}^2)/\pi]\sin(\pi z)+2\omega_{01}\cos(\pi z)+[2\omega_{01}+iS_m(z-1)(\pi^2-\omega_{01}^2)]\exp[i\omega_{01}S_m(z-1)]\}$, $f_m^-=f_m^{+*}$; (B3a)

$g_m^+=\{-R_{10}\pi/[C\sqrt{G}(\pi^2-\omega_{01}^2)]\}\{\pi\cos(\pi z)+i\omega_{01}S_m\sin(\pi z)+\pi\exp[i\omega_{01}S_m(z-1)]\}$, $g_m^-=g_m^{+*}$, $g_m=-R_{10}[\cos(\pi z)+1]/(C\sqrt{G})$; (B3b)

$f_{lp}^{(ij)} = -\langle H_{lp}^{(ij)} \exp(-h_{lp}^{(ij)} z)\rangle_1 \exp(-h_{lp}^{(ij)} z) + \langle H_{lp}^{(ij)} \exp(h_{lp}^{(ij)} z)\rangle_z \exp(-h_{lp}^{(ij)} z)$

$(i, j=0, 1, 2)$, (B3c)

where

$\langle f\rangle_z \equiv \int_0^z f dz$, $\langle f\rangle_1 \equiv \int_0^1 f dz$, $(h_{lp}^{(00)}, H_{lp}^{(00)}) \equiv \{0, [R_{00}\, a_{lp}^2 B_{lp}^{(s)} - \pi^2 \sin(2\pi z)(1-\psi_{lp})/G]/C\}$, $(h_{lp}^{(10)}, H_{lp}^{(10)}) \equiv \{-i\omega_{01} S_l, H_{lp}^{(00)}\}$, $(h_{lp}^{(20)}, H_{lp}^{(20)}) \equiv \{i\omega_{01} S_l, H_{lp}^{(00)}\}$, $(h_{lp}^{(01)}, H_{lp}^{(01)}) \equiv \{-i\omega_{01} S_p, [R_{00}\, a_{lp}^2 B_{lp}^{(o)} - \pi^2 \sin(2\pi z)(1-\psi_{lp})/G]/C\}$, $(h_{lp}^{(02)}, H_{lp}^{(02)}) \equiv \{h_{lp}^{(01)*}, H_{lp}^{(01)*}\}$, $(h_{lp}^{(11)}, H_{lp}^{(11)}) \equiv \{-i\omega_{01}(S_l+S_p), H_{lp}^{(01)}\}$, $(h_{lp}^{(12)}, H_{lp}^{(12)}) \equiv \{-i\omega_{01}(S_l-S_p), H_{lp}^{(01)*}\}$, $(h_{lp}^{(21)}, H_{lp}^{(21)}) \equiv \{h_{lp}^{(12)*}, H_{lp}^{(01)}\}$, $(h_{lp}^{(22)}, H_{lp}^{(22)}) \equiv \{h_{lp}^{(11)*}, H_{lp}^{(01)*}\}$. (B3d)