Appendix to "On the stability of a compressible axisymmetric rotating flow in a pipe"

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Appendix A. Derivation of perturbation equations (30) and (31)

Elimination of pressure from (19) and (20) by cross differentiation in terms of x and y, respectively, followed by subtraction gives a relationship between ψ_1, K_{1x} and ρ_1:

$$\left(\frac{\psi_{1xx}}{2y}+\psi_{1yy}\right)_t + w_0\left(\frac{\psi_{1xx}}{2y}+\psi_{1yy}\right)_x - w_{0yy}\psi_{1x} + \frac{\omega K_0\rho_0}{2y^2}K_{1x}$$
$$+\gamma M_0^2\left(\frac{\omega^2K_0^2}{4y^2}\rho_{1x} - 2w_0w_{0y}\rho_{1x} - w_0^2\rho_{1xy} - \int_0^x \rho_{1ytt}dx' - 2w_{0y}\rho_{1t} - 2w_0\rho_{1yt}\right) = 0. \quad \text{(A-1)}$$

Solving (A-1) for K_{1x} and substituting in linearized $\theta-$momentum equation (21) results in:

$$K_{1t} = \frac{\omega K_{0y}}{\rho_0}\psi_{1x} + \frac{2y^2w_0}{\omega K_0\rho_0}\left[\left(\frac{\psi_{1xx}}{2y}+\psi_{1yy}\right)_t + w_0\left(\frac{\psi_{1xx}}{2y}+\psi_{1yy}\right)_x - w_{0yy}\psi_{1x}\right.$$
$$\left.+\gamma M_0^2\left(\frac{\omega^2K_0^2}{4y^2}\rho_{1x} - 2w_0w_{0y}\rho_{1x} - w_0^2\rho_{1xy} - \int_0^x \rho_{1ytt}dx' - 2w_{0y}\rho_{1t} - 2w_0\rho_{1yt}\right)\right] \quad \text{(A-2)}$$

Elimination of K_1 from (A-1) and (A-2) by cross differentiation in terms of t and x, respectively, followed by subtraction, and multiplying by $\omega K_0\rho_0/(2y^2w_0)$, gives

$$2\left(\frac{\psi_{1xx}}{2y}+\psi_{1yy}\right)_{xt} + \frac{1}{w_0}\left(\frac{\psi_{1xx}}{2y}+\psi_{1yy}\right)_{tt} + w_0\left(\frac{\psi_{1xx}}{2y}+\psi_{1yy}\right)_{xx}$$
$$+\left(\frac{\omega^2K_0K_{0y}}{2y^2w_0} - w_{0yy}\right)\psi_{1xx} - \frac{w_{0yy}}{w_0}\psi_{1xt}$$
$$= -\gamma M_0^2\left[\frac{\omega^2K_0^2}{4y^2}\left(\rho_{1xx}+\frac{\rho_{1xt}}{w_0}\right) - 4w_{0y}\rho_{1xt} - 3\rho_{1ytt} - 3w_0\rho_{1xyt}\right.$$
$$\left.-\frac{1}{w_0}\int_0^x \rho_{1yttt}dx' - \frac{2w_{0y}}{w_0}\rho_{1tt} - 2w_0w_{0y}\rho_{1xx} - w_0^2\rho_{1xxy}\right]. \quad \text{(A-3)}$$

Differentiation of (A-3) with respect to x gives (30).

Differentiating (22) with respect to x gives

$$\gamma M_0^2(\rho_0T_{1xt}+\rho_0w_0T_{1xx}) = \frac{\gamma-1}{\gamma}\left[\sqrt{2y}P_{0y}u_{1x} + \gamma M_0^2(P_{1xt}+w_0P_{1xx})\right] - \rho_0\sqrt{2y}T_{0y}u_{1x} \quad \text{(A-4)}$$

From (16) we have $\rho_0 T_1 = P_1 - \rho_1 T_0$. Substituting this in (A-4) gives

$$M_0^2(P_{1xt} + w_0 P_{1xx}) - \gamma M_0^2 T_0(\rho_{1xt} + w_0\rho_{1xx}) = \left(\frac{\gamma-1}{\gamma}\frac{P_{0y}}{\rho_0} - T_{0y}\right)\sqrt{2y}\rho_0 u_{1x}. \quad \text{(A-5)}$$

Using (20) to express P_{1xt} and P_{1xx} in (A-5) and multiplying by $-\frac{1}{w_0}$ gives

$$\begin{aligned}\gamma M_0^2 &\left[\left(\frac{T_0}{w_0} - 3M_0^2 w_0\right)\rho_{1xt} + (T_0 - M_0^2 w_0^2)\rho_{1xx} - 3M_0^2\rho_{1tt} - \frac{M_0^2}{w_0}\int_0^x \rho_{1ttt}dx'\right] \\ &= \left(M_0^2 w_{0y} - \frac{T_{0y}}{w_0} + \frac{\gamma-1}{\gamma}\frac{P_{0y}}{\rho_0 w_0}\right)\psi_{1xx} - M_0^2 w_0\psi_{1xxy} \\ &\qquad -\frac{M_0^2}{w_0}\psi_{1ytt} + M_0^2\frac{w_{0y}}{w_0}\psi_{1xt} - 2M_0^2\psi_{1xyt}. \qquad \text{(A-6)}\end{aligned}$$

Differentiating (A-6) with respect to x gives (31).

Appendix B. Boundary conditions for (43) and (44)

The substitution of (42) into (32)-(41) gives boundary conditions for $\tilde{\phi}$ and $\tilde{\rho}$

$$\tilde{\phi}(x,0) = 0, \qquad \tilde{\phi}(x,1/2) = \tilde{\phi}(0,1/2) \qquad \text{(B-1)}$$

for $0 \le x \le x_0$ and

$$\tilde{\phi}_{xx}(0,y) = 0, \qquad \text{(B-2)}$$

$$\gamma M_0^2 w_0\tilde{\rho}(0,y) = \tilde{\phi}_y(0,y), \qquad \text{(B-3)}$$

$$\left(\frac{T_0}{w_0}\tilde{\phi}_y(0,y)\right)_y = \gamma M_0^2\left(\frac{\sigma\tilde{\phi}_x(0,y)}{2y} + \omega^2\frac{K_0^2}{4y^2 w_0}\tilde{\phi}_y(0,y)\right)$$
$$\text{with } \tilde{\phi}(0,0) = \tilde{\phi}_y(0,0) = 0, \qquad \text{(B-4)}$$

$$\begin{aligned}&\frac{\tilde{\phi}_{xxx}(0,y)}{2y} + \tilde{\phi}_{xyy}(0,y) + \left(\frac{\omega^2 K_0 K_{0y}}{2y^2 w_0^2} - \frac{w_{0yy}}{w_0}\right)\tilde{\phi}_x(0,y) + \sigma\frac{\tilde{\phi}_{yy}(0,y)}{w_0} \\ &= -\gamma M_0^2\left[\left(\frac{\omega^2 K_0^2}{4y^2 w_0} - 2w_{0y}\right)\tilde{\rho}_x(0,y) - w_0\tilde{\rho}_{xy}(0,y) - 2\sigma\frac{w_{0y}}{w_0}\tilde{\rho}(0,y) - 2\sigma\tilde{\rho}_y(0,y)\right], \quad \text{(B-5)}\end{aligned}$$

$$\begin{aligned}&\sigma\frac{\tilde{\phi}_{xxx}(0,y)}{yw_0} + \sigma\frac{2\tilde{\phi}_{xyy}(0,y)}{w_0} + \frac{\sigma^2}{w_0^2}\tilde{\phi}_{yy}(0,y) + \frac{\tilde{\phi}_{xxxx}(0,y)}{2y} - \sigma\frac{w_{0yy}}{w_0^2}\tilde{\phi}_x(0,y) \\ &= -\frac{\gamma M_0^2}{w_0}\left[\frac{\omega^2 K_0^2}{4y^2}\left(\tilde{\rho}_{xx}(0,y) + \sigma\frac{\tilde{\rho}_x(0,y)}{w_0}\right) - 4\sigma w_{0y}\tilde{\rho}_x(0,y) - 3\sigma^2\tilde{\rho}_y(0,y)\right. \\ &\qquad \left. -3\sigma w_0\tilde{\rho}_{xy}(0,y) - \sigma^2\frac{2w_{0y}}{w_0}\tilde{\rho}(0,y) - 2w_0 w_{0y}\tilde{\rho}_{xx}(0,y) - w_0^2\tilde{\rho}_{xxy}(0,y)\right], \quad \text{(B-6)}\end{aligned}$$

$$\begin{aligned}&\gamma\left[\sigma\left(\frac{T_0}{w_0} - 3M_0^2 w_0\right)\tilde{\rho}_x(0,y) + (T_0 - M_0^2 w_0^2)\tilde{\rho}_{xx}(0,y) - 3\sigma^2 M_0^2\tilde{\rho}(0,y)\right] \\ &\qquad = \sigma\frac{w_{0y}}{w_0}\tilde{\phi}_x(0,y) - \sigma^2\frac{1}{w_0}\tilde{\phi}_y(0,y) - 2\sigma\tilde{\phi}_{xy}(0,y), \quad \text{(B-7)}\end{aligned}$$

$$\tilde{\phi}_x(x_0,y) = 0, \qquad \sigma\tilde{\rho}(x_0,y) + w_0\tilde{\rho}_x(x_0,y) = 0 \qquad \text{(B-8)}$$

for $0 \le y \le 1/2$.

Appendix C. Analysis of imaginary parts of (43) and (44)

Substituting (60) into (43) and (44), collecting terms of the orders $\epsilon_I,\ \sigma_I,\ \epsilon_I\sigma_R,\ \epsilon_I\Delta\Omega$ and neglecting terms of the orders $O(\sigma_R^2,\ \sigma_I^2,\ \sigma_R\sigma_I,\ \sigma_I\epsilon_R,\ \sigma_I\Delta\Omega)$ and higher gives

$$\begin{aligned}
&\epsilon_I\left\{\frac{\phi_{Ixx}}{2y}+\phi_{Iyy}+\left(\frac{\Omega_1K_0K_{0y}}{2y^2w_0^2}-\frac{w_{0yy}}{w_0}\right)\phi_I+\gamma M_0^2\left[\left(\frac{\Omega_1K_0^2}{4y^2w_0}-2w_{0y}\right)\rho_I-w_0\rho_{Iy}\right]\right\}_{xxx}\\
&+\sigma_I\left\{\frac{2}{w_0}\left(\frac{\psi_{1cxx}}{2y}+\psi_{1cyy}\right)-\frac{w_{0yy}}{w_0^2}\psi_{1c}+\gamma M_0^2\left[\left(\frac{\Omega_1K_0^2}{4y^2w_0^2}-4\frac{w_{0y}}{w_0}\right)\rho_{1c}-3\rho_{1cy}\right]\right\}_{xx}\\
&+\epsilon_I\sigma_R\left\{\frac{2}{w_0}\left(\frac{\phi_{Ixx}}{2y}+\phi_{Iyy}\right)-\frac{w_{0yy}}{w_0^2}\phi_I+\gamma M_0^2\left[\left(\frac{\Omega_1K_0^2}{4y^2w_0^2}-4\frac{w_{0y}}{w_0}\right)\rho_I-3\rho_{Iy}\right]\right\}_{xx}\\
&\qquad+\epsilon_I\Delta\Omega\left\{\frac{K_0K_{0y}}{2y^2w_0^2}\phi_I+\gamma M_0^2\frac{K_0^2}{4y^2w_0}\rho_I\right\}_{xxx}=0,
\end{aligned}\tag{C-1}$$

$$\begin{aligned}
&\epsilon_I\left\{\rho_I-\frac{1}{\gamma M_0^2(T_0-M_0^2w_0^2)}\left[\left(M_0^2w_{0y}-\frac{T_{0y}}{w_0}+\frac{(\gamma-1)M_0^2\Omega_1K_0^2}{4y^2w_0}\right)\phi_I-M_0^2w_0\phi_{Iy}\right]\right\}_{xxx}\\
&\qquad+\frac{\sigma_I}{T_0-M_0^2w_0^2}\left\{\frac{T_0-3M_0^2w_0^2}{w_0}\rho_{1c}-\frac{w_{0y}}{\gamma w_0}\psi_{1c}+\frac{2}{\gamma}\psi_{1cy}\right\}_{xx}\\
&\qquad+\frac{\epsilon_I\sigma_R}{T_0-M_0^2w_0^2}\left\{\frac{T_0-3M_0^2w_0^2}{w_0}\rho_I-\frac{w_{0y}}{\gamma w_0}\phi_I+\frac{2}{\gamma}\phi_{Iy}\right\}_{xx}\\
&\qquad+\epsilon_I\Delta\Omega\left\{\frac{\gamma-1}{\gamma(T_0-M_0^2w_0^2)}\frac{K_0^2}{4y^2w_0}\phi_I\right\}_{xxx}=0.
\end{aligned}\tag{C-2}$$

From (B-1)-(B-8), the boundary conditions for these equations are:

$$\phi_I(x,0)=0,\qquad \phi_I(x,1/2)=\phi_I(0,1/2)\tag{C-3}$$

for $0 \le x \le x_0$ and

$$\phi_{Ixx}(0,y)=0,\tag{C-4}$$

$$\gamma M_0^2w_0\rho_I(0,y)=\phi_{Iy}(0,y),\tag{C-5}$$

$$\begin{aligned}
&\epsilon_I\left\{\left(\frac{T_0}{w_0}\phi_{Iy}(0,y)\right)_y-\gamma M_0^2\frac{\Omega_1K_0^2}{4y^2w_0}\phi_{Iy}(0,y)\right\}-\sigma_I\left\{\gamma M_0^2\frac{\psi_{1cx}(0,y)}{2y}\right\}\\
&-\epsilon_I\sigma_R\left\{\gamma M_0^2\frac{\phi_{Ix}(0,y)}{2y}\right\}-\epsilon_I\Delta\Omega\left\{\gamma M_0^2\frac{K_0^2}{4y^2w_0}\phi_{Iy}(0,y)\right\}=0\\
&\qquad\text{with}\ \ \phi_I(0,0)=\phi_{Iy}(0,0)=0,
\end{aligned}\tag{C-6}$$

$$\begin{aligned}
&\epsilon_I\left\{\frac{\phi_{Ixxx}(0,y)}{2y}+\phi_{Ixyy}(0,y)+\left(\frac{\Omega_1K_0K_{0y}}{2y^2w_0^2}-\frac{w_{0yy}}{w_0}\right)\phi_{Ix}(0,y)\right.\\
&\qquad\left.+\gamma M_0^2\left[\left(\frac{\Omega_1K_0^2}{4y^2w_0}-2w_{0y}\right)\rho_{Ix}(0,y)-w_0\rho_{Ixy}(0,y)\right]\right\}\\
&+\epsilon_I\sigma_R\left\{\frac{\phi_{Iyy}(0,y)}{w_0}-2\gamma M_0^2\left(\frac{w_{0y}}{w_0}\rho_I(0,y)+\rho_{Iy}(0,y)\right)\right\}
\end{aligned}$$

$$+\epsilon_I \Delta\Omega \left\{ \frac{K_0 K_{0y}}{2y^2 w_0^2} \phi_{Ix}(0,y) + \gamma M_0^2 \frac{K_0^2}{4y^2 w_0} \rho_{Ix}(0,y) \right\} = 0, \tag{C-7}$$

$$\epsilon_I \left\{ \frac{\phi_{Ixxxx}(0,y)}{2y} + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0} - 2w_{0y} \right) \rho_{Ixx}(0,y) - w_0 \rho_{Ixxy}(0,y) \right] \right\}$$
$$+\sigma_I \left\{ \frac{\psi_{1cxxx}(0,y)}{yw_0} + \frac{2\psi_{1cxyy}(0,y)}{w_0} - \frac{w_{0yy}}{w_0^2} \psi_{1cx}(0,y) \right.$$
$$\left. + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 4\frac{w_{0y}}{w_0} \right) \rho_{1cx}(0,y) - 3\rho_{1cxy}(0,y) \right] \right\}$$
$$+\epsilon_I \sigma_R \left\{ \frac{\phi_{Ixxx}(0,y)}{yw_0} + \frac{2\phi_{Ixyy}(0,y)}{w_0} - \frac{w_{0yy}}{w_0^2} \phi_{Ix}(0,y) \right.$$
$$\left. + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 4w_{0y} \right) \rho_{Ix}(0,y) - 3\rho_{Ixy}(0,y) \right] \right\}$$
$$+\epsilon_I \Delta\Omega \left\{ \gamma M_0^2 \frac{K_0^2}{4y^2 w_0} \rho_{Ixx}(0,y) \right\} = 0, \tag{C-8}$$

$$\epsilon_I \rho_{Ixx}(0,y) + \frac{\sigma_I}{T_0 - M_0^2 w_0^2} \left\{ \frac{T_0 - 3M_0^2 w_0^2}{w_0} \rho_{1cx}(0,y) - \frac{w_{0y}}{\gamma w_0} \psi_{1cx}(0,y) + \frac{2}{\gamma} \psi_{1cxy}(0,y) \right\}$$
$$+ \frac{\epsilon_I \sigma_R}{T_0 - M_0^2 w_0^2} \left\{ \frac{T_0 - 3M_0^2 w_0^2}{w_0} \rho_{Ix}(0,y) - \frac{w_{0y}}{\gamma w_0} \phi_{Ix}(0,y) + \frac{2}{\gamma} \phi_{Ixy}(0,y) \right\} = 0, \tag{C-9}$$

$$\phi_{Ix}(x_0,y) = 0, \qquad \sigma_I \rho_{1c}(x_0,y) + w_0 \epsilon_I \rho_{Ix}(x_0,y) + \epsilon_I \sigma_R \rho_I(x_0,y) = 0 \tag{C-10}$$

for $0 \le y \le 1/2$.

Two integrations with respect to x of (C-1) and (C-2) and the use of boundary conditions (C-4) and (C-7)-(C-10) result in

$$\epsilon_I \left\{ \frac{\phi_{Ixx}}{2y} + \phi_{Iyy} + \left(\frac{\Omega_1 K_0 K_{0y}}{2y^2 w_0^2} - \frac{w_{0yy}}{w_0} \right) \phi_I + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0} - 2w_{0y} \right) \rho_I - w_0 \rho_{Iy} \right] \right\}_x$$
$$+\sigma_I \left\{ \frac{2}{w_0} \left(\frac{\psi_{1cxx}}{2y} + \psi_{1cyy} \right) - \frac{w_{0yy}}{w_0^2} \psi_{1c} + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 4\frac{w_{0y}}{w_0} \right) \rho_{1c} - 3\rho_{1cy} \right] \right\}$$
$$+\epsilon_I \sigma_R \left\{ \frac{2}{w_0} \left(\frac{\phi_{Ixx}}{2y} + \phi_{Iyy} \right) - \frac{w_{0yy}}{w_0^2} \phi_I + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 4\frac{w_{0y}}{w_0} \right) \rho_I - 3\rho_{Iy} \right] \right\}$$
$$+\epsilon_I \Delta\Omega \left\{ \frac{K_0 K_{0y}}{2y^2 w_0^2} \phi_I + \gamma M_0^2 \frac{K_0^2}{4y^2 w_0} \rho_I \right\}_x = \epsilon_I \sigma_R f_1(y), \tag{C-11}$$

$$\epsilon_I \left\{ \rho_I - \frac{1}{\gamma M_0^2 (T_0 - M_0^2 w_0^2)} \left[\left(M_0^2 w_{0y} - \frac{T_{0y}}{w_0} + \frac{(\gamma-1) M_0^2 \Omega_1 K_0^2}{4y^2 w_0} \right) \phi_I - M_0^2 w_0 \phi_{Iy} \right] \right\}_x$$
$$+ \frac{\sigma_I}{T_0 - M_0^2 w_0^2} \left\{ \frac{T_0 - 3M_0^2 w_0^2}{w_0} \rho_{1c} - \frac{w_{0y}}{\gamma w_0} \psi_{1c} + \frac{2}{\gamma} \psi_{1cy} \right\}$$
$$+ \frac{\epsilon_I \sigma_R}{T_0 - M_0^2 w_0^2} \left\{ \frac{T_0 - 3M_0^2 w_0^2}{w_0} \rho_I - \frac{w_{0y}}{\gamma w_0} \phi_I + \frac{2}{\gamma} \phi_{Iy} \right\}$$
$$+\epsilon_I \Delta\Omega \left\{ \frac{\gamma - 1}{\gamma (T_0 - M_0^2 w_0^2)} \frac{K_0^2}{4y^2 w_0} \phi_I \right\}_x = \sigma_I f_2(y) + \epsilon_I \sigma_R f_3(y), \tag{C-12}$$

Here

$$f_1(y) = \frac{\phi_{Iyy}(0,y)}{w_0} - \frac{w_{0yy}}{w_0^2}\phi_I(0,y) + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 2\frac{w_{0y}}{w_0} \right) \rho_I(0,y) - \rho_{Iy}(0,y) \right],$$

$$f_2(y) = -\frac{1}{T_0 - M_0^2 w_0^2} \left(2M_0^2 w_0 S(y) + \frac{w_{0y}}{\gamma w_0}\Phi(y) - \frac{2}{\gamma}\Phi_y(y) \right), \tag{C-13}$$

$$f_3(y) = -\frac{1}{T_0 - M_0^2 w_0^2} \left(2M_0^2 w_0 \rho_I(x_0,y) + \frac{w_{0y}}{\gamma w_0}\phi_I(x_0,y) - \frac{2}{\gamma}\phi_{Iy}(x_0,y) \right).$$

Solution of (C-12) for $\epsilon_I \rho_{Ix}$, substitution of the result in (C-11), multiplication by $(T_0 - M_0^2 w_0^2)/T_0$ and an additional integration with respect to x gives (61).

Appendix D. Analysis of real parts of (43) and (44)

Substituting expansions (60) into (43) and (44) and neglecting terms of the orders $O(\sigma_R^2,\ \sigma_I^2,\ \sigma_R\epsilon_R,\ \sigma_I\epsilon_I,\ \sigma_R\Delta\Omega)$ and higher gives

$$\epsilon_R \left\{ \frac{\phi_{Rxx}}{2y} + \phi_{Ryy} + \left(\frac{\Omega_1 K_0 K_{0y}}{2y^2 w_0^2} - \frac{w_{0yy}}{w_0} \right) \phi_R + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0} - 2w_{0y} \right) \rho_R - w_0 \rho_{Ry} \right] \right\}_{xxx}$$
$$+ \sigma_R \left\{ \frac{2}{w_0} \left(\frac{\psi_{1cxx}}{2y} + \psi_{1cyy} \right) - \frac{w_{0yy}}{w_0^2}\psi_{1c} + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0^2} - 4\frac{w_{0y}}{w_0} \right) \rho_{1c} - 3\rho_{1cy} \right] \right\}_{xx}$$
$$+ \Delta\Omega \left\{ \frac{K_0 K_{0y}}{2y^2 w_0^2}\psi_{1c} + \gamma M_0^2 \frac{K_0^2}{4y^2 w_0}\rho_{1c} \right\}_{xxx} = 0, \tag{D-1}$$

$$\epsilon_R \left\{ \rho_R - \frac{1}{\gamma M_0^2 (T_0 - M_0^2 w_0^2)} \left[\left(M_0^2 w_{0y} - \frac{T_{0y}}{w_0} + (\gamma - 1) M_0^2 \frac{\Omega_1 K_0^2}{4y^2 w_0} \right) \phi_R - M_0^2 w_0 \phi_{Ry} \right] \right\}_{xxx}$$
$$+ \frac{\sigma_R}{T_0 - M_0^2 w_0^2} \left\{ \frac{T_0 - 3M_0^2 w_0^2}{w_0}\rho_{1c} - \frac{w_{0y}}{\gamma w_0}\psi_{1c} + \frac{2}{\gamma}\psi_{1cy} \right\}_{xx}$$
$$- \Delta\Omega \left\{ \frac{(\gamma - 1)}{\gamma (T_0 - M_0^2 w_0^2)} \frac{K_0^2}{4y^2 w_0}\psi_{1c} \right\}_{xxx} = 0. \tag{D-2}$$

From (B-1)-(B-8), the boundary conditions for these equations are:

$$\phi_R(x,0) = 0, \qquad \phi_R(x,1/2) = \phi_R(0,1/2) \tag{D-3}$$

for $0 \le x \le x_0$ and

$$\phi_{Rxx}(0,y) = 0, \tag{D-4}$$

$$\gamma M_0^2 w_0 \rho_R(0,y) = \phi_{Ry}(0,y), \tag{D-5}$$

$$\epsilon_R \left\{ \left(\frac{T_0}{w_0}\phi_{Ry}(0,y) \right)_y - \gamma M_0^2 \frac{\Omega_1 K_0^2}{4y^2 w_0}\phi_{Ry}(0,y) \right\} - \sigma_R \left\{ \gamma M_0^2 \frac{\psi_{1cx}(0,y)}{2y} \right\} = 0$$
$$\text{with} \ \ \phi_R(0,0) = \phi_{Ry}(0,0) = 0, \tag{D-6}$$

$$\epsilon_R \left\{ \frac{\phi_{Rxxx}(0,y)}{2y} + \phi_{Rxyy}(0,y) + \left(\frac{\Omega_1 K_0 K_{0y}}{2y^2 w_0^2} - \frac{w_{0yy}}{w_0} \right) \phi_{Rx}(0,y) \right.$$
$$\left. + \gamma M_0^2 \left[\left(\frac{\Omega_1 K_0^2}{4y^2 w_0} - 2w_{0y} \right) \rho_{Rx}(0,y) - w_0 \rho_{Rxy}(0,y) \right] \right\}$$

$$+\Delta\Omega\left\{\frac{K_0K_{0y}}{2y^2w_0^2}\psi_{1cx}(0,y)+\gamma M_0^2\frac{K_0^2}{4y^2w_0}\rho_{1cx}(0,y)\right\}=0, \qquad \text{(D-7)}$$

$$\epsilon_R\left\{\frac{\phi_{Rxxxx}(0,y)}{2y}+\gamma M_0^2\left[\left(\frac{\Omega_1K_0^2}{4y^2w_0}-2w_{0y}\right)\rho_{Rxx}(0,y)-w_0\rho_{Rxxy}(0,y)\right]\right\}$$
$$+\sigma_R\left\{\frac{\psi_{1cxxx}(0,y)}{yw_0}+\frac{2\psi_{1cxyy}(0,y)}{w_0}-\frac{w_{0yy}}{w_0^2}\psi_{1cx}(0,y)\right.$$
$$\left.+\gamma M_0^2\left[\left(\frac{\Omega_1K_0^2}{4y^2w_0^2}-4\frac{w_{0y}}{w_0}\right)\rho_{1cx}(0,y)-3\rho_{1cxy}(0,y)\right]\right\}=0, \qquad \text{(D-8)}$$

$$\epsilon_R\rho_{Rxx}(0,y)$$
$$+\frac{\sigma_R}{T_0-M_0^2w_0^2}\left\{\frac{T_0-3M_0^2w_0^2}{w_0}\rho_{1cx}(0,y)-\frac{w_{0y}}{\gamma w_0}\psi_{1cx}(0,y)+\frac{2}{\gamma}\psi_{1cxy}(0,y)\right\}=0, \qquad \text{(D-9)}$$

$$\phi_{Rx}(x_0,y)=0, \qquad \sigma_R\rho_{1c}(x_0,y)+\epsilon_Rw_0\rho_{Rx}(x_0,y)=0 \qquad \text{(D-10)}$$

for $0\le y\le 1/2$.

Two integrations of (D-1) and (D-2) with respect to x and the use of boundary conditions (D-4), (D-7)-(D-10) result in

$$\epsilon_R\left\{\frac{\phi_{Rxx}}{2y}+\phi_{Ryy}+\left(\frac{\Omega_1K_0K_{0y}}{2y^2w_0^2}-\frac{w_{0yy}}{w_0}\right)\phi_R+\gamma M_0^2\left[\left(\frac{\Omega_1K_0^2}{4y^2w_0}-2w_{0y}\right)\rho_R-w_0\rho_{Ry}\right]\right\}_x$$
$$+\sigma_R\left\{\frac{2}{w_0}\left(\frac{\psi_{1cxx}}{2y}+\psi_{1cyy}\right)-\frac{w_{0yy}}{w_0^2}\psi_{1c}+\gamma M_0^2\left[\left(\frac{\Omega_1K_0^2}{4y^2w_0^2}-4\frac{w_{0y}}{w_0}\right)\rho_{1c}-3\rho_{1cy}\right]\right\}$$
$$+\Delta\Omega\left\{\frac{K_0K_{0y}}{2y^2w_0^2}\psi_{1c}+\gamma M_0^2\frac{K_0^2}{4y^2w_0}\rho_{1c}\right\}_x=0, \qquad \text{(D-11)}$$

$$\epsilon_R\left\{\rho_R-\frac{1}{\gamma M_0^2(T_0-M_0^2w_0^2)}\left[\left(M_0^2w_{0y}-\frac{T_{0y}}{w_0}+(\gamma-1)M_0^2\frac{\Omega_1K_0^2}{4y^2w_0}\right)\phi_R-M_0^2w_0\phi_{Ry}\right]\right\}_x$$
$$+\frac{\sigma_R}{T_0-M_0^2w_0^2}\left\{\frac{T_0-3M_0^2w_0^2}{w_0}\rho_{1c}-\frac{w_{0y}}{\gamma w_0}\psi_{1c}+\frac{2}{\gamma}\psi_{1cy}\right\}$$
$$-\Delta\Omega\left\{\frac{(\gamma-1)}{\gamma(T_0-M_0^2w_0^2)}\frac{K_0^2}{4y^2w_0}\psi_{1c}\right\}_x=\sigma_Rf_2(y). \qquad \text{(D-12)}$$

Here $f_2(y)$ is defined in (C-13). Also, the conditions in (D-5) and (D-6) can be solved and show that

$$\epsilon_R\phi_R(0,y)=\sigma_R\gamma M_0^2\frac{\pi}{4x_0}\int_0^y \exp(\alpha(y'))\ [\int_0^{y'} g(y'')\exp(-\alpha(y''))\ dy'']\ dy',$$
$$\gamma M_0^2w_0\rho_R(0,y)=\sigma_R\gamma M_0^2\frac{\pi}{4x_0}\exp(\alpha(y))\ [\int_0^{y'} g(y'')\exp(-\alpha(y''))\ dy''] \qquad \text{(D-13)}$$

where

$$\alpha(y)=-\int_0^y p(y')dy', \quad p(y)=\frac{w_0}{T_0}\left(\frac{T_0}{w_0}\right)_y-\gamma M_0^2\Omega_1\frac{K_0^2}{4y^2T_0}, \quad g(y)=\frac{\Phi(y)}{y}\frac{w_0(y)}{T_0(y)}.$$

Note that $\phi_R(0,y)=0$ when $M_0=0$. Also, note that in the general case $\phi_R(0,1/2)$ is now determined and may not be zero. For example, in the case of a solid-body rotation profile where $K_0 = 2y$ and $w_0 = T_0 = 1$ we find $p(y) = -\gamma M_0^2 \Omega_1$, $\alpha(y) = \gamma M_0^2 \Omega_1 y$, and then

$$\phi_R(0, y) = \gamma M_0^2 \frac{\pi}{4x_0} \int_0^y \exp(\gamma M_0^2 \Omega_1 y) \left[\int_0^{y'} \frac{\Phi(y)}{y} \exp(-\gamma M_0^2 \Omega_1 y'') \, dy''\right] dy'. \quad \text{(D-14)}$$

Examples of calculating $\phi_R(0, y)$ according to (D-14) are shown in Fig. D-1 for various Mach numbers. It is clear that $\phi_R(0, 1/2)$ is not zero.

Solving (D-12) for $\epsilon_R \rho_{Rx}$ and substituting in (D-11), multiplying by $(T_0 - M_0^2 w_0^2)/T_0$, and integrating again with respect to x gives (63).

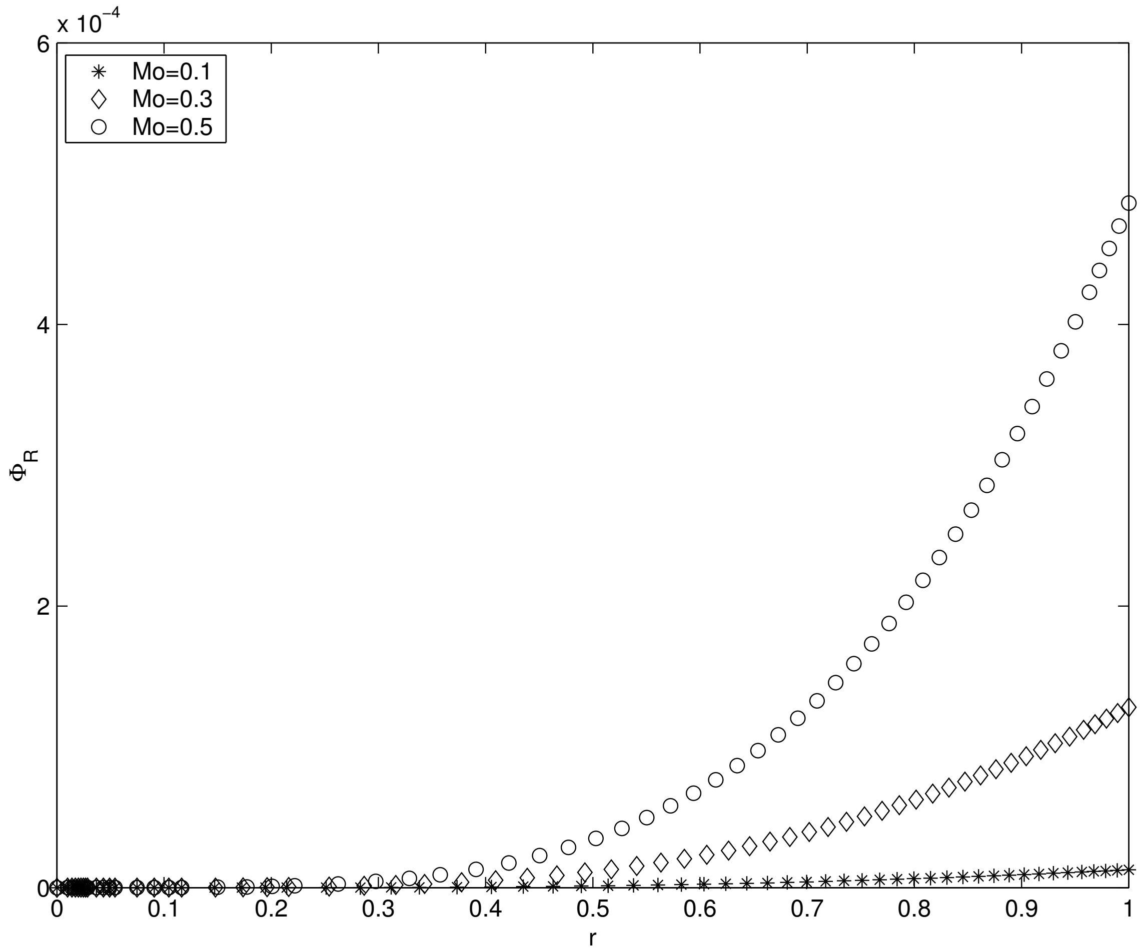

Figure 1: D-1: Solutions of $\phi_R(0, y)$ for a solid-body rotation flow at various Mach numbers.