Generalized Taylor dispersion in Suspensions of gyrotactic swimming micro-organisms

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Appendix A

Numerical solution

Making use of Galerkin's method, we numerically compute the steady orientational distribution of the particles and the corresponding vector **b**-field. Once these are obtained, we calculate the requisite macroscopic phenomenological coefficients, namely the average swimming velocity vector $\bar{\mathbf{U}}$ and the Taylor dispersivity dyadic $\bar{\mathsf{D}}$.

Towards this end, $P_0^\infty(\theta,\phi)$ is expanded in a series of surface harmonics

$$P_0^\infty(\theta,\phi) = \Sigma_{n=0}^{\infty}\{\frac{1}{2}A_n P_n(\cos\theta) + \Sigma_{m=1}^{n}[A_n^m \cos m\phi + B_n^m \sin m\phi]P_n^m(\cos\theta)\}, \quad \text{(A.1)}$$

where P_n^m denote the associated Legendre function of order n and degree m. Making use of the normalization condition (2.22) we obtain

$$A_0 = 1/2\pi\ . \quad \text{(A.2)}$$

Truncation of the expansion after $n = N$ and substitution of (A.1)-(A.2) into (2.21) lead to a system of $N^2 + 2N$ linear algebraic equations for the constant coefficients A_n, A_n^m and B_n^m, numerically solved.

We similarly expand the scalar components of $\mathbf{b}(\theta,\phi)$

$$b_i(\theta,\phi) = \Sigma_{n=0}^{\infty}\{\frac{1}{2}C_n^{(i)} P_n(\cos\theta) + \Sigma_{m=1}^{n}[C_n^{m(i)} \cos m\phi + D_n^{m(i)} \sin m\phi]P_n^m(\cos\theta)\},$$
$$(i = 1,2,3)\ . \quad \text{(A.3)}$$

Making use of the homogeneous normalization condition (2.24) we obtain

$$C_0^{(i)} = 0\ ,\ (i = 1,2,3). \quad \text{(A.4)}$$

Using the same procedure (i.e. truncation of the expansion after $n = N$ and substitution of (A.3)-(A.4) into (2.23)) lead to a system of $N^2 + 2N$ linear algebraic equations for the constant coefficients of the corresponding components of the vector b-field, namely $C_n^{(i)}$, $C_n^{m(i)}$ and $D_n^{m(i)}$. The solution is made first for b_1 and b_3 and then for b_2 (since the forcing term of the latter's equation is depenedent on b_1 in the case of shear field (2.19) discussed here).

Making use of the orthogonality relations satisfied by the associated Legendre functions, we obtain for the mean swimming velocity (see (2.11))

$$\bar{\mathbf{U}} = U(-\frac{4\pi}{3}A_1^1,\ -\frac{4\pi}{3}B_1^1,\ \frac{2\pi}{3}A_1). \tag{A.5}$$

The Taylor dispersivity dyadic is given by (2.12). The first term on the right hand side of (2.12) depends on just a small number of the coefficients calculated

$$\int_{S_2} \langle \mathbf{be} \rangle^s d^2\mathbf{e} = \frac{\pi}{3} \begin{pmatrix} -4C_1^{1(1)} & -2(D_1^{1(1)} + C_1^{1(2)}) & 0 \\ -2(D_1^{1(1)} + C_1^{1(2)}) & -4D_1^{1(2)} & 0 \\ 0 & 0 & 2C_1^{(3)} \end{pmatrix}, \tag{A.6}$$

whereas calculation of the second term on the right-hand involves appropriate numerical quadratures and is not compactly expressible as the above.