Supplementary material for the paper 'Nonlinear steady convection in rotating mushy layers'

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The additional material is provided in two Appendices A and B, which are given below

Appendix A

The expressions for the coefficients $V_{01}^*(z)$, $\psi_{01}^*(z)$, $\theta_{01}^*(z)$ and $\phi_{01}^*(z)$ are given below

$(V_{01}^*, \theta_{01}^*)=(b_1, c_1)\, z\sin(\pi z)+(b_2, c_2)z\cos(\pi z)+(b_3, c_3)z^2\cos(\pi z)+(b_4, c_4)\sinh(rz)+(b_5, c_5)\cosh(rz)+(b_6, c_6)\cos(\pi z)+(b_7, c_7)\sin(\pi z),$ (A1a)

$\psi_{01}^*=T\{b_1[\sin(\pi z)+\pi z\cos(\pi z)]+b_2[\cos(\pi z)-\pi z\sin(\pi z)]+b_3[2z\cos(\pi z)-\pi z^2\sin(\pi z)]+b_4 r\cosh(rz)+b_5 r\sinh(rz)-b_6\pi\sin(\pi z)+\pi[b_7+(1-z)(\pi^2+a^2)/(CR_{00}a^2)]\cos(\pi z)\},$ (A1b)

$\phi_{01}^*=b_0+(R_{00}a^2/C)\{b_1[-(z/\pi)\cos(\pi z)+(1/\pi^2)\sin(\pi z)]+b_2[(z/\pi)\sin(\pi z)+(1/\pi^2)\cos(\pi z)]+b_3[(z^2/\pi)\sin(\pi z)+(2/\pi^2)z\cos(\pi z)-(2/\pi^3)\sin(\pi z)]+(b_4/r)\cosh(rz)+(b_5/r)\sinh(rz)+(b_6/\pi)\sin(\pi z)-(b_7/\pi)\cos(\pi z)\}-[(2R_{01}+R_{00})a^2(\pi^2+a^2)/(2C\pi R_{00}a^2)+1/(\pi C^2)](\pi^2+a^2)\cos(\pi z)+(1/C)[1-(\pi^2+a^2)/(C\pi^2)]\sin(\pi z)+(-1+1/C)(\pi^2+a^2)[1/(\pi C)][-z\cos(\pi z)+(1/\pi)\sin(\pi z)]-[(\pi^2+a^2)/(\pi C^2)]z,$ (A1c)

where

$b_2=(\pi^2+a^2)[2R_{01}+R_{00}/2-2\pi^2T^2(\pi^2+a^2)/(Ca^2R_{00})]/d_1,\ d_1\equiv2\pi[2\pi^2(1+T^2)+a^2(2+T^2)],$ (A1d)

$r=\{[a^2(2+T^2)+d_0^{0.5}]/(2+2T^2)\}^{0.5},\ d_0\equiv a^2[a^2(2+T^2)^2-4(a^2-R_{00}^2)(1+T^2)],$ (A1e)

$b_3=(\pi^2+a^2)[-R_{00}+2\pi^2T^2(\pi^2+a^2)/(Ca^2R_{00})]/(2d_1),$ (A1f)

$b_1=\{-R_{00}+2\pi T^2(\pi^2+a^2)(a^2+3\pi^2)/(a^2R_{00}C)-b_3[2a^2(2+T^2)+12\pi^2(1+T^2)]\}/d_1,$ (A1g)

$b_4=(\pi^2 d_2+d_3)/[(\pi^2+r^2)\sinh(r)]$, $d_2\equiv b_3+[1+\cosh(r)](2\pi b_1+2b_3-d_4)/(\pi^2+r^2)$, $d_4\equiv 2\pi T^2(\pi^2+a^2)/[R_{00}a^2C(1+T^2)]$, $d_3\equiv -d_4+2\pi b_1+(2-\pi^2)b_3+[\pi^2-r^2\cosh(r)](d_4-2\pi b_1-2b_3)/(\pi^2+r^2)$, (A1h)

$$b_5=(d_4-2\pi b_1-2b_3)/(\pi^2+r^2)=-b_6,\ b_7=1, \quad (A1i)$$

$$c_1=[-\pi(1+T^2)(\pi b_1+4b_3)-a^2 b_1+2\pi^2T^2(\pi^2+a^2)/(a^2CR_{00})]/R_{00}, \quad (A1j)$$

$$(c_2, c_3)=(b_2, b_3)[-\pi^2(1+T^2)-a^2]/R_{00}, \quad (A1k)$$

$$(c_4, c_5)=(b_4, b_5)[r^2(1+T^2)-a^2]/R_{00}, \quad (A1l)$$

$$c_6=[(1+T^2)(-\pi^2 b_6+2\pi b_1+2b_3)-a^2 b_6-2\pi T^2(\pi^2+a^2)/(a^2CR_{00})]/R_{00}, \quad (A1m)$$

$$c_7=[(1+T^2)(-\pi^2-2\pi b_2)-a^2+R_{01}-2\pi^2T^2(\pi^2+a^2)/(a^2CR_{00})]/R_{00}. \quad (A1n)$$

The expressions for $V_{10}^*(z)$, $\psi_{10}^*(z)$, $\theta_{10}^*(z)$, $\phi_{10}^*(z)$, $V_{10}^\wedge(z, \Phi_{lp})$, $\psi_{10}^\wedge(z, \Phi_{lp})$, $\theta_{10}^\wedge(z, \Phi_{lp})$ and $\phi_{10}^\wedge(z, \Phi_{lp})$ are given below

$(V_{10}^*, \theta_{10}^*)=(b_8, c_8)z\cos(\pi z)+(b_9, c_9)\sinh(rz)$, $(b_8, c_8)\equiv R_{10}\{1/[2\pi(1+T^2)], -(\pi^2+a^2)/(2\pi R_{00})\}$, $b_9\equiv b_8/\sinh(r)$, $c_9\equiv -b_9R_{00}a^2/(r^2+a^2)$, (A2a)

$$\psi_{10}^*=T[rb_9\cosh(rz)+b_8\cos(\pi z)-\pi b_8 z\sin(\pi z)], \quad (A2b)$$

$\phi_{10}^*=(R_{00}a^2/C)\{(b_8/\pi^2)[1+\pi z\sin(\pi z)+\cos(\pi z)]+b_9[\cosh(rz)-\cosh(r)]/r\}-R_{10}(\pi^2+a^2)[1+\cos(\pi z)]/(\pi CR_{00})$, (A2c)

$V_{10}^\wedge=b_{10}\sin(\pi z)+b_{11}\sin(2\pi z)$, $b_{10}\equiv(a_{lp}^2+\pi^2)(2\pi T^2/C)(\pi^2+a^2)^2/\{2a^2R_{00}[(a_{lp}^2+\pi^2)(a_{lp}^2+\pi^2+\pi^2T^2)-a_{lp}^2R_{00}^2]\}$, $a_{lp}^2\equiv 2a^2(1+\Phi_{lp})$, $b_{11}\equiv[\pi(T^2/C)(\pi^2+a^2)^2(a_{lp}^2+4\pi^2)/(a^2R_{00})-\pi R_{00}(\pi^2+a^2)(1-\Phi_{lp})/2]/\{(a_{lp}^2+4\pi^2)[a_{lp}^2+4\pi^2(1+T^2)]-R_{00}^2a_{lp}^2\}$, (A2d)

$\theta_{10}^\wedge=c_{10}\sin(\pi z)+c_{11}\sin(2\pi z)$, $c_{10}\equiv -R_{00}a_{lp}^2 b_{10}/(a_{lp}^2+\pi^2)$, $c_{11}\equiv[\pi(\pi^2+a^2)(1-\Phi_{lp})/2-R_{00}a_{lp}^2 b_{11}]/(a_{lp}^2+4\pi^2)$, (A2e)

$$\psi_{10}^\wedge=T\{-(\pi^2+a^2)^2\cos(\pi z)[1+\cos(\pi z)]/(2CR_{00}a^2)+\pi b_{10}\cos(\pi z)+2\pi b_{11}\cos(2\pi z)\}, \quad (A2f)$$

$\phi_{10}^\wedge=[R_{00}a_{lp}^2/(\pi C)]\{-[1+\cos(\pi z)]b_{10}+[1-\cos(2\pi z)]b_{11}/2\}+(\pi^2+a^2)(1-\Phi_{lp})[\cos(2\pi z)-1]/(4C)$. (A2g)

The expressions for the coefficients F_{11} and G_{11} are given below

$$F_{11}=[-R_{00}/(\pi^2+a^2)]\{K_c(\pi^2+a^2)^3[\pi^2(1-T^2)(1+\Phi_{lp})+a_{lp}^2]/(2\pi CR_{00}^2a^2)+K_2(\pi^2+a^2)^3[(\pi^2+a_{lp}^2)(9\pi^2+16)/(18\pi^2)+20/9+(28/9+\pi^2/2)\Phi_{lp}-\pi^2T^2(1+\Phi_{lp})(9\pi^2+56)/(18\pi^2)]+\pi T^2(\pi^2+a^2)^3(1+\Phi_{lp})[1+56/(9\pi^2)-2\pi C^2E_5/(\pi^2+a^2)]+T(\pi^2+a^2)^2[(E_4+\pi TE_1+\pi TE_2)(1+\Phi_{lp})+\pi E_6/(\pi^2a^2+a^4)]/(CR_{00})-\pi(\pi^2+a^2)(1+\Phi_{lp}/2)E_3-R_{00}\pi a^2E_2(1/2+\Phi_{lp})+2c_{11}/(3\pi)-R_{01}(\pi^2+a^2)c_{10}a_{lp}^2/(2a^2R_{00})+(S_tR_{00}/C+R_{01})b_{10}a_{lp}^2/2+8R_{00}a_{lp}^2b_{11}/(9\pi^2)-T^2(\pi^2+a^2)\pi a_{lp}^2[\pi b_{10}/4+20b_{11}/(9\pi)]/(CR_{00}a^2)\}, \tag{A3a}$$

$$G_{11}=[-R_{00}/(\pi^2+a^2)]\{\pi TE_7(\pi^2+a^2)/(CR_{00}a^2)-R_{01}(\pi^2+a^2)[c_9E_8-c_8/(4\pi)]-R_{10}(\pi^2+a^2)E_9/R_{00}-T^2(\pi^2+a^2)\pi[b_8/4+rb_9(E_{10}-E_{11})]/(CR_{00})+\pi c_8/4-rc_9E_{12}-S_tR_{00}a^2b_9E_8/r+S_tR_{10}(\pi^2+a^2)a^2/(2C^2)+R_{00}a^2[b_8/(4\pi)+b_9(E_8-2E_{13})]/2+R_{01}a^2[-b_8/(4\pi)+b_9E_8]+R_{10}a^2E_1\}, \tag{A3b}$$

where

$$E_1=\langle V_{01}^*\sin(\pi z)\rangle,\ E_2=\langle V_{01}^*\sin(2\pi z)\rangle,\ E_3=\langle\theta_{01}^*\sin(2\pi z)\rangle,\ E_4=\langle\psi_{01}^*\cos(\pi z)[1+\cos(\pi z)]\rangle,\ E_5=\langle\phi_{01}^*\cos^2(\pi z)\rangle,\ E_6=a_{lp}^2\langle(z-1)\psi_{10}^{\wedge}\cos(\pi z)\rangle,\ E_7=a^2\langle(z-1)\psi_{10}^*\cos(\pi z)\rangle,\ E_8=\langle\sin(\pi z)\sinh(rz)\rangle,\ E_9=\langle\theta_{01}^*\sin(\pi z)\rangle,\ E_{10}=\langle\sin(\pi z)\cosh(rz)\rangle,\ E_{11}=\langle z\cos(\pi z)\cosh(rz)\rangle,\ E_{12}=\langle\sin(\pi z)\cosh(rz)\rangle,\ E_{13}=\langle z\sin(\pi z)\sinh(rz)\rangle. \tag{A3c}$$

The expressions for F_{20}, H_{20} and G_{20} are given below

$$F_{20}=[-R_{00}/(\pi^2+a^2)]\{K_2(\pi^2+a^2)^4[-5\pi^2-10a^2(3/2+\Phi_{ml}+\Phi_{mp}+\Phi_{lp})-2\pi^2(1+7\Phi_{lp})+7T^2R_{00}\pi^2(\Phi_{ml}+\Phi_{mp})+7T^2\pi^2]/(8C^2R_{00}^2a^2\pi^2)-T^2(\pi^2+a^2)\pi^2[2(1+\Phi_{ml}+\Phi_{mp})E_{14}+7(\pi^2+a^2)^2(1+\Phi_{ml}+\Phi_{mp})/(8\pi^2C^2)+a^2R_{00}(2+2\Phi_{lp}+\Phi_{ml}+\Phi_{mp})E_{15}/(T\pi^2Ca_{lp}^2)-R_{00}(a_{lp}^2+\Phi_{ml}+\Phi_{mp})E_{16}/(\pi^2C)+7(\pi^2+a^2)^2(1+\Phi_{ml}+\Phi_{mp})/(\pi^2C^2)]-\pi(\pi^2+a^2)E_{17}(2+\Phi_{ml}+\Phi_{mp})/2-\pi R_{00}a^2E_{18}(1+\Phi_{lp}+\Phi_{ml}+\Phi_{mp})\}, \tag{A4a}$$

$$H_{20}=[-R_{00}/(\pi^2+a^2)]\{R_{10}a_{lp}^2[b_{10}-(\pi^2+a^2)c_{10}/(a^2R_{00})]/2-T(\pi^2+a^2)^2(1+\Phi_{lp})[E_{19}+2T\pi^2CE_{20}/R_{00}-Ta^2\pi(E_{21}+E_{22})]-\pi(\pi^2+a^2)(1+\Phi_{lp}/2)E_{23}-\pi R_{00}a^2E_{22}(1/2+\Phi_{lp})\}, \tag{A4b}$$

$$G_{20}=[-R_{00}R_{10}a^2/(\pi^2+a^2)][-b_8/(4\pi)+\pi b_9\sinh(r)/(\pi^2+r^2)]+R_{10}[-c_8/(4\pi)+\pi\sinh(r)/(\pi^2+r^2)], \tag{A4c}$$

where

E_{14}=<$\phi_{10}^{\wedge}\cos^2(\pi z)$>, $E_{15}=-a_{lp}^2$< $\psi_{10}^{\wedge}\cos(\pi z)[1+\cos(\pi z)]$>, E_{16}=<$(dV_{10}^{\wedge}/dz)\cos(\pi z)$ $[1+\cos(\pi z)]$>, E_{17}=<$\theta_{10}^{\wedge}\sin(2\pi z)$>, E_{18}=<$V_{10}^{\wedge}\sin(2\pi z)$>, $E_{19}=-a^2$<$\psi_{10}^{*}\cos(\pi z)[1+\cos(\pi z)]$>, E_{20}=<$\phi_{10}^{*}\cos^2(\pi z)$>, E_{21}=<$V_{10}^{*}\sin(\pi z)$>, E_{22}=<$V_{10}^{*}\sin(2\pi z)$>, E_{23}=<$\theta_{10}^{*}\sin(2\pi z)$>. (A4d)

Appendix B

The stability system for the present problem is given below

$$\nabla^2\{\varepsilon[K_1\phi'+K_2(2\phi_B+2\varepsilon\phi)\phi']\Delta_2 V+K(\phi_B+\varepsilon\phi)\Delta_2 V'\}+(\partial/\partial z)\{\varepsilon\mathbf{\Omega}V.\nabla[K_1\phi'+K_2(2\phi_B+2\varepsilon\phi)\phi']+\mathbf{\Omega}V'\cdot\nabla K(\phi_B+\varepsilon\phi)\}+(\partial/\partial z)\{\varepsilon[(\partial/\partial x)(K_1\phi'+2K_2\phi_B\phi'+2K_2\varepsilon\phi\phi')(\partial\psi/\partial y)-(\partial/\partial y)(K_1\phi'+2K_2\phi_B\phi'+2K_2\varepsilon\phi\phi')(\partial\psi/\partial x)]+[(\partial/\partial x)K(\phi_B+\varepsilon\phi)(\partial\psi'/\partial y)-(\partial/\partial y)K(\phi_B+\varepsilon\phi)(\partial\psi'/\partial x)]\}-R\Delta_2\theta'+T(\partial/\partial z)\{(\partial/\partial y)[(\partial\psi'/\partial y+\partial^2V'/\partial x\partial z)/(1-\phi_B-\varepsilon\phi)+\varepsilon(\partial\psi/\partial y+\partial^2V/\partial x\partial z)(1+2(\phi_B+\varepsilon\phi)+3(\phi_B+\varepsilon\phi)^2)\phi'+(\partial/\partial x)[(\partial\psi/\partial x-\partial^2V/\partial y\partial z)(1+2(\phi_B+\varepsilon\phi)+3(\phi_B+\varepsilon\phi)^2)\phi']\}=0, \quad \text{(B1a)}$$

$$\varepsilon[K_1\phi'+2K_2(\phi_B+\varepsilon\phi)\phi']\Delta_2\psi+K(\phi_B+\varepsilon\phi)\Delta_2\psi'+\varepsilon(\partial^2V/\partial x\partial z+\partial\psi/\partial y)(\partial/\partial y)[K_1\phi'+2K_2(\phi_B+\varepsilon\phi)\phi']+(\partial^2V'/\partial x\partial z+\partial\psi'/\partial y)(\partial/\partial y)K(\phi_B+\varepsilon\phi)-(\partial^2V'/\partial y\partial z-\partial\psi'/\partial x)(\partial/\partial x)K(\phi_B+\varepsilon\phi)-\varepsilon(\partial^2V/\partial y\partial z-\partial\psi/\partial x)(\partial/\partial x)[K_1\phi'+2K_2(\phi_B+\varepsilon\phi)\phi']-T\{(\partial/\partial x)[(\partial\psi'/\partial y+\partial^2V'/\partial x\partial z)/(1-\phi_B-\varepsilon\phi)+\varepsilon(\partial\psi/\partial y+\partial^2V/\partial x\partial z)(1+2(\phi_B+\varepsilon\phi)+3(\phi_B+\varepsilon\phi)^2)\phi']+(\partial/\partial y)[(-\partial\psi'/\partial x+\partial^2V'/\partial y\partial z)/(1-\phi_B-\varepsilon\phi)+\varepsilon(-\partial\psi/\partial x+\partial^2V/\partial y\partial z)(1+2(\phi_B+\varepsilon\phi)+3(\phi_B+\varepsilon\phi)^2)\phi']\}=0, \quad \text{(B1b)}$$

$$(\partial/\partial t-\delta\partial/\partial z)(-\theta'+S_t\phi')+R(d\theta_B/dz)\Delta_2V'+\nabla^2\theta'=\varepsilon R[(\mathbf{\Omega}V+\mathbf{E}\psi).\nabla\theta'+(\mathbf{\Omega}V'+\mathbf{E}\psi').\nabla\theta], \quad \text{(B1c)}$$

$$(\partial/\partial t-\delta\partial/\partial z)[(-1+\phi_B)\theta'+\theta_B\phi'+\varepsilon\phi\theta'+\varepsilon\phi'\theta-C\phi'/\delta]+R(d\theta_B/dz)\Delta_2V'=R\varepsilon[(\mathbf{\Omega}V+\mathbf{E}\psi).\nabla\theta'+(\mathbf{\Omega}V'+\mathbf{E}\psi').\nabla\theta], \quad \text{(B1d)}$$

$$V'=\theta'=0 \text{ at } z=0, \quad \text{(B1e)}$$

$$V'=\theta'=\phi' \text{ at } z=1. \qquad \text{(B1f)}$$

The expressions for L_{10}, L_{11}, L_{20} and $F\square_{20}$ are given below

$$L_{10}= -\pi[(\pi^2+a^2)^2/(CR_{00}a^2)][T^2(1+\Phi_{lp})+(T/2)(1+T^2)\Psi_{lp}], \qquad \text{(B2a)}$$

$$L_{11}=[F_{11}(\Phi_{lp})+H_{11}\Psi_{lp}],\ H_{11}=\{K_c\,\pi(\pi^2+a^2)^2T/(2CR_{00}a^2)+ TK_2\pi(\pi^2+a^2)^2\,[1+C/2+56/(9\pi^2)]$$
$$/(C^2\,R_{00}a^2)-\pi T(\pi^2+a^2)^2[1/2+28/(9\pi^2)]/(C^2R_{00}a^2)-T(\pi^2+a^2)E_{16}/C-T^3\pi^2(\pi^2+a^2)\,E_5/(R_{00}a^2)\,+$$
$$T^3\pi(\,\pi^2+a^2)^2[1/2+28/(9\pi^2)]/(C^2R_{00}a^2)+T^2(\pi^2+a^2)E_4/C+[R_{00}{}^2a^2E_{24}/(\pi^2+a^2)-\pi TE_3R_{00}/2$$
$$+\pi TE_5\}, \qquad \text{(B2b)}$$

$$L_{20}=[H_{20}(\Phi_{lp})+H\square_{20}\Psi_{lp}],\ H\square_{20}=\{T(\pi^2+a^2)\pi^2E_{20}/(R_{00}a^2)+T(\pi^2+a^2)\pi(E_{21}+E_{22})/C+$$
$$\pi R_{00}TE_{23}/2-R_{00}{}^2E_{25}/(\pi^2+a^2)+T^2(\pi^2+a^2)E_{19}/(Ca^2)-\,\pi^2T^3(\,\pi^2+a^2)E_{20}/(R_{00}a^2)\},$$

(B2c)

$$F\square_{20}=\{-7T(\pi^2+a^2)^3(K_2+CT^2R_{00}/2)/(4C^2R_{00}{}^2a^2)+T\pi^2(\pi^2+a^2)(1-$$
$$T^2E_{14})/(R_{00}a^2)+T(\pi^2+a^2)(E_{16}\ \ +TE_{15}/a_{lp}{}^2)/C+\pi TR_{00}E_{17}/2-R_{00}{}^2a^2E_{26}/[(\pi^2+a^2)a_{lp}{}^2]\},$$

(B2d)

Where

$$E_{24}=<\psi_{01}{}^*\sin^2(\pi z)>,\ E_{25}=-a^2<\psi_{10}{}^*\sin^2(\pi z)>,\ E_{26}=-a_{lp}{}^2<\psi_{10}{}^\wedge\sin^2(\pi z)>. \qquad \text{(B2e)}$$