Appendix A

In the periodic case (2.9a) the initial fields F^{od}, G^{od} are periodic in x and can be represented as

$$\left(F^{od}, G^{od}\right) = \sum_{m=-\infty}^{m=\infty} \left(f_m^{od}(y), g_m^{od}(y)\right) e^{imx}. \tag{A.1}$$

where f_m^{od}, g_m^{od} are also odd functions. The solution to the problem (3.11), (3.13), (A.1) is represented in the form

$$\tilde{v}_{0f} = \sum_{m=-\infty}^{m=\infty} \tilde{v}_{0m}(y,t) e^{imx}. \tag{A.2}$$

Substitution of (A.2) into (3.11), (3.13) gives

$$-\frac{\partial^2 \tilde{v}_{0m}}{\partial t^2} + \frac{\partial^2 \tilde{v}_{0m}}{\partial y^2} - (m^2+1)\tilde{v}_{0m} = 0, \quad \left(\tilde{v}_{0m}, \frac{\partial \tilde{v}_{0m}}{\partial t}\right)_{t=0} = \left(f_m^{od}, g_m^{od}\right). \tag{A.3a,b}$$

The system (A.3) is solved using the Fourier transformation:

$$\tilde{v}_{0m} = \int_{-\infty}^{\infty} \hat{v}_{0m}(l,t) e^{ily} dl, \quad \hat{v}_{0m} = A_m^{(+)} e^{i\omega_{ml}t} + A_m^{(-)} e^{-i\omega_{ml}t} \tag{A.4b,c}$$

where ω_{ml} is the frequency of IG waves,

$$\omega_{ml} = \sqrt{m^2 + l^2 + 1}, \tag{A.5}$$

and

$$A_m^{(\pm)} = \frac{1}{2}\left(\hat{f}_m \mp \frac{i\hat{g}_m}{\omega_{ml}}\right), \quad \left(f_m^{od}, g_m^{od}\right) = \int_{-\infty}^{\infty} \left(\hat{f}_m(l), \hat{g}_m(l)\right) e^{ily} dl. \tag{A.6a,b}$$

It is of importance for the following estimates that $A_m^{(\pm)}(l)$ is an odd function of the wavenumber l, i.e.

$$A_m^{(\pm)}(l) = O(l), \quad l \to 0. \tag{A.7}$$

The formulae (A.2), (A.4) to (A.6) give the solution for $\tilde{v}_{0f}$. The function $\tilde{v}_{00}$ has the form analogous (A.2), (A.4) but with unknown amplitudes $A_m^{(\pm)}$ depending on the slow times T_n and equal to zero for $T_1 = T_2 = \ldots 0$.

In the "step" case the solution $\tilde{v}_{0f}$ is, generally, neither periodic nor localised in x unless the initial zonal motion at $y = \pm\infty$ is geostrophic. Therefore to represent $\tilde{v}_{0f}$ as a superposition of linear waves we find the Fourier representation for the localised derivative $\frac{\partial \tilde{v}_{0f}}{\partial x}$, and also for the limiting fields $\tilde{v}_{0f}^{(\pm)}(y,t)$ ($\tilde{v}_{0f} \to \tilde{v}_{0f}^{(\pm)}(y,t), \quad y \to \pm\infty$) to compose the desired solution for $\tilde{v}_{0f}$. The problem for $\frac{\partial \tilde{v}_{0f}}{\partial x}$ is written as

$$L_w\left(\frac{\partial \tilde{v}_{0f}}{\partial x}\right) = 0, \quad \left.\frac{\partial \tilde{v}_{0f}}{\partial x}\right|_{y=0} = 0, \quad \left(\frac{\partial \tilde{v}_{0f}}{\partial x}, \frac{\partial^2 \tilde{v}_{0f}}{\partial x \partial t}\right)_{t=0} = \left(\frac{\partial F}{\partial x}, \frac{\partial G}{\partial x}\right), \qquad \text{(A.8a,b,c)}$$

where L_W is the linear operator in (3.11). The solution to (A.8) is readily written in the form of two-dimensional Fourier integrals (see, e.g. RZB),

$$\frac{\partial \tilde{v}_{0f}}{\partial x} = \int_{-\infty}^{\infty} \hat{v}_{0x}(k,l,t) e^{i(kx+ly)} dk dl, \quad \hat{v}_{0x} = A^{(+)} e^{i\omega_{kl} t} + A^{(-)} e^{-i\omega_{kl} t}, \qquad \text{(A.9a,b)}$$

where

$$\omega_{kl} = \sqrt{k^2 + l^2 + 1}, \quad A^{(\pm)} = \frac{1}{2}\left(\hat{F}_x(k,l) \mp \frac{i\hat{G}_x(k,l)}{\omega_{kl}}\right), \qquad \text{(A.10a,b)}$$

and the functions $\hat{F}_x, \hat{G}_x$ are Fourier transformations of the functions F^{od}, G^{od}. Obviously, $\hat{F}_x, \hat{G}_x$, and, therefore, $A^{(\pm)}$ are also odd functions in l,

$$A^{(\pm)} = O(l), \quad l \to 0. \qquad \text{(A.11)}$$

The system for $\tilde{v}_{0f}^{(\pm)}(y,t)$ coincides with (A.3) if we put $m = 0$, $f_m^{od} = 0$ (the function F is localised), and $g_m^{od} = G^{(\pm)}(y)$ where

$$G^{(\pm)}(y) = \lim G^{od}(x,y), \quad x \to \pm\infty. \qquad \text{(A.12)}$$

The solution $\tilde{v}_{0f}^{(\pm)}(y,t)$ has the form

$$\tilde{v}_{0f}^{(\pm)} = \int_{-\infty}^{\infty} \hat{v}_0^{(\pm)}(l,t) e^{ily} dl, \quad \hat{v}_0^{(\pm)} = A_p^{(\pm)} e^{i\omega_{0l} t} + A_m^{(\pm)} e^{-i\omega_{0l} t}, \qquad \text{(A.13a,b)}$$

where

$$A_p^{(\pm)} = -\frac{i\hat{G}^{(\pm)}}{2\omega_{0l}}, \quad A_m^{(\pm)} = \frac{i\hat{G}^{(\pm)}}{2\omega_{0l}}. \qquad \text{(A.14a,b)}$$

To find $\tilde{v}_{0f}$ we use a simple identity

$$\tilde{v}_{0f} = \tilde{v}_{01} + \tilde{v}_{0f}^{(\pm)} - \tilde{v}_{01}^{(\pm)}, \tag{A.15}$$

where

$$\tilde{v}_{01} = \int_0^x \frac{\partial \tilde{v}_{0f}}{\partial x} dx, \quad \tilde{v}_{01}^{(\pm)}(y,t) = \lim \tilde{v}_{01},\ x \to \pm\infty. \tag{A.16}$$

Representing $\tilde{v}_{01}$ in the form

$$\tilde{v}_{01} = -\int_{-\infty}^{\infty} \hat{v}_{0x}(k,l,t) e^{ily} \frac{1 - e^{ikx}}{ik} dkdl \tag{A.17}$$

and calculating $\tilde{v}_{01}^{(\pm)}$ from (A.17) we arrive at the equation

$$\tilde{v}_{0f} = -i \int_{-\infty}^{\infty} dl ⨍ \frac{\hat{v}_{0x}(k,l,t)}{k} e^{i(kx+ly)} dk + \int_{-\infty}^{\infty} \hat{g}_v(l,t) e^{ily} dl, \tag{A.18}$$

$$\hat{g}_v = \hat{v}_{0f}^{(-)}(l,t) + \pi \hat{v}_{0x}(0,l,t); \tag{A.19}$$

where $⨍$ denotes the Cauchy principal value integral over k Equation (A.18) expresses $\tilde{v}_{0f}$ as a superposition of plane IG waves. Again, the slow time-dependent part $\tilde{v}_{00}$ has a form analogous to (A.18) but with unknown amplitudes $A^{(\pm)}$ in (A.9b) and $\hat{g}_v$ depending on slow times and equal to zero at $T_1 = T_2 = ...0$.

For the localised case (2.9c) the solution $\tilde{v}_{0f}$ has the form (A.9), (A.10):

$$\tilde{v}_{0f} = \int_{-\infty}^{\infty} \hat{v}_{0f}(k,l,t) e^{i(kx+ly)} dkdl, \quad \hat{v}_{0f} = A^{(+)} e^{i\omega_{kl}t} + A^{(-)} e^{-i\omega_{kl}t}, \tag{A.20a,b}$$

$$A^{(\pm)} = \frac{1}{2}\left(\hat{F}(k,l) \mp \frac{i\hat{G}(k,l)}{\omega_{kl}} \right). \tag{A.21}$$

Of course, the property (A.11) for $A^{(\pm)}$ holds in this case too.

The fields $\tilde{u}_{01}, \tilde{h}_{01}$ in (3.16) can be also expressed in the form of a superposition of IG waves using the representations (3.17). For the periodic case we have:

$$(\tilde{u}_{01}, \tilde{h}_{01}) = \sum_{m=-\infty}^{m=\infty} (\tilde{u}_{01}^{(m)}(y,t), \tilde{h}_{01}^{(m)}(y,t))\, e^{imx}, \tag{A.22a}$$

where

$$(\tilde{u}_{01}^{(m)}(y,t), \tilde{h}_{01}^{(m)}(y,t)) = \int_{-\infty}^{\infty} (\hat{u}_{01}^{(m)}(l,t), \hat{h}_{01}^{(m)}(l,t)) e^{ily} dl, \tag{A.22b}$$

$$\hat{u}_{01}^{(m)} = \frac{1}{l^2+1}\left(ml\hat{v}_{0m} - \frac{\partial \hat{v}_{0m}}{\partial t}\right), \quad \hat{h}_{01}^{(m)} = \frac{i}{l^2+1}\left(m\hat{v}_{0m} + l\frac{\partial \hat{v}_{0m}}{\partial t}\right). \qquad \text{(A.22c,d)}$$

For the "step" case (2.9b)

$$(\tilde{u}_{01}, \tilde{h}_{01}) = -i\int_{-\infty}^{\infty} dl ⨍ \frac{(\hat{u}_{0x}, \hat{h}_{0x})}{k} e^{i(kx+ly)} dk + \int (\hat{g}_u, \hat{g}_h)\, e^{ily} dl, \qquad \text{(A.23a)}$$

$$\hat{u}_{0x} = \frac{1}{l^2+1}\left(kl\hat{v}_{0x} - \frac{\partial \hat{v}_{0x}}{\partial t}\right), \quad \hat{h}_{0x} = \frac{i}{l^2+1}\left(k\hat{v}_{0x} + l\frac{\partial \hat{v}_{0x}}{\partial t}\right), \qquad \text{(A.23a,b)}$$

$$(\hat{g}_u, \hat{g}_h) = \frac{(-1, il)}{l^2+1}\frac{\partial}{\partial t}[\hat{v}_{0f}^{(-)}(l,t) + \pi\hat{v}_{0x}(0,l,t)]. \qquad \text{(A.23c)}$$

Finally, in the localised case (2.9c)

$$(\tilde{u}_{01}, \tilde{h}_{01}) = \int_{-\infty}^{\infty} (\hat{u}_{01}, \hat{h}_{01})\, e^{i(kx+ly)} dkdl, \qquad \text{(A.24a)}$$

$$\hat{u}_{01} = \frac{1}{l^2+1}\left(kl\hat{v}_{0f} - \frac{\partial \hat{v}_{0f}}{\partial t}\right), \quad \hat{h}_{01} = \frac{i}{l^2+1}\left(k\hat{v}_{0f} + l\frac{\partial \hat{v}_{0f}}{\partial t}\right). \qquad \text{(A.24b,c)}$$

Appendix B

All estimates will be made for the simplest case of localised initial conditions; the generalisation to the periodic and "step" cases can be readily done using the representation of the fast fields in these cases as a superposition of harmonic IG waves and Kelvin waves (see Appendix A).

Estimates for $\tilde{v}_{0f}, \tilde{u}_0, \tilde{h}_0$

Estimation of $\tilde{v}_{0f}$ in (A.20a) at large times is reduced to the estimation of the integrals,

$$J^{(\pm)} = \int_{-\infty}^{\infty} A^{(\pm)}(k,l) e^{i(kx+ly)\pm\omega_{kl}t} dkdl \qquad \text{(B.1)}$$

where $A^{(\pm)}$ is given by (A.21). We will consider $J^{(+)}$; the integral $J^{(-)}$ can be estimated in the same way. It is convenient to write (B.1) in the form

$$J^{(+)} = \int_0^\infty dk \int_0^\infty A_1^{(+)}(k,l,x,y) e^{i\omega_{kl}t} dl, \tag{B.2}$$

where

$$A_1^{(+)} = [A^{(+)}(k,l)e^{ily} + A^{(+)}(k,-l)e^{-ily}]e^{ikx} + \\ [A^{(+)}(-k,l)e^{ily} + A^{(+)}(-k,-l)e^{-ily}]e^{-ikx} \tag{B.3}$$

We now estimate the internal integral in (B.2) for $t \to \infty,\ x, y$ fixed,

$$J_l = \int_0^\infty A_1^{(+)}(k,l,x,y) e^{i\omega_{kl}t} dl \tag{B.4}$$

using that

$$A_1^{(+)} = O(l^2),\ \ l \to 0 \tag{B.5}$$

due to the oddness of $A^{(+)}$ in l (see Appendix A). Integrating by parts, taking into account of (B.5) gives,

$$J_l = \frac{1}{t}\int_0^\infty A_2^{(+)}(k,l,x,y) e^{i\omega_{kl}t} dl \tag{B.6}$$

where

$$A_2^{(+)} = i\left(\frac{A_1^{(+)}}{\omega'_{kl}}\right)' = O(1),\ \ l \to 0. \tag{B.7}$$

The prime denotes differentiation with respect to l.

The integral in (B.6) is estimated using the standard stationary phase method (e.g. Olver, 1974). The stationary point (corresponding to zero group velocity) here is $l = 0$; as a result we have

$$J_l = O\left(\frac{1}{t^{3/2}}\right),\ \ t \to \infty,\ \ x, y \text{fixed}. \tag{B.8}$$

The final estimate for the integral $J^{(\pm)}$ is

$$J^{(\pm)} = O\left(\frac{1}{t^2}\right),\ \ t \to \infty,\ \ x, y \text{fixed}. \tag{B.9}$$

Faster decay of $J^{(\pm)}$ in comparison with J_l is explained by the additional integrating over k of the integrand containing an exponent $e^{i\omega_{k0}t}$. In the periodic and "step" cases this integration is absent, and therefore the integral $J^{(\pm)}$ behaves like J_l in these cases. Analogous estimates for $\tilde{u}_{01}, \tilde{h}_{01}$ are performed in the same way using the representations (A.24), so in all cases the estimates (3.27) hold.

Estimates of the forced solution to the problem (5.13a), (5.14)

Interaction IG wave - IG wave

The typical forcing term $\Phi_1^{(ig)}\Phi_2^{(ig)}$ can be written as

$$\Phi_1^{(ig)}\Phi_2^{(ig)} = \int_{-\infty}^{\infty} A_1(k_1,l_1)A_2(k_2,l_2)e^{i[(k_1+k_2)x+(l_1+l_2)y+\Omega^{(\pm)}t]}dk_{12}dl_{12} \tag{B.10}$$

where

$$dk_{12} = dk_1dk_2,\ \ dl_{12} = dl_1dl_2,\ \ \Omega^{(\pm)} = \omega_{k_1l_1} \pm \omega_{k_2l_2} \tag{B.11}$$

and

$$A_i(k_i,l_i) = O(l_i),\ \ l_i \to 0,\ \ i = 1,2. \tag{B.12}$$

The response $\tilde{v}_{ww}$ to this forcing satisfying the equation

$$L_w\tilde{v}_{ww} = \Phi_1^{(ig)}\Phi_2^{(ig)},\quad L_w = -\frac{\partial^2}{\partial t^2} + \nabla^2 - 1 \tag{B.13}$$

can be written as

$$\tilde{v}_{ww} = \int_{-\infty}^{\infty} A_1(k_1,l_1)A_2(k_2,l_2)e^{i[(k_1+k_2)x+(l_1+l_2)y]}\frac{e^{i\Omega^{(\pm)}t}}{\left(\Omega^{(\pm)}\right)^2 - \omega_{kl}^2}dk_{12}dl_{12} \tag{B.14}$$

where $(k,l) = (k_1+k_2, l_1+l_2)$.

Triad interactions of IG waves are impossible therefore the denominator in the integrand in (B.14) never vanishes. Integration by parts over l_1 and l_2 in (B.14) using (B.12) gives the following estimate

$$\tilde{v}_{ww} = O\left(\frac{1}{t^3}\right),\ \ t \to \infty,\ \ x,y \text{ fixed.} \tag{B.15}$$

Interaction IG wave – slow motion

This estimate is the most tedious. The forcing term $\Phi_0^{(s)}\Phi_0^{(ig)}$ has the form

$$\Phi_0^{(s)}\Phi_0^{(ig)} = \int_{-\infty}^{\infty} A_1(k_1,l_1)A_2(k_2,l_2)e^{i[(k_1+k_2)x+(l_1+l_2)y\pm\omega_{k_1l_1}t]}dk_{12}dl_{12} \tag{B.16}$$

where A_1, A_2 satisfy (B.12). It is convenient to introduce the new variables $k_1, l_1, k = k_1 + k_2$, $l = l_1 + l_2$ and then to reduce the integration over l, l_1 to integration over $l > 0, l_1 > 0$:

$$\Phi_0^{(s)}\Phi_0^{(ig)} = \int_{-\infty}^{\infty} dk_{01} e^{ikx} \int_0^{\infty} F(k_1, l_1, k, l, y) e^{\pm i\omega_{k_1 l_1} t} dl_{01} \tag{B.17}$$

where $dk_{01} = dkdk_1,\ dl_{01} = dldl_1$ and

$$F = [A_1(k_1, l_1)A_2(k - k_1, l - l_1) + A_1(k_1, -l_1)A_2(k - k_1, l + l_1)]e^{ily} +$$
$$[A_1(k_1, l_1)A_2(k - k_1, -l - l_1) + A_1(k_1, -l_1)A_2(k - k_1, -l + l_1)]e^{-ily} \tag{B.18}$$

The integrand in (B.17) contains resonant harmonics therefore the response $\tilde{v}_{sw}$ is represented in the form

$$\tilde{v}_{sw} = \int_{-\infty}^{\infty} dk_{01} e^{ikx} \int_0^{\infty} F(k_1, l_1, k, l, y) \frac{e^{i\omega_{k_1 l_1} t} - e^{i\omega_{kl} t}}{\omega_{k_1 l_1}^2 - \omega_{kl}^2} dl_{01}. \tag{B.19}$$

We have chosen the + sign in the exponent in the integrand of (B.17) for definiteness.

Introducing the new variables $X = \omega_{kl}, Y = \omega_{k_1 l_1}$ instead of l, l_1,

$$l = \sqrt{X^2 - X_B^2},\ l_1 = \sqrt{Y^2 - Y_B^2},\ X_B = \sqrt{k^2 + 1},\ Y_B = \sqrt{k_1^2 + 1} \tag{B.20}$$

one obtains

$$\tilde{v}_{sw} = \int dk_{01} e^{ikx} f_h(k_1, k, t) \tag{B.21}$$

where

$$f_h(k_1, k, t) = \int_{X_B}^{\infty} dX \int_{Y_B}^{\infty} \frac{F(k_1, l_1, k, l, y) XY}{\sqrt{X^2 - X_B^2}\sqrt{Y^2 - Y_B^2}(X + Y)} \frac{e^{iYt} - e^{iXt}}{Y - X} dY. \tag{B.22}$$

For definiteness we put

$$X_B > Y_B \tag{B.23}$$

and represent f_h in the form

$$f_h = f_h^{(1)} + f_h^{(2)} + f_h^{(3)} + f_h^{(4)} + f_R \tag{B.24a}$$

where

$$f_h^{(1)} = \int_{X_B}^{X_B+\varepsilon} dX \int_{Y_B}^{X_B-\varepsilon} (\ ) dY,\quad f_h^{(2)} = \int_{X_B}^{X_B+\varepsilon} dX \int_{X+\varepsilon}^{\infty} (\ ) dY, \tag{B.24b,c}$$

$$f_h^{(3)} = \int_{X_B}^{X_B+\varepsilon} dX \int_{X_B-\varepsilon}^{X+\varepsilon} (\ ) dY,\quad f_h^{(4)} = \int_{X_B+\varepsilon}^{\infty} dX \int_{X-\varepsilon}^{X+\varepsilon} (\ ) dY. \tag{B.24d,e}$$

The term f_R in (B.24a) corresponds to the integration over "non-resonant" domain outside the vicinity of the resonance line $Y = X$ and of the point $X = X_B$ (correponding to zero group velocity). Simple estimate shows that

$$f_R = O\left(\frac{1}{t}\right), \quad t \to \infty. \tag{B.25}$$

The "resonant" term $f_h^{(4)}$ can be represented in the form

$$f_h^{(4)} \cong \int_{X_B+\varepsilon}^{\infty} \frac{F(k_1,l_1,k,l,y)_{X=Y} X e^{iXt}}{2\sqrt{X^2 - X_B^2}\sqrt{X^2 - Y_B^2}} dX \int_{X-\varepsilon}^{X+\varepsilon} \frac{e^{i(Y-X)t} - 1}{Y - X} dY. \tag{B.26}$$

We have for the internal integral:

$$J_1 = \int_{X-\varepsilon}^{X+\varepsilon} \frac{e^{i(Y-X)t} - 1}{Y - X} dY = \int_{-\varepsilon}^{\varepsilon} \frac{e^{iut} - 1}{u} du = i\pi + O\left(\frac{1}{t}\right), \quad t \to \infty, \tag{B.27}$$

therefore integration by parts over X in (B.26) gives:

$$f_h^{(4)} = O\left(\frac{1}{t}\right), \quad t \to \infty. \tag{B.28}$$

We now write the term $f_h^{(3)}$ as

$$f_h^{(3)} \cong \frac{F(k_1,l_1,k,l,y)_{X=Y=X_B}\sqrt{X_B}}{2\sqrt{2}\sqrt{X_B^2 - Y_B^2}} \int_{X_B}^{X_B+\varepsilon} dX \int_{X_B-\varepsilon}^{X+\varepsilon} \frac{e^{iYt} - e^{iXt}}{(Y - X)\sqrt{X - X_B}} dY \tag{B.29}$$

and estimate the integral term

$$J_2 = \int_{X_B}^{X_B+\varepsilon} dX \int_{X_B-\varepsilon}^{X+\varepsilon} \frac{e^{iYt} - e^{iXt}}{(Y - X)\sqrt{X - X_B}} dY = e^{iX_Bt} \int_0^{\varepsilon} \frac{e^{iut}}{\sqrt{u}} du \int_{-\varepsilon}^{u+\varepsilon} \frac{e^{i(v-u)t} - 1}{v - u} du. \tag{B.30}$$

for $t \to \infty$. Resulting estimate for $f_h^{(3)}$ is written as:

$$f_h^{(3)} = \frac{e^{iX_Bt}}{\sqrt{t}} \left\{ i\pi C_0 \left[\frac{F(k_1,l_1,k,l,y)\sqrt{X_B}}{2\sqrt{2}\sqrt{X_B^2 - Y_B^2}} \right]_{X=Y=X_B} + O\left(\frac{\ln t}{\sqrt{t}}\right) \right\} \tag{B.31a}$$

where

$$C_0 = \int_0^{\infty} \frac{e^{iz}}{\sqrt{z}} dz. \tag{B.31b}$$

The term $f_h^{(1)}$ in (B.24a) is estimated in a similar way:

$$f_h^{(1)} = f_h^{(11)} - f_h^{(12)}, \tag{B.32a}$$

$$f_h^{(11)} = \int_{X_B}^{X_B+\varepsilon} dX \int_{Y_B}^{X_B-\varepsilon} \frac{F(k_1,l_1,k,l,y)XY}{\sqrt{X^2-X_B^2}\sqrt{Y^2-Y_B^2}(X+Y)} \frac{e^{iYt}}{Y-X} dY = O\left(\frac{1}{t}\right), \tag{B.32b}$$

$$f_h^{(12)} = \int_{X_B}^{X_B+\varepsilon} dX \int_{Y_B}^{X_B-\varepsilon} \frac{F(k_1,l_1,k,l,y)XY}{\sqrt{X^2-X_B^2}\sqrt{Y^2-Y_B^2}(X+Y)} \frac{e^{iYt}}{Y-X} dY \cong$$
$$\sqrt{\frac{X_B}{2}} \int_{X_B}^{X_B+\varepsilon} \frac{e^{iXt}dX}{\sqrt{X-X_B}} \int_{Y_B}^{X_B-\varepsilon} \frac{F(k_1,l_1,k,l,y)Y}{(Y^2-X_B^2)\sqrt{Y^2-Y_B^2}} dY = \tag{B.32c}$$
$$\left[\sqrt{\frac{X_B}{2}} \int_{Y_B}^{X_B-\varepsilon} \frac{F(k_1,l_1,k,l,y)_{X=X_B} Y}{(Y^2-X_B^2)\sqrt{Y^2-Y_B^2}} dY\right] C_0 \frac{e^{iX_B t}}{\sqrt{t}} + O\left(\frac{1}{t}\right).$$

The resulting estimate for $f_h^{(1)}$ takes the form

$$f_h^{(1)} = -\left[\sqrt{\frac{X_B}{2}} \int_{Y_B}^{X_B-\varepsilon} \frac{F(k_1,l_1,k,l,y)_{X=X_B} Y}{(Y^2-X_B^2)\sqrt{Y^2-Y_B^2}} dY\right] C_0 \frac{e^{iX_B t}}{\sqrt{t}} + O\left(\frac{1}{t}\right). \tag{B.33}$$

The term $f_h^{(2)}$ is estimated in exactly the same way as $f_h^{(1)}$:

$$f_h^{(2)} = -\left[\sqrt{\frac{X_B}{2}} \int_{X_B+\varepsilon}^{\infty} \frac{F(k_1,l_1,k,l,y)_{X=X_B} Y}{(Y^2-X_B^2)\sqrt{Y^2-Y_B^2}} dY\right] C_0 \frac{e^{iX_B t}}{\sqrt{t}} + O\left(\frac{1}{t}\right). \tag{B.34}$$

Collecting the estimates (B.25), (B.28), (B.31), (B.33), and (B.34) we obtain for f_h:

$$f_h = E(k,k_1)\frac{e^{iX_B t}}{\sqrt{t}} + O\left(\frac{\ln t}{t}\right), \tag{B.35a}$$

$$E = -C_0 \frac{e^{iX_B t}}{\sqrt{t}} ⨍_{X_B} \frac{F(k_1,l_1,k,l,y)_{X=X_B} Y}{(Y^2-X_B^2)\sqrt{Y^2-Y_B^2}} dY + i\pi C_0 \frac{F(k_1,l_1,k,l,y)_{X=Y=X_B}}{2\sqrt{2}\sqrt{X_B^2-Y_B^2}}. \tag{B.35b}$$

Here $⨍_{X_B}$ denotes the Cauchy principal value integral from Y_B to infinity with the singular point $Y = Y_B$.

The same estimates for f_h are obtained for the case $X_B \le Y_B$ therefore the resulting estimate for $\tilde{v}_{sw}$ is

$$\tilde{v}_{sw} = O\left(\frac{1}{t}\right), \; t \to \infty, \; x, y \text{ fixed.} \tag{B.36}$$

We note that $\tilde{v}_{sw}$ in (B.36) decays faster than f_h due to additional integration over *k* of the integrand proportional to $e^{it\sqrt{k^2+1}}$. In the periodic and "step" cases this additional integration is absent therefore

$$\tilde{v}_{sw} = O\left(\frac{1}{\sqrt{t}}\right), \; t \to \infty, x, y \text{ fixed.} \tag{B.37}$$

Interaction IG wave - Kelvin wave

Representing the lowest-order Kelvin wave in the form

$$K_w^{(0)} e^{-y} = e^{-y} \int_{-\infty}^{\infty} \hat{K}_w(k) e^{ik(x-t)} dk \tag{B.38}$$

one can write the forcing term $\Phi_0^{(k)} \Phi_3^{(ig)}$ in (5.14) as follows

$$\Phi_0^{(k)} \Phi_3^{(ig)} = \int_{-\infty}^{\infty} A_1(k_1, l_1) \hat{K}_w(k_2) e^{i[(k_1+k_2)x + (l_1+i)y + (\pm\omega_{k_1 l_1} - k_2)t]} dk_{12} dl_1. \tag{B.39}$$

Then the corresponding response is

$$\tilde{v}_{kw} = -e^{-y} \int_{-\infty}^{\infty} dk_{12} \hat{K}_w(k_2) e^{i[(k_1+k_2)x - k_2)t]} \int_0^{\infty} dl_1 F(k_1, k_2, l_1, y) e^{\pm i\omega_{k_1 l_1} t}, \tag{B.40}$$

where

$$F = \frac{A(k_1, l_1) e^{il_1 y}}{D(k_1, k_2, l_1)} + \frac{A(k_1, -l_1) e^{-il_1 y}}{D(k_1, k_2, -l_1)} = O(l_1^2), \; l_1 \to 0 , \tag{B.41}$$

$$D = (k_1 + k_2)^2 + l_1^2 + 2il_1 - (\omega_{k_1 l_1} \pm k_2)^2. \tag{B.42}$$

Resonances are possible if $D = 0$, i.e. $l_1 = 0$, but this point is also not dangerous because of (B.41). It can be shown by integration by parts that

$$\tilde{v}_{kw} = O\left(\frac{1}{t^{3/2}}\right), \; t \to \infty. \tag{B.43}$$

We note that really $\tilde{v}_{kw}$ decays much faster than in (B.43) since the estimate (B.43) does not take into account the integration over k_2 in (B.40).

Interaction slow motion - Kelvin wave

In the localised case the response $\tilde{v}_{ks}$ is given by formula (B.40) in which $\omega_{k_1 l_1}$ is replaced by zero and D takes the form

$$D = k_2^2 + 2k_1k_2 + l_1^2 + 2il_1. \qquad \text{(B.44)}$$

Integrating by parts over k_2 one can show that

$$\tilde{v}_{ks} = O\left(\frac{1}{t^\infty}\right), \ \ t \to \infty. \qquad \text{(B.45)}$$

The estimate (B.45) is physically reasonable since the *localised* rapidly propagating Kelvin wave cannot efficiently interact with a localised slow motion.

In the "step" case we have the estimate

$$\tilde{v}_{ks} = O\left(\frac{1}{t}\right), \ \ t \to \infty, \qquad \text{(B.46)}$$

and in the periodic case the response does not decay in time,

$$\tilde{v}_{ks} = O(1), \ \ t \to \infty \qquad \text{(B.47)}$$

but always

$$\langle \tilde{v}_{ks} \rangle_t = 0 \qquad \text{(B.48)}$$

which means that the interaction is not resonant.