APPENDICES – FOR JFM FILE

APPENDIX A

- Solution for Eq. 14 with negligible cover inertia and axial force.

Neglecting the ice cover inertia and axial force terms, Eq. **14** reduced to

$$A+\left(1-\frac{U}{c}\right)d+\frac{3}{4d_o}d^2+\frac{l^4}{2}d''''=0 \qquad \text{(A1)}$$

It is known that cnoidal waves with slowly varying amplitude are observed in rivers (Debnath 1994). By assuming a solution of the following form a cnoidal wave solution for Eq. A1 can be obtained.

$$d(\zeta)=-acn^4(\alpha\zeta,\kappa)+b \qquad \text{(A2)}$$

where a, which is the wave height, and α, κ and b are constants to be determined.

Noting that

$$\frac{d^4}{dx^4}cn^4(x,\kappa)=840\kappa^4cn^8(x,\kappa)+1040(-2\kappa^4+\kappa^2)cn^6(x,\kappa)$$

$$+32(53\kappa^4-53\kappa^2+8)cn^4(x,\kappa)$$

$$+240(-2\kappa^4+3\kappa^2-1)cn^2(x,\kappa)+24(1-\kappa^2)^2 \qquad \text{(A3)}$$

the following relationships between a, b, α, κ, and U can be obtained by equating the coefficients of the polynomial of cn on both sides of Eq. A1,.

$$\frac{l^4\alpha^4}{2}a840\kappa^4=\frac{3a^2}{4d_0} \qquad \text{(A4)}$$

$$\frac{l^4\alpha^4}{2}a1040(-2\kappa^4+\kappa^2)=0 \qquad \text{(A5)}$$

$$\frac{l^4\alpha^4}{2}a32(53\kappa^4-53\kappa^2+8)=(1-\frac{U}{c}+\frac{3}{2d_0}b)(-a) \qquad \text{(A6)}$$

$$\frac{l^4\alpha^4}{2}a240(-2\kappa^4+3\kappa^2-1)=0 \tag{A7}$$

$$\frac{l^4\alpha^4}{2}a24(1-\kappa^2)^2 = A+(1-\frac{U}{c})b+\frac{3}{4d_0}b^2 \tag{A8}$$

Equations A5 and A7 only give

$$\kappa^2 = \frac{1}{2} \tag{A9}$$

Equation A4 gives

$$\alpha^4 = \frac{a}{140d_0l^4} \tag{A10}$$

The constant A in Eq. A1 can be determined by the condition

$$\int_0^{L_w} d(x,t)dx = 0 \tag{A11}$$

where L_w is the wave length. The wavelength of $cn^4(\alpha x,\kappa)$ is $2\alpha^{-1}K(\kappa)$ (Lawden 1989, Byrd and Friedman 1971). Hence

$$\int_0^{2K(k)} -acn^4(y,\kappa)dy + 2K(\kappa)b = 0 \tag{A12}$$

Equations A10 and A12 gives $b = a/3$, and Eq. A6 gives

$$U = (1-\frac{1}{10}\frac{a}{d_0})c \tag{A13}$$

Based on the above, the wave length is

$$L_W = 2l(\frac{140d_0}{a})^{\frac{1}{4}}K(\sqrt{\frac{1}{2}}) \tag{A14}$$

The constant A is obtained from Eq. A8 as $A = -\frac{23a^2}{210d_0}$.

Based on Eq. A2, the wave profile is obtained as

$$d(x,t) = a\left(\frac{1}{3} - cn^4\left(\left(\frac{a}{140d_o}\right)^{\frac{1}{4}}\frac{(x-Ut)}{l}, \sqrt{\frac{1}{2}}\right)\right) \tag{A15}$$

APPENDIX B

- Solution of Eq. 14.

By letting $X = \alpha\frac{x-Ut}{l}$, Equation 13 can be rewritten as

$$-[A + \left(1 - \frac{U}{c}\right)d + \frac{3}{4d_o}d^2] = \frac{1}{2}\alpha^4 d_{XXXX} + \beta\alpha^2 d_{XX} \tag{A16}$$

where $\beta = \frac{\rho_i}{2\rho}\frac{d_0\eta}{l^2}(1+F_r)^2 + \frac{N}{2\rho g l^2}$ which represents ice inertia and axial force.

The solution of Eq. A16 is assumed to be a Cnoidal wave in the following form

$$d(\zeta) = -acn^4(\alpha\frac{x-Ut}{l},\kappa) + bcn^2(\alpha\frac{x-Ut}{l},\kappa) + g \tag{A17}$$

Where a, b, g, α, κ, and U are constants to be determined.

Using Eq. A3 and the following three equations

$$\frac{d^2}{dx^2}cn^2(x,\kappa) = -6\kappa^2 cn^4(x,\kappa) + 4(-1+2\kappa^2)cn^2(x,\kappa) + 2(1-\kappa^2) \tag{A18}$$

$$\frac{d^4}{dx^4}cn^2(x,\kappa) = 120\kappa^4 cn^6(x,\kappa) + 120\kappa^2(-2\kappa^2+1)cn^4(x,\kappa)$$

$$+8(17\kappa^4 - 17\kappa^2 + 2)cn^2(x,\kappa) + 8(-2\kappa^4 + 3\kappa^2 - 1) \tag{A19}$$

$$\frac{d^2}{dx^2}cn^4(x,\kappa) = -20\kappa^2 cn^6(x,\kappa) + 16(-1+2\kappa^2)cn^4(x,\kappa) + 12(1-\kappa^2)cn^2(x,\kappa)$$

(A20)

five relationships between a, b, g, α, κ and U, are obtained by equating the coefficients of polynomials of cn on both sides of Eq. A16 as well as Eq. A11 for A gives

$$\frac{a}{d_0} = 560\kappa^4\alpha^4 \quad \text{(A21)}$$

$$\frac{b}{d_0} = -\left(\frac{14\times 80}{3}(1-2\kappa^2)\alpha^2 - \frac{14\times 40}{13\times 3}\beta\right)\kappa^2\alpha^2 \quad \text{(A22)}$$

$$\frac{g}{a} = \frac{-1}{39\kappa^2\dfrac{\alpha^2}{\beta}}\left(\frac{E(\kappa)}{K(\kappa)\kappa^2} - \frac{1-\kappa^2}{\kappa^2}\right) + \frac{1-\kappa^2}{3\kappa^2} \quad \text{(A23)}$$

$$\frac{U}{c} = 1 + \frac{3a}{2d_0}\frac{g}{a} + Q \quad \text{(A24)}$$

$$C_3\left(\frac{\alpha^2}{\beta}\right)^3 + C_2\left(\frac{\alpha^2}{\beta}\right)^2 + C_1\left(\frac{\alpha^2}{\beta}\right) + C_0 = 0 \quad \text{(A25)}$$

where

$$Q = \left(16\left(53\kappa^4 - 53\kappa^2 + 8\right) + \left(40 - \frac{70\times 8}{3}\right)\left(1-2\kappa^2\right)^2\right)\alpha^4$$
$$+\left(-20 + \frac{500}{3\times 13}\right)\left(-2\kappa^2+1\right)\beta\alpha^2 - \frac{2\times 31}{3\times 13^2}\beta^2 \quad \text{(A26)}$$

$$C_3 = 3\times 120(-2\kappa^4 + 3\kappa^2 - 1)\kappa^2 + 8(17\kappa^4 - 17\kappa^2 + 2)(-2\kappa^2+1)$$
$$-2(-2\kappa^2+1)\left(16(53\kappa^4 - 53\kappa^2 + 8) + (40 - \frac{70\times 8}{3})(1-2\kappa^2)^2\right) \quad \text{(A27)}$$

$$C_2 = 3\kappa^2 12(1-\kappa^2) - \frac{4}{13}(17\kappa^4 - 17\kappa^2 + 2) - 8(-2\kappa^2+1)^2$$
$$-2(-20 + \frac{500}{3\times 13})(-2\kappa^2+1)^2$$
$$+\left(16(53\kappa^4 - 53\kappa^2 + 8) + (40 - \frac{70\times 8}{3})(1-\kappa^2)^2\right)/13 \quad \text{(A28)}$$

$$C_1 = [\frac{4}{13} + (-20 + \frac{500}{3 \times 13})\frac{1}{13} + \frac{4 \times 31}{3 \times 13^2}](-2\kappa^2 + 1) \quad \text{(A29)}$$

and

$$C_0 = -\frac{2 \times 31}{3 \times 13^2} \quad \text{(A30)}$$

Where $E(\kappa)$ is the complete elliptic integral of the second kind.