Ageostrophic dynamics of an intense localized vortex on a β-plane

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APPENDICES A, B and C

Appendix A. First-order solution

The first-order solution analysis starts from the relationships (4.20) rewritten here for convenience

$$\tilde{q}_0 = 0,\ \tilde{r}_1 = 0,\ \overline{R}_1^{(1)} = 0. \tag{A.1a,b,c}$$

For $n=1$ we have in (4.1):

$$F(U)_1 = F(V)_1 = 0,\ F(q)_1 = -r\Omega_0\cos\theta,\ \ F(\varsigma)_1 = F(r)_1 = F(B)_1 = 0, \tag{A.2a}$$

$$D(U)_1 = D(V)_1 = D(h)_1 = D(h')_1 = 0 \tag{A.2b}$$

The equation (4.5c) for $\overline{q}_0$ takes the form

$$\frac{\partial \overline{q}_0}{\partial T_1} + \Omega_0 \frac{\partial \overline{q}_0}{\partial \theta} = -r\Omega_0 \cos\theta. \tag{A.3}$$

By virtue of (A.1a) the vorticity $q_0 = \overline{q}_0$, and, as follows from (2.24), q_0 is zero at the initial moment

$$q_0 = 0 \text{ at } T_1 = 0. \tag{A4}$$

The solution to (A.3), (A4) is given by the equations (5.2a,b), (5.3).

Since $F(U)_1 = F(V)_1 = F(\varsigma)_1 = 0$ the problem (4.6), (4.5k,m) for $\overline{h}_1$ is reduced to the system:

$$\nabla^2 \overline{h}_1 - \overline{h}_1 = q_0, \quad \left[\overline{h}_1\right] = 0,\ \left[\frac{\partial \overline{h}_1}{\partial r}\right] = -\left[V'_{I0}\right]\overline{r}_1. \tag{A.5}$$

In accordance with (4.7) the function $\overline{h}_1$ is represented as

$$\overline{h}_1 = \overline{h}_{11} + \overline{h}_{12}, \tag{A.6a}$$

$$\nabla^2 \bar{h}_{11} - \bar{h}_{11} = q_0, \quad [\bar{h}_{11}] = \left[\frac{\partial \bar{h}_{11}}{\partial r}\right] = 0, \tag{A.6b}$$

$$\nabla^2 \bar{h}_{12} - \bar{h}_{12} = 0, \qquad [\bar{h}_{12}] = 0, \quad \left[\frac{\partial \bar{h}_{12}}{\partial r}\right] = -[V'_{I0}]\,\bar{r}_1. \tag{A.6c}$$

The solution to (A.6b) has the form:

$$\bar{h}_{11} = A_{1s}\sin\theta + A_{1c}\cos\theta\,; \tag{A.7}$$

the quantity $A = A_{1s} + iA_{1c}$ is given by (5.5).

The function $\bar{h}_{12}$ is related to r_1 (see by (4.8) for $n = 1$):

$$\bar{h}_{12} = -\gamma\sum_{m=0}^{\infty}\Phi_m\,\mathrm{Im}(R_1^{(m)}e^{im\theta}), \qquad r_1 = \sum_{m=0}^{\infty}\mathrm{Im}(R_1^{(m)}e^{im\theta}). \tag{A.8a,b}$$

where we take into account that $r_1 = \bar{r}_1$ by virtue of (A.1b).

The following equations for the coefficients $\bar{G}_1^{(m)}$ in (4.12a) are readily derived from (A.2a), and (A.7):

$$\bar{G}_1^{(1)} = -iA|_b; \quad \bar{G}_1^{(m)} = 0, \;\; m \neq 1. \tag{A.9a,b}$$

Using (A.9a) and (A.1c) one can find from (4.10) the zero-order slow translation speed (5.7).

Also due to (A.9b), the zero initial conditions (2.24) for r_1, and (A.1b,c) we have from (4.11) that $\bar{R}_1^{(k)} = 0,\; k = 0,1,2,...$ whence (5.1) follows and, therefore,

$$\bar{h}_{12} = 0\,, \quad \bar{h}_1 = A_{1s}\sin\theta + A_{1c}\cos\theta\,. \tag{A.10a,b}$$

The analysis of the first-order slow component is completed by the calculation of $\bar{R}_2^{(1)}$ using (4.13), (A.1c), (A.2a), and (A.10b):

$$\bar{R}_2^{(1)} = -\int_0^1 r^2 A\,dr\,. \tag{A.11}$$

By virtue of (A.1a) and (A.2a) the vorticity equation (4.14c) gives

$$\tilde{q}_1 = 0\,, \tag{A.12}$$

and the equation (4.17a) for $\widetilde{h}_1$ takes the form:

$$-\frac{\partial^2 \widetilde{h}_1}{\partial t^2} + \nabla^2 \widetilde{h}_1 - \widetilde{h}_1 = 0 \ . \tag{A.13}$$

The fields $\overline{h}_1$, $\overline{U}_1 = -\partial \overline{h}_1 / r\partial\theta$, $\overline{V}_1 = \partial \overline{h}_1 / \partial r$ are zero at the initial moment (see (A.10b), (5.5), (5.2) , and (A.4)), therefore we have from (4.18), (4.19), (A.1a), and (A.2a) that

$$\widetilde{h}_1\Big|_{t=0} = \frac{\partial \widetilde{h}_1}{\partial t}\Big|_{t=0} = 0. \tag{A.14}$$

The conditions for $\widetilde{h}_1$ at $r=1$ are obtained from (4.14k,m) using (A.1b), (A.2b):

$$\left[\widetilde{h}_1\right] = \left[\frac{\partial \widetilde{h}_1}{\partial r}\right] = 0.$$

(A.15)

Obviously, the homogeneous problem (A.13) to (A.15) has a zero solution

$$\widetilde{h}_1 = 0 . \tag{A.16}$$

Then, by virtue of (4.14a,b) for $n = 1$, (A.2a), and zero initial conditions for $\widetilde{U}_1$, $\widetilde{V}_1$ (because of (2.24) and zero initial conditions for $\overline{U}_1$, $\overline{V}_1$, see above) we have also that

$$\widetilde{U}_1 = \widetilde{V}_1 = 0 \ . \tag{A.17}$$

The function $\widetilde{R}_2^{(1)}$ also vanishes which follows from (4.14g), (A.2a), (A.1b), and (A.16), i.e.

$$\widetilde{R}_2^{(1)} = 0. \tag{A.18}$$

One can readily see from (4.14f), (A.2a), (A.1b), (A.17), and (A.18) that

$$\widetilde{X}_0 = \widetilde{Y}_0 = 0 \ , \tag{A.19}$$

and

The calculation of next approximation starts from the equations (A.11), (A.12), and (A.20).

Apendix B. Second-order correction

In the second-order approximation we have

$$F(U)_2 = -\frac{\partial U_1}{\partial T_1} + \frac{V_{I0}}{r} V_1^* - \frac{V_{I0}}{r}\left(\frac{\partial U_1}{\partial \theta} - V_1\right) + yV_{I0}, \tag{B.1a}$$

$$F(V)_2 = -\frac{\partial V_1}{\partial T_1} - V_{I0}' U_1^* - \frac{V_{I0}}{r}\left(\frac{\partial V_1}{\partial \theta} + U_1\right), \tag{B.1b}$$

$$F(q)_2 = -\left[\frac{\partial q_0}{\partial T_2} + U_1^* \frac{\partial q_0}{\partial r} + (V_{I1} + V_1^*)\frac{1}{r}\frac{\partial q_0}{\partial \theta} + U_1 \sin\theta + (V_{I1} + V_1)\cos\theta - h_{I0} V_{I0} \cos\theta\right], \tag{B.1c}$$

$$F(\varsigma)_2 = \gamma h_1 H(1-r) + h_{I0} q_0,$$

(B.1d)

$$F(r)_2 = F(B)_2 = 0, \; D(U)_2 = D(V)_2 = D(h)_2 = D(h')_2 = 0. \tag{B.1e}$$

Here

$$U_1^* = U_1 - \dot{X}_0 \cos\theta - \dot{Y}_0 \sin\theta, \quad V_1^* = V_1 + \dot{X}_0 \sin\theta - \dot{Y}_0 \cos\theta\,. \tag{B.2}$$

For $n = 2$ the slow vorticity equation (4.5c) takes the form, noting that $F(q)_2 = \overline{F}(q)_2$ since all quantities in the right-hand side part of (B.1c) are slow,

$$\frac{\partial \overline{q}_1}{\partial T_1} + \Omega_0 \frac{\partial \overline{q}_1}{\partial \theta} = F(q)_2. \tag{B.3}$$

To prevent secular growth $\overline{q}_1$ in T_1 we remove from $F(q)_2$ the corresponding secular terms i.e. we put

The solution to (B.4) satisfying zero initial conditions and compatible with (5.2), (5.3) has the form (5.2a,b) where φ_0, C_0 are replaced by

$$\varphi_1 = \Omega_0 T_1 + \Omega_1 T_2, \quad C_1 = C_1(r, T_3, \ldots), \quad C_1 = 0 \text{ for } T_3 = 0 \tag{B.5}$$

respectively.

The solution to (B.3) is conveniently represented in the form

$$\overline{q}_1 = -h_{I0} q_0 + q_{12} \tag{B.6}$$

where the equation for q_{12} coincides (within notations) with equation (5.6e) of RGB,

$$\frac{\partial q_{12}}{\partial T_1} + \Omega_0 \frac{\partial q_{12}}{\partial \theta} = -J(h_1 + \dot{X}_0 y - \dot{Y}_0 x, q_0) - \frac{\partial h_1}{\partial x}, \tag{B.7}$$

and has the solution which was analyzed in detail in RGB (c.f. the solution (6.2), (6.3) of RGB):

$$q_{12} = q_{20} + q_{2s} \sin 2\theta + q_{2c} \cos 2\theta \ , \tag{B.8a}$$

$$q_{20} = \operatorname{Im} \int_0^{T_1} \frac{1}{2r} \left(\overline{A}^* \overline{Q}_0 \right)' dT_1 - \int_0^{T_1} \dot{Y}_0 dT_1 \ , \tag{B.8b}$$

$$q_{2s} + i q_{2c} = \frac{1}{2r} e^{-2i\Omega_0 T_1} \int_0^{T_1} \left(\overline{A}\,\overline{Q}_0' - \overline{A}'\,\overline{Q}_0 \right) e^{2i\Omega_0 T_1} dT_1 \ , \tag{B.8c}$$

$$\overline{A} = \overline{A}_{s1} + i\overline{A}_{c1} = A(r,t) - rA(1,t) \ , \ \overline{Q}_0 = Q_0 + r \tag{B.8d}$$

Using (B.1a,b,d) the equation (4.6) for the slow elevation $\overline{h}_2$ is written as follows

$$\nabla^2 \overline{h}_2 - \overline{h}_2 = \overline{F}(h)_2 = \gamma h_1 H(1-r) + \frac{1}{r}\left[2(V_{I0} V_1)' - 2V_{I0}' \frac{\partial U_1}{\partial \theta} + (r^2 V_{I0})' \sin\theta \right] + q_{12}. \tag{B.9}$$

We now represent $\overline{h}_2$ in the form (see (4.7), (B.1e)),

$$\overline{G}_2 = -\left(\frac{\partial \overline{h}_{21}}{\partial \theta} - \overline{F}(V)_2 \right)\Bigg|_b . \qquad \text{(B.15)}$$

By virtue of (B.11), (B.14), and (B.1b) the function $\overline{G}_2$ includes the dipole and quadrupole terms only. Calculating $\overline{G}_2^{(1)}$ in (4.12a) and using $\overline{R}_2^{(1)}$ from (A.11) we determine from (4.10) the translation speed components $\dot{\overline{X}}_1$, $\dot{\overline{Y}}_1$ given by (5.20).

Also knowing $\overline{G}_2^{(2)}$ one can calculate the coefficients $R_2^{(2)}$ in the Fourier representation (5.21a) for r_2. The resulting solutions for r_2 and $\overline{h}_{22}$ are presented by the equations (5.21a,b), (5.22) (we recall that $\tilde{r}_2 = 0$ by virtue of (A.20) and $R_2^{(1)}$ are determined by (A.11)).

Now we know the function $\overline{h}_2 = \overline{h}_{21} + \overline{h}_{22}$ and can determine the slow velocity components $\overline{U}_2$, $\overline{V}_2$ from (4.5a,b):

$$\overline{V}_2 = \frac{\partial \overline{h}_2}{\partial r} - \overline{F}(U)_2, \quad \overline{U}_2 = -\frac{1}{r}\frac{\partial \overline{h}_2}{\partial \theta} + \overline{F}(V)_2 . \qquad \text{(B.16a,b)}$$

Finally, the first azimuthal component $\overline{R}_3^{(1)}$ is determined from (4.13) using (A.11), (B.1e), and the function $\overline{h}_2$:

$$\overline{R}_3^{(1)} = \frac{i}{\pi}\int_0^{2\pi} \overline{r}_3 e^{-i\theta} d\theta = -\frac{i}{\pi}\int_0^{2\pi} e^{-i\theta} d\theta \int_0^1 r^2 \overline{h}_2 dr - h_{10}\big|_b \overline{R}_2^{(1)} . \qquad \text{(B.17)}$$

The function $F(q)_2$ in (B.1c) is slow and $\tilde{q}_1 = 0$ (see (A.12)) therefore we have from (4.14c) for $n = 2$ that

$$\tilde{q}_2 = 0 . \qquad \text{(B.18)}$$

Since the functions $F(U)_2$, $F(V)_2$, $F(\varsigma)_2$ in (B.1a,b,d) are slow, equation (4.17) for $\tilde{h}_2$ takes the form:

$$-\frac{\partial^2 \tilde{h}_2}{\partial t^2} + \nabla^2 \tilde{h}_2 - \tilde{h}_2 = 0 . \qquad \text{(B.19)}$$

Initial conditions for (B.19) follow from (4.18), (4.19):

$$\tilde{h}_2 = -\overline{h}_2, \quad \frac{\partial \tilde{h}_2}{\partial t} = -\frac{\partial \varsigma_1}{\partial T_1} - V_{I0}\cos\theta \quad \text{for } t = 0; \qquad \text{(B.20a,b)}$$

when writing (B.20) we use (B.16), (A.12), and the fact that $\tilde{F}(U)_2 = \tilde{F}(V)_2 = \tilde{F}(\varsigma)_2 = 0$. The problem (B.19), (B.20) is well-defined and $\tilde{h}_2$ can be readily found, for example, using the Fourier transform. Known $\tilde{h}_2$ the velocity $\tilde{\mathbf{U}}_2 = \tilde{U}_2 + i\tilde{V}_2$ is determined from the system

$$\frac{\partial \tilde{\mathbf{U}}_2}{\partial t} + i\tilde{\mathbf{U}}_2 = -\left(\frac{\partial \tilde{h}_2}{\partial r} + \frac{i}{r}\frac{\partial \tilde{h}_2}{\partial \theta}\right), \quad \tilde{\mathbf{U}}_2\Big|_{t=0} = -(\overline{U}_2 + i\overline{V}_2)\Big|_{t=0}. \qquad \text{(B.21a,b)}$$

One can readily show that all the right-hand side parts in (B.20), (B.21) have a dipole structure therefore $\tilde{h}_2$ and $\tilde{\mathbf{U}}_2 = \tilde{U}_2 + i\tilde{V}_2$ are also dipoles i.e.

$$\tilde{h}_2 = \tilde{h}_{2s}\sin\theta + \tilde{h}_{2c}\cos\theta, \quad \tilde{\mathbf{U}}_2 = (\tilde{U}_{2s} + i\tilde{V}_{2s})\sin\theta + (\tilde{U}_{2c} + i\tilde{V}_{2c})\cos\theta. \qquad \text{(B.22a,b)}$$

To determine the fast components of translation speed $\dot{\tilde{X}}_1$, $\dot{\tilde{Y}}_1$ we calculate $\tilde{R}_3^{(1)}$ using (4.14g):

$$\tilde{R}_3^{(1)} = -\int_0^1 r^2 \tilde{H}_2 dr, \quad \tilde{H}_2 = \tilde{h}_{2s} + i\tilde{h}_{2c}. \qquad \text{(B.23a,b)}$$

Substituting (B.23a), (B.22b) into (4.14f) we obtain two equations:

$$\dot{\tilde{Y}}_1 + i\dot{\tilde{X}}_1 = \int_0^1 r^2 \frac{\partial \tilde{H}_2}{\partial t} dr + (\tilde{U}_{2s} + i\tilde{U}_{2c})\Big|_b, \qquad \text{(B.24)}$$

$$\tilde{R}_3^{(n)} = 0, \quad n \neq 1. \qquad \text{(B.25)}$$

The first of these equations determines the fast components of translation speed $\dot{\tilde{X}}_1$, $\dot{\tilde{Y}}_1$, the second one together with (B.23a) - the fast boundary perturbation $\tilde{r}_3$. Knowing $\tilde{r}_3$, $\tilde{q}_2$, $\overline{R}_3^{(1)}$ (see (B.17), (B.18)) one can calculate the next approximation.

Appendix C

The functions $F(U)_3$, $F(V)_3$ have the form

$$F(U)_3 = -\frac{\partial U_2}{\partial T_1} - \frac{\partial U_1}{\partial T_2} - U_1^* \frac{\partial U_1}{\partial r} - \left(\frac{\partial U_2}{\partial \theta} - 2V_2 - \dot{X}_1 \sin\theta + \dot{Y}_1 \cos\theta - rY_0\right)\frac{V_{I0}}{r} - \frac{V_1^* + V_{I1}}{r}\left(\frac{\partial U_1}{\partial \theta} - V_1\right) - \frac{V_{I1}V_1^*}{r} + y(V_{I1} + V_1); \quad \text{(C.1a)}$$

$$F(V)_3 = -\frac{\partial V_2}{\partial T_1} - \frac{\partial V_1}{\partial T_1} - U_1^* \frac{\partial (V_1 + V_{I1})}{\partial r} - U_2^* V_{I0}' - \frac{V_1^* + V_{I1}}{r}\left(\frac{\partial V_1}{\partial \theta} + U_1\right) - \frac{V_{I0}}{r}\left(\frac{\partial V_2}{\partial \theta} + U_2\right) - yU_1. \quad \text{(C.1b)}$$

Let us average the equations (7.2a-d) with respect to θ; using (B.14), (B.8) we have,

$$\frac{\partial U_3^{(0)}}{\partial t} - V_3^{(0)} = -\frac{\partial h_3^{(0)}}{\partial r} + F(U)_3^{(0)}, \quad \frac{\partial V_3^{(0)}}{\partial t} + U_3^{(0)} = F(V)_3^{(0)}, \quad \text{(C.2a,b)}$$

$$\varsigma_3^{(0)} - h_3^{(0)} = \gamma < h_2 > H(1-r) + < h_{I0} q_{20} > + < h_1 q_0 > + < q_2 >, \quad \text{(C.2c)}$$

$$\varsigma_3^{(0)} = \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_3^{(0)}) - \frac{\partial U_3^{(0)}}{\partial \theta}\right]. \quad \text{(C.2d)}$$

where the notation

$$K_3^{(0)} = < K_3 > = \frac{1}{2\pi}\int_0^{2\pi} K_3 d\theta$$

denotes the azimuthal average.

The fast components of the fields U_2, V_2, and h_2 are dipolar (see Sec 5 and Appendix B) and q_2 does not depend on the fast time by virtue of (B.18). Therefore $F(U)_3^{(0)}$, $F(V)_3^{(0)}$, and the right-hand side of (C.2c) are slow functions of the time; one can readily check also that these functions are equal to zero for $t = 0$. We now reduce the equations (C.2) to a single equation for $h_3^{(0)}$; simple analysis of this equation shows that $h_3^{(0)}$ and therefore $U_3^{(0)}$, $V_3^{(0)}$ do not depend on the fast time. Then by virtue of (C.2b) we have after some transformations

$$U_3^{(0)}\Big|_b = F(V)_3^{(0)}\Big|_b = -\frac{\partial V_{20}}{\partial T_1}\bigg|_b - < U_1\big|_b \sin\theta > \tag{C.3}$$

where V_{20} is related to the axisymmetric part of the geostrophic second-order elevation correction $\overline{h}_{2g}^c$,

$$V_{20} = \frac{\partial < \overline{h}_{2g}^c >}{\partial r} = \frac{\partial B_{20}}{\partial r}. \tag{C.4}$$

Using the equations (B.14b), (B.8), (5.6a), and (5.4b) we come to (7.4), (7.5).