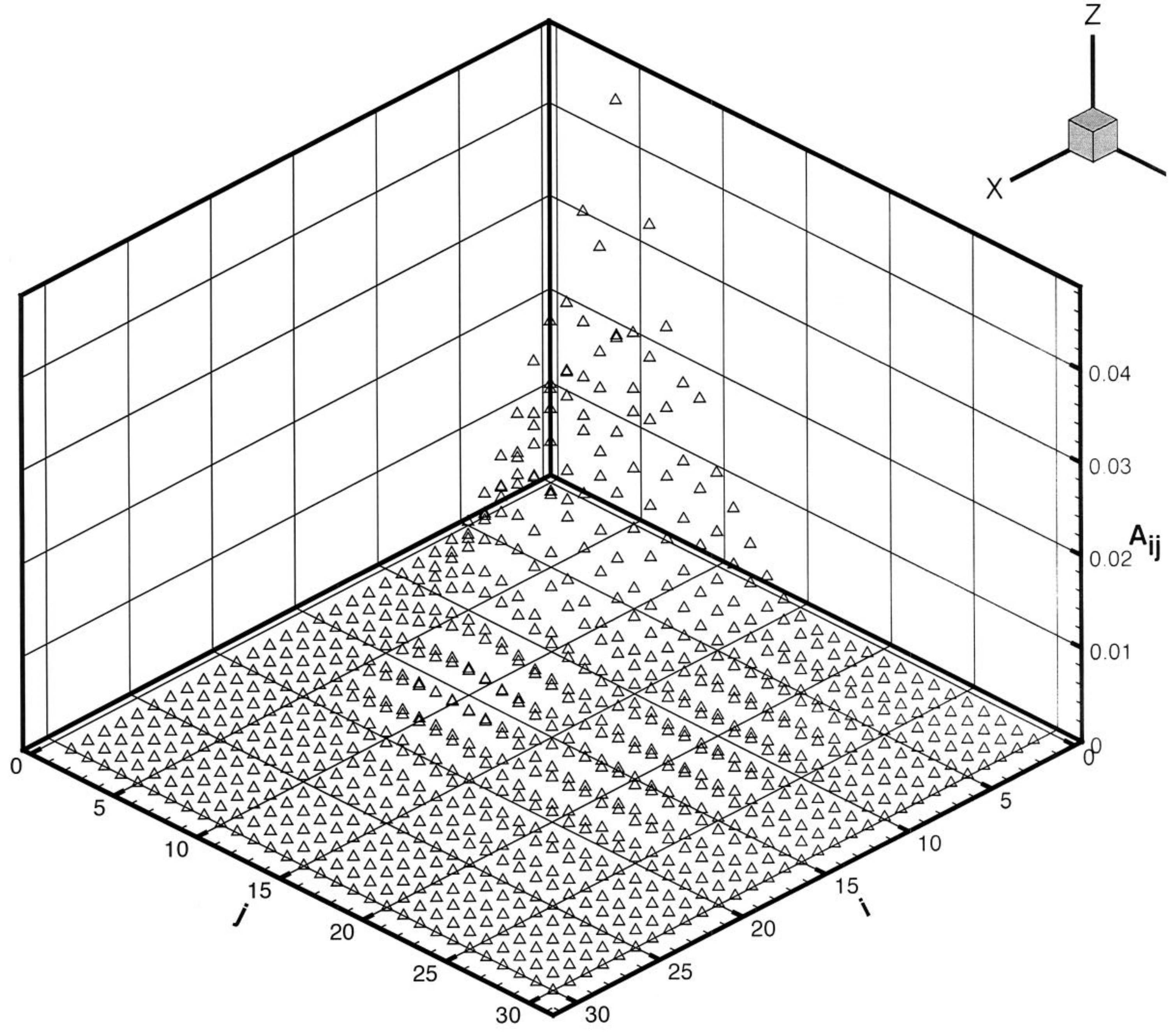

Fig.7a. Coefficients A_{ij} for steady state at Re = 2471 and $\gamma = 1$

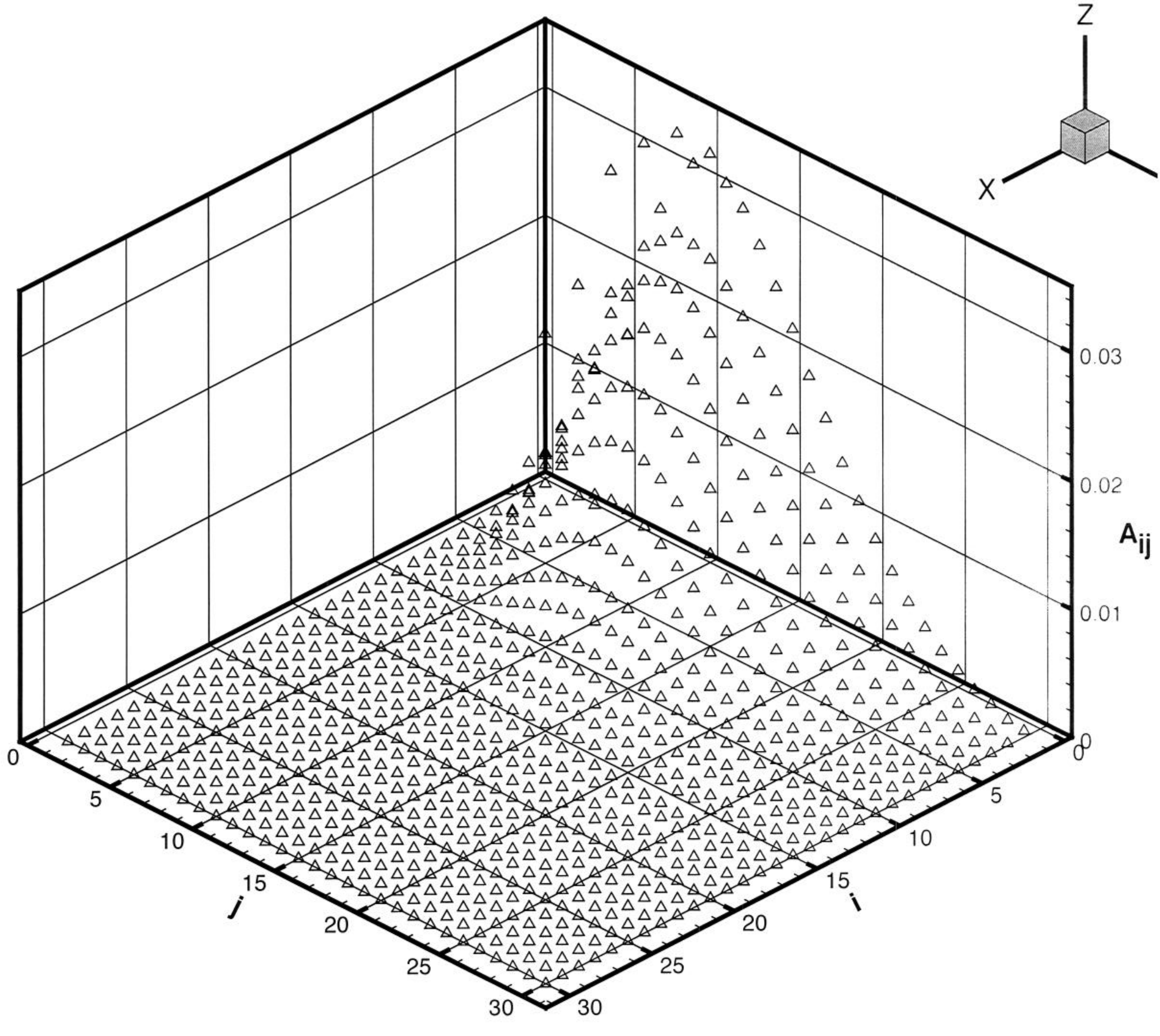

Fig.7b. Coefficients A_{ij} for steady state at Re = 2132 and $\gamma = 3.5$

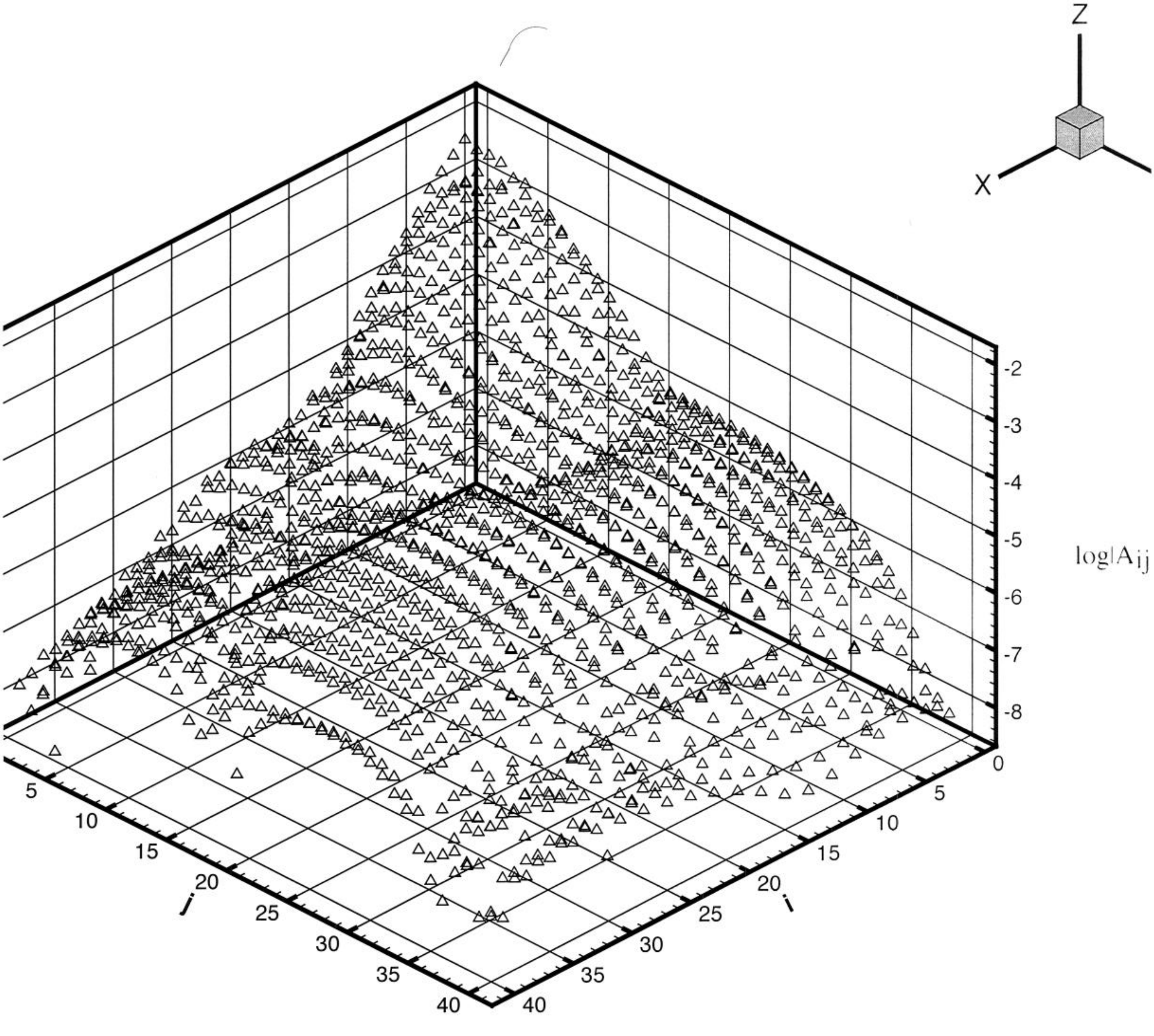

Fig.7c. Coefficients A_{ij} for steady state at Re = 2471 and γ = 1

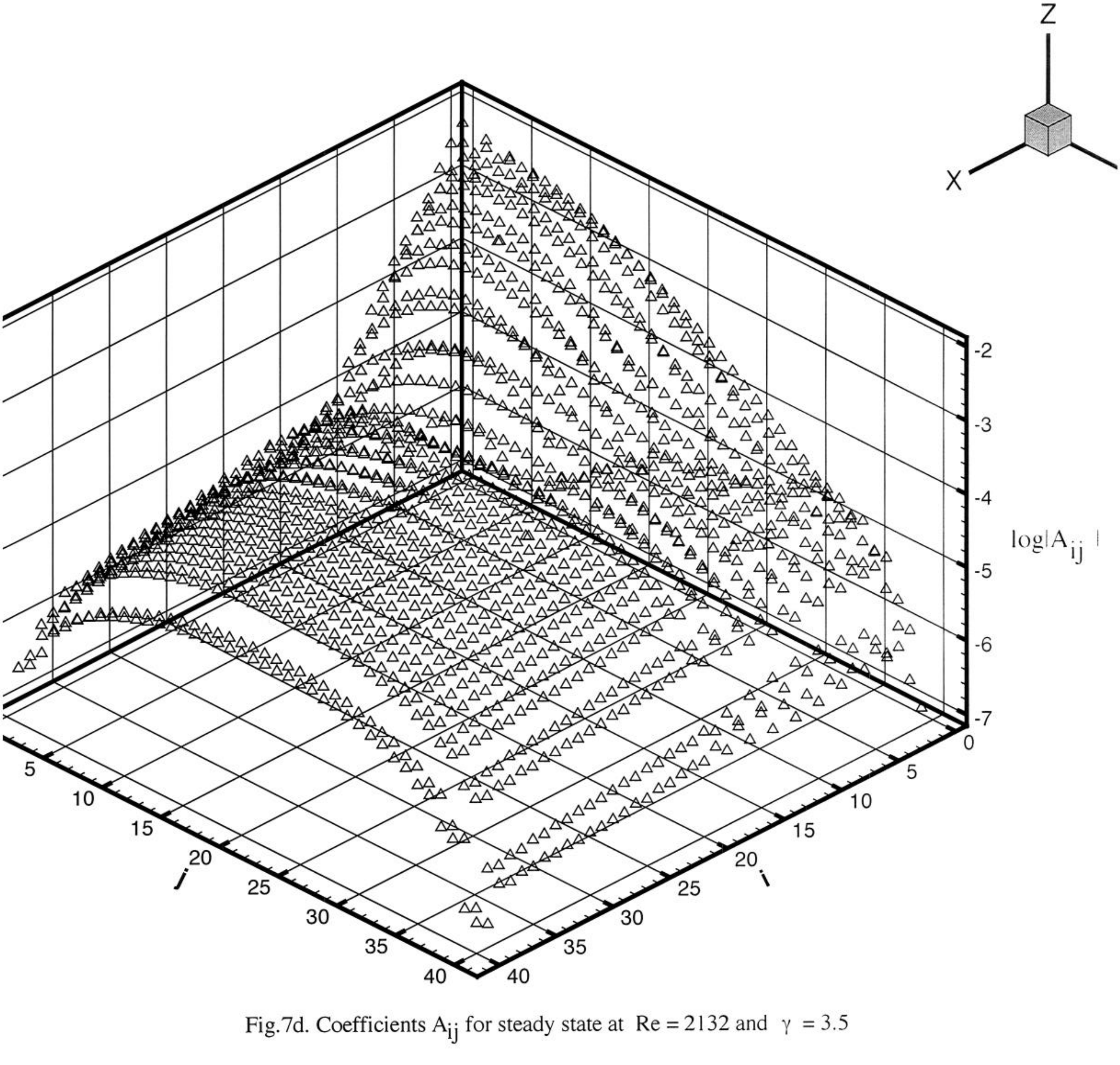

Fig.7d. Coefficients A_{ij} for steady state at Re = 2132 and γ = 3.5

Appendix I

Projection of linear and bilinear terms

I.1. Projection of linear terms

The projection of the Laplacian operator of velocity on the bases $\mathbf{V}_{ij}$ and $\mathbf{W}_{ij}$ can be divided into four inner products: $\langle \Delta \mathbf{V}_{ij}, \mathbf{V}_{pq} \rangle$, $\langle \Delta \mathbf{V}_{ij}, \mathbf{W}_{pq} \rangle$, $\langle \Delta \mathbf{W}_{ij}, \mathbf{V}_{pq} \rangle$, and $\langle \Delta \mathbf{W}_{ij}, \mathbf{W}_{pq} \rangle$. Using (31) and (32) these inner products may be expressed as

$$\langle \Delta \mathbf{V}_{ij}, \mathbf{V}_{pq} \rangle = \sum_{l=0}^{4} \sum_{m=0}^{4} c_{il} c_{pm} R_{i+l,p+m}^{11} \cdot \sum_{l=0}^{4} \sum_{m=0}^{4} d_{jl} d_{qm} Z_{j+l,q+m}^{11} + \\ + \sum_{l=0}^{4} \sum_{m=0}^{4} c_{il} c_{pm} Q_{i+l,p+m}^{11} \cdot \sum_{l=0}^{4} \sum_{m=0}^{4} d_{jl} d_{qm} Y_{j+l,q+m}^{11} \tag{I.1}$$

$$\langle \Delta \mathbf{V}_{ij}, \mathbf{W}_{pq} \rangle = \sum_{l=0}^{4} \sum_{m=0}^{4} c_{il} e_{pm} R_{i+l,p+m}^{12} \cdot \sum_{l=0}^{4} \sum_{m=0}^{4} d_{jl} f_{qm} Z_{j+l,q+m}^{12} + \\ + \sum_{l=0}^{4} \sum_{m=0}^{4} c_{il} e_{pm} Q_{i+l,p+m}^{12} \cdot \sum_{l=0}^{4} \sum_{m=0}^{4} d_{jl} f_{qm} Y_{j+l,q+m}^{12} \tag{I.2}$$

$$\langle \Delta \mathbf{W}_{ij}, \mathbf{V}_{pq} \rangle = \sum_{l=0}^{4} \sum_{m=0}^{4} e_{il} c_{pm} R_{i+l,p+m}^{21} \cdot \sum_{l=0}^{4} \sum_{m=0}^{4} f_{jl} d_{qm} Z_{j+l,q+m}^{21} + \\ + \sum_{l=0}^{4} \sum_{m=0}^{4} e_{il} c_{pm} Q_{i+l,p+m}^{21} \cdot \sum_{l=0}^{4} \sum_{m=0}^{4} f_{jl} d_{qm} Y_{j+l,q+m}^{21} \tag{I.3}$$

$$Y_{jq}^{12} = \left\langle T_j''(z), U_{q-1}(z) \right\rangle \tag{I.12}$$

$$\begin{aligned} R_{ip}^{21} = & \left\langle (\alpha+1) r^{3+\alpha} T_i''(r) + 5(\alpha+1) r^{2+\alpha} T_i'(r) + \left[3(\alpha+1) + k^2(1-\alpha) \right] r^{1+\alpha} T_i(r), T_p(r) \right\rangle + \\ & + 2p \left\langle r^{4+\alpha} T_i''(r) + 5r^{3+\alpha} T_i'(r) + \left(3 - k^2\right)(1+\alpha) r^{2+\alpha} T_i(r), U_{p-1}(r) \right\rangle \end{aligned} \tag{I.13}$$

$$Z_{jq}^{21} = \left\langle U_{j-1}(z), T_q(z) \right\rangle \tag{I.14}$$

$$Q_{ip}^{21} = \left\langle (\alpha+1) r^{3+\alpha} T_i(r), T_p(r) \right\rangle + 2i \left\langle r^{4+\alpha} T_i(r), U_{p-1} \right\rangle \tag{I.15}$$

$$Y_{jq}^{21} = \left\langle U_{j-1}''(z), T_q(z) \right\rangle \tag{I.16}$$

$$\hat{R}_{ip}^{22} = \left\langle r^5 T_i''(r) + 5r^4 T_i'(r) + \left(3 - k^2\right) r^3 T_i(r), T_p(r) \right\rangle \tag{I.17}$$

$$\hat{Z}_{jq}^{22} = \left\langle U_{j-1}(z), U_{q-1}(z) \right\rangle \tag{I.18}$$

$$\hat{Q}_{ip}^{22} = \left\langle r^5 T_i(r), T_p(r) \right\rangle \tag{I.19}$$

$$\hat{Y}_{jq}^{22} = \left\langle U_{j-1}''(z), U_{q-1}(z) \right\rangle \tag{I.20}$$

$$\tilde{R}_{ip}^{22} = k^2 \left\langle r^3 T_i''(r) + 3r^2 T_i'(r) + \left(1 - k^2\right) T_i(r), T_p(r) \right\rangle \tag{I.21}$$

$$\tilde{Z}_{jq}^{22} = \frac{k^2}{4jq} \left\langle T_j(z), T_q(z) \right\rangle \tag{I.22}$$

$$\tilde{Q}_{ip}^{22} = \left\langle r^3 T_i(r), T_p(r) \right\rangle \tag{I.23}$$

$$\tilde{Y}_{jq}^{22} = \frac{k^2}{4jq} \left\langle T_j''(z), T_q(z) \right\rangle \tag{I.24}$$

In a similar way one can express projections of the Laplacian of the temperature:

$$\begin{aligned} \left\langle \Delta\theta_{ij}, \theta_{pq} \right\rangle = & \sum_{l=0}^{2} \sum_{m=0}^{2} \sigma_{il} \sigma_{pm} R_{i+l,p+m}^{\theta} \cdot \sum_{l=0}^{2} \sum_{m=0}^{2} \delta_{jl} \delta_{qm} Z_{j+l,q+m}^{\theta} + \\ & + \sum_{l=0}^{2} \sum_{m=0}^{2} \sigma_{il} \sigma_{pm} Q_{i+l,p+m}^{\theta} \cdot \sum_{l=0}^{2} \sum_{m=0}^{2} \delta_{jl} \delta_{qm} Y_{j+l,q+m}^{\theta} \end{aligned} \tag{I.25}$$

that the azimuthal wavenumber is included only as even powers of k, and for $\pm k$ there are two complex conjugate eigenvalues $\pm i\omega$. In the case $V \neq 0$ the azimuthal wave can propagate with or opposite to the direction of V. Thus, for each azimuthal wavenumber $+k$ or $-k$ there exists only one eigenvalue $+i\omega$ or $-i\omega$.

To denote the Galerkin projections of the bilinear terms $\left\langle \left(\mathbf{V}_1^{3D} \cdot \nabla \right) \mathbf{V}^{2D}, \mathbf{V}_2^{3D} \right\rangle +$ $\left\langle \left(\mathbf{V}^{2D} \cdot \nabla \right) \mathbf{V}_1^{3D}, \mathbf{V}_2^{3D} \right\rangle$ we will use symbols B_{ijnmpq} such that the indices i and j correspond to a series of $\mathbf{V}_1^{3D}$, indices n and m correspond to the series of axisymmetric steady state $\mathbf{V}^{2D}$, and indices p and q correspond to the series of the projection vector $\mathbf{V}_2^{3D}$ (according to series (24)). The symbols B_{ijnmpq} will be divided into eight parts to denote real and imaginary parts of the bilinear terms, and projections on two different basis systems (31) and (32). Thus, similarly to (I.1)-(I.2),

$$B_{ijnmpq} = \left[\hat{B}_{ijnmpq}^{11} + \hat{B}_{ijnmpq}^{12} + \hat{B}_{ijnmpq}^{21} + \hat{B}_{ijnmpq}^{22} \right] + i \left[\tilde{B}_{ijnmpq}^{11} + \tilde{B}_{ijnmpq}^{12} + \tilde{B}_{ijnmpq}^{21} + \tilde{B}_{ijnmpq}^{22} \right] \quad \text{(I.29)}$$

The six-indices symbols are defined as scalar products (V_{nm} represents the n,m-term in the series (37), $\mathbf{e}_\varphi$ is the unit vector in the azimuthal direction):

$$\hat{B}_{ijnmpq}^{11} = \left\langle \left(\mathbf{V}_{ij} \cdot \nabla \right) \mathbf{U}_{nm} + \left(\mathbf{U}_{nm} \cdot \nabla \right) \mathbf{V}_{ij}, \mathbf{V}_{pq} \right\rangle, \quad \tilde{B}_{ijnmpq}^{11} = \left\langle \left(\mathbf{V}_{ij} \cdot \nabla \right) V_{nm} \mathbf{e}_\varphi + \left(V_{nm} \mathbf{e}_\varphi \cdot \nabla \right) \mathbf{V}_{ij}, \mathbf{V}_{pq} \right\rangle \quad \text{(I.30)}$$

$$\hat{B}_{ijnmpq}^{12} = \left\langle \left(\mathbf{V}_{ij} \cdot \nabla \right) \mathbf{U}_{nm} + \left(\mathbf{U}_{nm} \cdot \nabla \right) \mathbf{V}_{ij}, \mathbf{W}_{pq} \right\rangle, \quad \tilde{B}_{ijnmpq}^{12} = \left\langle \left(\mathbf{V}_{ij} \cdot \nabla \right) V_{nm} \mathbf{e}_\varphi + \left(V_{nm} \mathbf{e}_\varphi \cdot \nabla \right) \mathbf{V}_{ij}, \mathbf{W}_{pq} \right\rangle \quad \text{(I.31)}$$

$$\hat{B}_{ijnmpq}^{21} = \left\langle \left(\mathbf{W}_{ij} \cdot \nabla \right) \mathbf{U}_{nm} + \left(\mathbf{U}_{nm} \cdot \nabla \right) \mathbf{W}_{ij}, \mathbf{V}_{pq} \right\rangle, \tilde{B}_{ijnmpq}^{21} = \left\langle \left(\mathbf{W}_{ij} \cdot \nabla \right) V_{nm} \mathbf{e}_\varphi + \left(V_{nm} \mathbf{e}_\varphi \cdot \nabla \right) \mathbf{W}_{ij}, \mathbf{V}_{pq} \right\rangle \quad \text{(I.32)}$$

$$\hat{B}_{ijnmpq}^{22} = \left\langle \left(\mathbf{W}_{ij} \cdot \nabla \right) \mathbf{U}_{nm} + \left(\mathbf{U}_{nm} \cdot \nabla \right) \mathbf{W}_{ij}, \mathbf{W}_{pq} \right\rangle, \tilde{B}_{ijnmpq}^{22} = \left\langle \left(\mathbf{W}_{ij} \cdot \nabla \right) V_{nm} \mathbf{e}_\varphi + \left(V_{nm} \mathbf{e}_\varphi \cdot \nabla \right) \mathbf{W}_{ij}, \mathbf{W}_{pq} \right\rangle \quad \text{(I.33)}$$

Substitution of (30)-(32) and (37) into (I.30)-(I.33) gives:

$$\hat{Y}^{11}_{jmq} = -\frac{1}{2m}\left\langle T_j'(z)T_m(z), T_q(z)\right\rangle \tag{1.39}$$

$$\hat{S}^{11}_{inp} = \hat{X}^{11}_{jmq} = \hat{F}^{11}_{inp} = \hat{G}^{11}_{jmq} = \hat{F}^{21}_{inp} = \hat{G}^{21}_{jmq} = \hat{F}^{12}_{inp} = \hat{G}^{12}_{jmq} = 0 \tag{1.40}$$

$$\hat{R}^{21}_{inp} = \frac{1}{2}k^2\left\langle r^{3+\alpha}T_i(r)T_n(r), T_p(r)\right\rangle \tag{1.41}$$

$$\hat{Z}^{21}_{jmq} = \frac{1}{2j}\left\langle T_j(z)U_{m-1}'(z), T_q(z)\right\rangle \tag{1.42}$$

$$\begin{aligned}\hat{Q}^{21}_{inp} &= \frac{1}{2}(\alpha+1)\left\langle r^{\alpha+4}T_i'(r)T_n(r) + 3r^{\alpha+3}T_i(r)T_n(r), T_p\right\rangle(r) + \\ &+ p\left\langle r^{\alpha+5}T_i'(r)T_n(r) + 3r^{\alpha+4}T_i(r)T_n(r), U_{p-1}(r)\right\rangle\end{aligned} \tag{1.43}$$

$$\hat{Y}^{21}_{jmq} = \left\langle U_{j-1}(z)U_{m-1}(z), T_q(z)\right\rangle \tag{1.44}$$

$$\begin{aligned}\hat{S}^{21}_{inp} &= (\alpha+1)\left\langle r^{\alpha+3}T_i(r)T_n(r) + nr^{\alpha+4}T_i(r)U_{n-1}(r), T_p(r)\right\rangle + \\ &+ 2p\left\langle r^{\alpha+4}T_i(r)T_n(r) + nr^{\alpha+5}T_i(r)U_{n-1}(r), U_{p-1}(r)\right\rangle\end{aligned} \tag{1.45}$$

$$\hat{X}^{21}_{jmq} = -\frac{1}{2m}\left\langle U_{n-1}'(z)T_m(z), T_q(z)\right\rangle \tag{1.46}$$

$$\hat{R}^{12}_{inp} = \frac{1}{2}\left\langle(\alpha+1)^2 r^{\alpha+3}T_i(r)T_n(r) + 2ir^{\alpha+5}U_{i-1}'(r)T_n(r) + 2i(2\alpha+3)r^{\alpha+4}U_{i-1}(r)T_n(r), T_p(r)\right\rangle \tag{1.47}$$

$$\hat{Z}^{12}_{jmq} = \left\langle T_j(z)U_{m-1}(z), U_{q-1}(z)\right\rangle \tag{1.48}$$

$$\begin{aligned}\hat{Q}^{12}_{inp} &= \frac{1}{2}\left\langle(\alpha+1)r^{\alpha+3}T_i(r)T_n(r) + 2ir^{\alpha+4}U_{i-1}(r)T_n(r), T_p(r)\right\rangle \\ &+ n\left\langle(\alpha+1)r^{\alpha+4}T_i(r)U_{n-1}(r) + 2ir^{\alpha+5}U_{i-1}(r)U_{n-1}(r), T_p(r)\right\rangle\end{aligned} \tag{1.49}$$

$$\hat{Y}^{12}_{jmq} = -\frac{1}{2m}\left\langle T_j'(z)T_m(z), U_{q-1}(z)\right\rangle \tag{1.50}$$

$$\hat{S}^{12}_{inp} = n\left\langle 3r^{\alpha+2}T_i(r)U_{n-1}(r) + r^{\alpha+3}T_i(r)U_{n-1}'(r), T_p(r)\right\rangle \tag{1.51}$$

$$\widetilde{Q}_{inp}^{21} = -ik\left[\left\langle(\alpha+1)r^{\alpha+3}T_i(r)T_n(r),T_p(r)\right\rangle + 2p\left\langle r^{\alpha+4}T_i(r)T_n(r),U_{p-1}(r)\right\rangle\right] \quad (I.66)$$

$$\widetilde{Y}_{jmq}^{21} = \frac{1}{2j}\left\langle T_j(z)T_m^{'}(z),T_q(z)\right\rangle \quad (I.67)$$

$$\widetilde{R}_{inp}^{12} = ik\left\langle -r^{\alpha+4}T_i(r)T_n^{'}(r),T_p(r) + (\alpha-1)r^{\alpha+3}T_i(r)T_n(r) + 2nr^{\alpha+4}T_i(r)U_{n-1}(r),T_p(r)\right\rangle \quad (I.68)$$

$$\widetilde{Z}_{jmq}^{12} = \left\langle T_j(z)T_m(z),U_{q-1}(z)\right\rangle \quad (I.69)$$

$$\widetilde{R}_{inp}^{22} = ik\left\langle r^5T_i(r)T_n(r),T_p(r)\right\rangle \quad (I.70)$$

$$\widetilde{Z}_{jmq}^{22} = \left\langle U_{j-1}(z)T_m(z) - \frac{1}{2j}T_j(z)T_m^{'}(z),U_{q-1}(z)\right\rangle \quad (I.71)$$

$$\widetilde{Q}_{inp}^{22} = ik^3\left\langle r^3T_i(r)T_n(r),T_p(r)\right\rangle \quad (I.72)$$

$$\widetilde{Y}_{jmq}^{22} = \frac{1}{2j}\frac{1}{2m}\left\langle T_j(z)T_m(z),U_{q-1}(z)\right\rangle \quad (I.73)$$

In the case $\Omega(r,z) \neq 0$, additional linear terms have to be added to the final Galerkin system. These terms can be expressed for $\tau,\rho = 1,2$, using (38) and (I.35) as

$$\begin{aligned}\Phi_{ijpq}^{\tau\rho} = \sum_{n=0}^{N_r}\sum_{m=0}^{N_z}\Omega_{nm}\Bigg[&\sum_{\xi=0}^{4}\sum_{\eta=0}^{4}c_{i\xi}c_{p\eta}\widetilde{R}_{i+\xi,n,p+\eta}^{\tau\rho}\cdot\sum_{\xi=0}^{4}\sum_{\eta=0}^{4}d_{j\xi}d_{q\eta}\widetilde{Z}_{j+\xi,m,q+\eta}^{\tau\rho} + \\ &+\sum_{\xi=0}^{4}\sum_{\eta=0}^{4}c_{i\xi}c_{p\eta}\widetilde{Q}_{i+\xi,n,p+\eta}^{\tau\rho}\cdot\sum_{\xi=0}^{4}\sum_{\eta=0}^{4}d_{j\xi}d_{q\eta}\widetilde{Y}_{j+\xi,m,q+\eta}^{\tau\rho}\Bigg]\end{aligned} \quad (I.74)$$

The convective terms in the energy equation (16) can be expressed in a similar way. We denote the projection of the convective terms of eq. (16) on the basis defined in (36) as E_{ijnmpq}:

In the case $\Omega(r,z) \neq 0$ additional linear terms have to be added to the final Galerkin system corresponding to the energy equation in a manner similar to (I.74). These terms can be expressed, using (38) and (I.75), as

$$\Phi^{\theta}_{ijpq} = \sum_{n=0}^{N_r} \sum_{m=0}^{N_z} \Omega_{nm} \sum_{\xi=0}^{2} \sum_{\eta=0}^{2} \sigma_{i\xi} \sigma_{p\eta} P^3_{i+\xi,n,p+\eta} \cdot \sum_{\xi=0}^{2} \sum_{\eta=0}^{2} \delta_{j\xi} \delta_{q\eta} R^3_{j+\xi,m,q+\eta} \qquad (I.86)$$

Appendix II

Test calculations

II.1. Validation of linear terms

To validate the Galerkin projections of the linear terms of equations (1)-(5) we consider the problem of the stability of the quiescent state in a vertical cylinder that rotates with an angular velocity Ω and is heated from below (Rayleigh-Benard instability in a rotating cylinder). For constant temperatures on the top and the bottom of the cylinder, the initial no-flow state is a linear distribution of temperature along the axis and a solid-body rotation of the whole system:

$$\Theta = 1 - z, \qquad V = \frac{\overline{R}^2}{\overline{\nu}}\overline{\Omega}\mathbf{e}_\varphi, \qquad U = W = 0 \tag{II.1}$$

where the temperature is scaled as $\Theta = \left(\overline{\Theta}^* - \overline{\Theta}_{cold}\right)\left(\overline{\Theta}_{hot} - \overline{\Theta}_{cold}\right)$. The linearized stability problem for infinitely small perturbations of the velocity, the temperature, and the pressure, in the coordinate system rotating with the angular velocity $\overline{\Omega}$, is defined by:

$$\frac{\partial \mathbf{v}}{\partial t} + Ta\mathbf{e}_\varphi \times \mathbf{v} = -\nabla p + \Delta \mathbf{v} + Ra\theta\mathbf{e}_z \tag{II.2}$$

$$Pr\frac{\partial \theta}{\partial t} + w = \Delta\theta \tag{II.3}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{II.4}$$

where $Ra = \overline{g}\overline{b}\left(\overline{\theta}_{hot} - \overline{\theta}_{cold}\right)\overline{R}^3 / \overline{\nu\chi}$ is the Rayleigh number and $Ta = 2\overline{\Omega}\overline{R}^2 / \overline{\nu}$ is the Taylor number. Since equations (II.2)-(II.4) are linear, the solution of this problem is a good test for the validation of the Galerkin approximations of the linear terms of equations (1)-(5).

We consider two sets of the boundary conditions for the system (II.2)-(II.4). The first one (no-slip and no perturbation of temperature at all boundaries) corresponds to the calculations of Hardin et al. (1990) for $Ta = 0$:

$$u = v = w = 0, \quad \theta = 0 \qquad at \quad z = 0, \gamma \ and \quad at \quad r = 1 \tag{II.5}$$

The second one (stress-free top and bottom with no perturbation of temperature, no-slip sidewall with no perturbation of heat flux) corresponds to the analytical solution of Jones & Moore (1979) (see also Goldstein et al. 1993) for $Ta \neq 0$:

$$\left\langle (\mathbf{v}\cdot\nabla)\mathbf{v},\mathbf{v}\right\rangle = \int_{\Im} (\mathbf{v}\cdot\nabla)\mathbf{v}\cdot\mathbf{v}d\Im = \int_{\Im}\left(\nabla\left[\frac{\mathbf{v}^2}{2}\right] - \mathbf{v}\times[\nabla\times\mathbf{v}]\right)\cdot\mathbf{v}d\Im = \int_{\Im}\nabla\left[\frac{\mathbf{v}^2}{2}\right]\cdot\mathbf{v}d\Im =$$
$$= \int_{\Im}\left(\nabla\cdot\left[\mathbf{v}\frac{\mathbf{v}^2}{2}\right] - \frac{\mathbf{v}^2}{2}\nabla\cdot\mathbf{v}\right)d\Im = \int_{\Gamma}(\mathbf{v}\cdot\mathbf{n})\frac{\mathbf{v}^2}{2}d\Gamma = 0 \qquad \text{(II.7)}$$

which holds for incompressible flow ($\nabla\cdot\mathbf{v} = 0$) with no-throughflow ($\mathbf{v}\cdot\mathbf{n} = 0$) on the boundaries. The corresponding relation for the energy equation is

$$\left\langle (\mathbf{v}\cdot\nabla)\theta,\theta\right\rangle = \int_{\Im}(\mathbf{v}\cdot\nabla)\theta\cdot\theta d\Im = \int_{\Im}\mathbf{v}\cdot\nabla\left(\frac{\theta^2}{2}\right)d\Im = \int_{\Im}\left[\nabla\cdot\left(\mathbf{v}\frac{\theta^2}{2}\right) - \frac{\theta^2}{2}\nabla\cdot\mathbf{v}\right]d\Im = \int_{\Gamma}(\mathbf{v}\cdot\mathbf{n})\frac{\theta^2}{2}d\Gamma = 0 \qquad \text{(II.8)}$$

Similar relations exist for each pair of basis functions of velocity or a pair of basis functions of velocity and temperature. The Galerkin projection of the integrals (II.7) and (II.8) can be extracted from the dynamical system (39) and written as

$$\hat{N}_{ijl}Z_iZ_jZ_l = 0 \qquad \text{(II.9)}$$

for each vector $\mathbf{Z}$ (summation over repeated indices is assumed). Here $\hat{N}_{ijk}$ is a matrix containing projections of all bilinear terms of (1)-(5). A particular case when $Z_i=1$ for all i leads to the relation

$$\sum_{ijk}\hat{N}_{ijl} = 0 \qquad \text{(II.10)}$$

which can be used as a test for calculated values of $\hat{N}_{ijk}$. This relation was used in Gelfgat et al., 1996 for the validation of bilinear terms of the axisymmetric part of the problem.

The relation (II.10) cannot be used directly in the present case when only axisymmetry-breaking bifurcations are the object of the study. There are two reasons for this. First, we do not need the whole matrix $\hat{N}_{ijk}$, but only its part corresponding to the linearized system of equations (20)-(24). The second reason is the complex form of the basis functions (31), (32) and (36) which leads to the following definition of the scalar product:

$$\left\langle \mathbf{u},\mathbf{v}\right\rangle = \int_{\Im}\mathbf{u}\cdot\mathbf{v}\; d\Im = \int_0^{2\pi}\int_0^{\gamma}\int_0^{1}\mathbf{u}\cdot\mathbf{v}^* r\,dr\,dz\,d\varphi \qquad \text{(II.11)}$$

where the overbar means complex conjugation. The presence of a complex conjugate in (II.11) does not allow a direct use of (II.7) and (II.8) for the validation of bilinear terms.

$$\sum_{ijk} Real\left(\mathrm{N}_{ijl}\right) = -\sum_{ijk} Real\left(\tilde{\mathrm{N}}_{ijl}\right) \tag{II.19}$$

A similar relation may be obtained for that part of the matrix N_{ijl} in (II.9), which corresponds to the energy equation. Denoting by Θ the axisymmetric solution, θ the three-dimensional perturbation, and $\hat{\theta} = \theta + \overline{\theta}$, we obtain:

$$\begin{aligned} 0 = \left\langle \left(\left[\mathbf{V} + \hat{\mathbf{v}} \right] \cdot \nabla \right) \left[\Theta + \hat{\theta} \right], \left[\Theta + \hat{\theta} \right] \right\rangle = \left\langle \left(\mathbf{V} \cdot \nabla \right) \Theta, \Theta \right\rangle + \left\langle \left(\hat{\mathbf{v}} \cdot \nabla \right) \hat{\theta}, \hat{\theta} \right\rangle + \\ + \left\langle \left(\mathbf{V} \cdot \nabla \right) \Theta, \hat{\theta} \right\rangle + \left\langle \left(\hat{\mathbf{v}} \cdot \nabla \right) \hat{\theta}, \Theta \right\rangle + \\ + \left\langle \left(\mathbf{V} \cdot \nabla \right) \hat{\theta}, \Theta \right\rangle + \left\langle \left(\mathbf{V} \cdot \nabla \right) \hat{\theta}, \hat{\theta} \right\rangle + \\ + \left\langle \left(\hat{\mathbf{v}} \cdot \nabla \right) \Theta, \Theta \right\rangle + \left\langle \left(\hat{\mathbf{v}} \cdot \nabla \right) \Theta, \hat{\theta} \right\rangle \end{aligned} \tag{II.20}$$

According to (II.8):

$$\left\langle \left(\mathbf{V} \cdot \nabla \right) \Theta, \Theta \right\rangle = \left\langle \left(\hat{\mathbf{v}} \cdot \nabla \right) \hat{\theta}, \hat{\theta} \right\rangle = 0 \tag{II.21}$$

and due to orthogonality, as in (II.14) :

$$\left\langle \left(\mathbf{V} \cdot \nabla \right) \hat{\theta}, \Theta \right\rangle = \left\langle \left(\mathbf{V} \cdot \nabla \right) \Theta, \hat{\theta} \right\rangle = \left\langle \left(\hat{\mathbf{v}} \cdot \nabla \right) \Theta, \Theta \right\rangle = 0 \tag{II.22}$$

Two parts of the matrix N_{ijl} in (II.9), which correspond to the energy equation, are defined as

$$\mathrm{N}_{ijl}^{(1)} = \left\langle \left(\mathbf{V}_j \cdot \nabla \right) \theta_i, \theta_l \right\rangle, \qquad \mathrm{N}_{ijl}^{(2)} = \left\langle \left(\mathbf{v}_i \cdot \nabla \right) \Theta_j, \theta_l \right\rangle \tag{II.23}$$

We can rewrite (II.20) as

$$\begin{aligned} 2\sum_{ijk} Real\left(\mathbf{N}_{ijl}^{(1)} + \mathbf{N}_{ijl}^{(2)} \right) = \left\langle \left(\mathbf{V} \cdot \nabla \right) \hat{\theta}, \hat{\theta} \right\rangle + \left\langle \left(\hat{\mathbf{v}} \cdot \nabla \right) \Theta, \hat{\theta} \right\rangle = \\ = -\left\langle \left(\hat{\mathbf{v}} \cdot \nabla \right) \hat{\theta}, \Theta \right\rangle = -2Real\left\langle \left(\mathbf{v} \cdot \nabla \right) \theta, \Theta \right\rangle \end{aligned} \tag{II.24}$$

If we define a new matrix $\mathbf{N}_{ijl}^{(3)}$ as

$$\mathbf{N}_{ijl}^{(3)} = \left\langle \left(\mathbf{v}_i \cdot \nabla \right) \overline{\theta}_j, \Theta_l \right\rangle \tag{II.25}$$

then the test relation for the real parts of $\mathbf{N}_{ijl}^{(1)}$ and $\mathbf{N}_{ijl}^{(2)}$ is

$$\sum_{ijk} Real\left(\mathbf{N}_{ijl}^{(1)} + \mathbf{N}_{ijk}^{(2)} \right) = -\sum_{ijl} Real\left(\mathbf{N}_{ijl}^{(3)} \right) \tag{II.26}$$

The imaginary part of the matrix N_{ijl} in (II.9) is not zero if the azimuthal component of the basic axisymmetric flow is not zero. To validate the imaginary part of N_{ijl} we consider the following integral relation (V is the azimuthal component of the axisymmetric basic state $\mathbf{V}$):

$$I = ik\int_{\Im} \frac{V}{r}\left[|u|^2 + |w|^2\right] d\Im +$$

$$+\int_{\Im}\left[-V\overline{v}\frac{\partial u}{\partial r} - \frac{1}{r}V\overline{v}u - Vu\frac{\partial \overline{v}}{\partial r} - 2\frac{1}{r}V\overline{u}v + \frac{1}{r}Vu\overline{v} - V\overline{v}\frac{\partial w}{\partial z} - Vw\frac{\partial \overline{v}}{\partial z} - V\overline{u}\frac{\partial v}{\partial r} - \frac{V}{r}\overline{u}v - V\overline{w}\frac{\partial v}{\partial z}\right] d\Im =$$

$$= ik\int_{\Im} \frac{V}{r}\left[|u|^2 + |w|^2\right] d\Im + \int_{\Im} V\left[-\overline{v}\left(\frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right) - u\frac{\partial \overline{v}}{\partial r} - \overline{u}\frac{\partial v}{\partial r} - w\frac{\partial \overline{v}}{\partial z} - \overline{w}\frac{\partial v}{\partial z} + \frac{1}{r}u\overline{v} - 3\frac{1}{r}\overline{u}v\right] d\Im \tag{II.31}$$

Now, using equation (18), $\frac{ik}{r}v = -\left(\frac{u}{r} + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial z}\right)$ and

$$I = ik\int_{\Im} \frac{V}{r}\left[|u|^2 + |v|^2 + |w|^2\right] d\Im + \int_{\Im} V\left[-u\frac{\partial \overline{v}}{\partial r} - \overline{u}\frac{\partial v}{\partial r} - w\frac{\partial \overline{v}}{\partial z} - \overline{w}\frac{\partial v}{\partial z} + \frac{1}{r}u\overline{v} - 3\frac{1}{r}\overline{u}v\right] d\Im \tag{II.32}$$

it follows from (29)-(32),(37),(38) that

$$\langle u\overline{v}, V\rangle = -\langle \overline{u}v, V\rangle, \qquad \left\langle u\frac{\partial \overline{v}}{\partial r}, V\right\rangle = -\left\langle \overline{u}\frac{\partial v}{\partial r}, V\right\rangle, \qquad \left\langle w\frac{\partial \overline{v}}{\partial z}, V\right\rangle = -\left\langle \overline{w}\frac{\partial v}{\partial z}, V\right\rangle \tag{II.33}$$

and

$$I = ik\int_{\Im} \frac{V}{r}\left[|u|^2 + |v|^2 + |w|^2\right] d\Im - 4\int_{\Im}\left[\frac{1}{r}V\overline{u}v\right] d\Im \tag{II.34}$$

Finally, for the imaginary part of N_{ijl}, which is defined as

$$iIm\left(N_{ijl}\right) = \left\langle \left(V_i \mathbf{e}_\varphi \cdot \nabla\right)\mathbf{v}_j, \mathbf{v}_l\right\rangle + \left\langle \left(\mathbf{v}_j \cdot \nabla\right)V_i \mathbf{e}_\varphi, \mathbf{v}_l\right\rangle \tag{II.35}$$

we obtain the following relation:

$$\sum_{ijl} Im\left(N_{ijl}\right) = \sum_{ijl} Im\left(\tilde{N}_{ijl}\right) \tag{II.36}$$

where

$$Im\left(\tilde{N}_{ijl}\right) = Im\left[k\left\langle \frac{1}{r}\left(\mathbf{v}_j \cdot \mathbf{v}_l\right)V_i\right\rangle - 4\left\langle \overline{u}_j v_l, V_i\right\rangle\right] \tag{II.37}$$

Relations similar to (I.29)-(I.73) are obtained for the calculation of $\tilde{N}_{ijl}$ and $N^{(3)}_{ijl}$. For the validation of the bilinear terms we require that the relations (II.19), (II.26) and (II.36) be satisfied analytically for any number of the basis functions $N_r \times N_z$ and for each azimuthal wavenumber k. For example, if $N_r \times N_z = 10$, the relations (II.19), (II.26) and (II.36) request an analytical equality of sums of 10^6 numbers, which is a rather strong requirement. Note that the relations (II.19), (II.26) and (II.36) can be used for validation of the conservative properties of other numerical methods as well.