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MATHEMATICS OF STRUCTURE-FUNCTION EQUATIONS OF ALL ORDERS

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CONTENTS

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ABSTRACT. Exact equations are derived that relate velocity structure functions of arbitrary order with other statistics. "Exact" means that no approximations are used except that the Navier-Stokes equation and incompressibility condition are assumed to be accurate. The exact equations are used to determine the structure-function equations of all orders for locally homogeneous but anisotropic turbulence as well as for the locally isotropic case. These equations can be used for investigating the approach to local homogeneity and to local isotropy as well as the balance of the equations and identification of scaling ranges.

1. INTRODUCTION

Full mathematical exposition on the topic of structure-function equations is given here. A brief summary of results derived here will appear in the Journal of Fluid Mechanics in the paper "Equations relating structure functions of all orders." The two sections below are sufficiently similar to that paper so as to guide the reader to the relevant mathematical details, much of which resides in the appendixes herein. The two sections below contain more mathematical detail than does that paper. Derivation of the results in "Equations relating structure functions of all orders" produces substantial mathematical detail. This is true for reduction of the viscous term and for the term involving the pressure gradient when deriving the exact equations. Applying isotropic formulas for structure functions of arbitrary order requires the invention of new notation and much use of combinatorial analysis. For these purposes, the book by Abramowitz and Stegun (1964) is useful and is cited in the appendixes. The divergence and Laplacian operating on isotropic formulas necessarily appear in the equations; evaluation of which requires the derivation of many identities. Finally, matrix-based algorithms are invented such that the isotropic formulas for the divergence and Laplacian of isotropic tensors of any order can be generated by computer.

There is some difference in notation between the paper and this document. In the paper, a component of a structure function is denoted by $D_{[N_1,N_2,N_3]}$, whereas here it is denoted by the more complicated notation $D_{[N:N_1,N_2,N_3]}$. The reason for the more complicated notation here is to avoid ambiguity at several places in the mathematics. In the paper, the components of the tensor $\left\{\mathbf{W}_{[N-2P]}\left(\mathbf{r}\right)\delta_{[2P]}\right\}$ are denoted by $\left\{\mathbf{W}_{[N-2P]}\left(\mathbf{r}\right)\delta_{[2P]}\right\}_{[N_1,N_2,N_3]}$. Here, there is no symbolic distinction between the tensor $\left\{\mathbf{W}_{[N-2P]}\left(\mathbf{r}\right)\delta_{[2P]}\right\}$ and its components. The distinction is implied by the context.

2. EXACT TWO-POINT EQUATIONS

The Navier-Stokes equation for velocity component $u_i(\mathbf{x},t)$ and the incompressibility condition are

$$\partial_t u_i(\mathbf{x},t) + u_n(\mathbf{x},t)\partial_{x_n} u_i(\mathbf{x},t) = -\partial_{x_i} p(\mathbf{x},t) + \nu \partial_{x_n}\partial_{x_n} u_i(\mathbf{x},t) \text{ , and } \partial_{x_n} u_n(\mathbf{x},t) = 0, \quad (1)$$

where $p(\mathbf{x},t)$ is the pressure divided by the density (density is constant), ν is kinematic viscosity, and ∂ denotes partial differentiation with respect to its subscript variable. Summation is implied by repeated Roman indexes. Consider another point $\mathbf{x}'$ such that $\mathbf{x}'$ and $\mathbf{x}$ are independent variables. For brevity, let $u_i = u_i(\mathbf{x},t)$, $u_i' = u_i(\mathbf{x}',t)$, etc. Require that $\mathbf{x}$ and $\mathbf{x}'$ have no relative motion. Then $\partial_{x_i} u_j' = 0$, $\partial_{x_i'} u_j = 0$, etc., and ∂_t is performed with both held $\mathbf{x}$ and $\mathbf{x}'$ fixed. Subtracting (1) at $\mathbf{x}'$ from (1) at $\mathbf{x}$ and using the aforementioned properties gives

$$\partial_t v_i + u_n \partial_{x_n} v_i + u_n' \partial_{x_n'} v_i = -P_i + \nu\left(\partial_{x_n}\partial_{x_n} v_i + \partial_{x_n'}\partial_{x_n'} v_i\right), \quad (2)$$

$$\text{where } v_i \equiv u_i - u_i' \text{ , } P_i \equiv \left(\partial_{x_i} p - \partial_{x_i'} p'\right). \quad (3)$$

Change independent variables from $\mathbf{x}$ and $\mathbf{x}'$ to the sum and difference independent variables:

$$\mathbf{X} \equiv (\mathbf{x}+\mathbf{x}')/2 \quad \text{and} \quad \mathbf{r} \equiv \mathbf{x}-\mathbf{x}', \quad \text{and define } r \equiv |\mathbf{r}|. \tag{4}$$

The relationship between the partial derivatives is

$$\partial_{x_i} = \partial_{r_i} + \frac{1}{2}\partial_{X_i} \ , \ \partial_{x'_i} = -\partial_{r_i} + \frac{1}{2}\partial_{X_i} \ \ , \partial_{X_i} = \partial_{x_i} + \partial_{x'_i} \ \ , \partial_{r_i} = \frac{1}{2}\left(\partial_{x_i} - \partial_{x'_i}\right). \tag{5}$$

The change of variables organizes the equations in a revealing way because of the following properties. In the case of homogeneous turbulence, ∂_{X_i} operating on a statistic produces zero because that derivative is the rate of change with respect to the place where the measurement is performed. Consider a term in an equation composed of ∂_{X_i} operating on a statistic. For locally homogeneous turbulence, that term becomes negligible as r is decreased relative to the integral scale. For the homogeneous and locally homogeneous cases, the statistical equations retain their dependence on $\mathbf{r}$, which is the displacement vector of two points of measurement. Using (5), (2) becomes

$$\partial_t v_i + U_n \partial_{X_n} v_i + v_n \partial_{r_n} v_i = -P_i + \nu\left(\partial_{x_n}\partial_{x_n} v_i + \partial_{x'_n}\partial_{x'_n} v_i\right), \tag{6}$$

$$\text{where } U_n \equiv (u_i + u'_i)/2. \tag{7}$$

Now multiply (6) by the product $v_j v_k \cdots v_l$, which contains $N-1$ factors of velocity difference, each factor having a distinct index. Sum the N such equations as required to produce symmetry under interchange of each pair of indexes, excluding the summation index n. French braces, i.e., $\{\circ\}$, denote the sum of all terms of a given type that produce symmetry under interchange of each pair of indexes. The differentiation chain rule gives

$$\{v_j v_k \cdots v_l \partial_t v_i\} = \partial_t (v_j v_k \cdots v_l v_i), \tag{8}$$

$$\{v_j v_k \cdots v_l U_n \partial_{X_n} v_i\} = U_n \partial_{X_n} (v_j v_k \cdots v_l v_i) = \partial_{X_n} (U_n v_j v_k \cdots v_l v_i), \tag{9}$$

$$\{v_j v_k \cdots v_l v_n \partial_{r_n} v_i\} = v_n \partial_{r_n} (v_j v_k \cdots v_l v_i) = \partial_{r_n} (v_n v_j v_k \cdots v_l v_i). \tag{10}$$

The right-most expressions in (9) and (10) follow from the incompressibility property obtained from (5) and the fact that $\partial_{x_i} u'_j = 0$, $\partial_{x'_i} u_j = 0$, namely

$$\partial_{X_n} U_n = 0, \partial_{X_n} v_n = 0, \partial_{r_n} U_n = 0, \partial_{r_n} v_n = 0. \tag{11}$$

The viscous term in (6) produces $\nu \left\{ v_j v_k \cdots v_l \left(\partial_{x_n} \partial_{x_n} v_i + \partial_{x'_n} \partial_{x'_n} v_i \right) \right\}$; this expression is treated in Appendix A. These results give

$$\partial_t (v_j \cdots v_i) + \partial_{X_n} (U_n v_j \cdots v_i) + \partial_{r_n} (v_n v_j \cdots v_i) =$$
$$- \{v_j \cdots v_l P_i\} + 2\nu \left[\left(\partial_{r_n} \partial_{r_n} + \frac{1}{4} \partial_{X_n} \partial_{X_n} \right) (v_j \cdots v_i) - \{v_k \cdots v_l e_{ij}\} \right], \tag{12}$$

$$\text{where } e_{ij} \equiv (\partial_{x_n} u_i)(\partial_{x_n} u_j) + \left(\partial_{x'_n} u'_i\right)\left(\partial_{x'_n} u'_j\right) = (\partial_{x_n} v_i)(\partial_{x_n} v_j) + \left(\partial_{x'_n} v_i\right)\left(\partial_{x'_n} v_j\right). \tag{13}$$

The quantity $\{v_j \cdots v_l P_i\}$ can be expressed differently on the basis that (5) allows P_i to be written as $P_i = \partial_{X_i} (p - p')$. The derivation is in Appendix B; the alternative formula is

$$\{v_j v_k \cdots v_l P_i\} = \{\partial_{X_i} [v_j v_k \cdots v_l (p - p')]\} - (N-1)(p - p') \left\{ \left(s_{ij} - s'_{ij} \right) v_k \cdots v_l \right\}, \tag{14}$$

where the rate of strain tensor s_{ij} is defined by

$$s_{ij} \equiv \left(\partial_{x_i} u_j + \partial_{x_j} u_i \right) / 2. \tag{15}$$

3. AVERAGED EQUATIONS

Consider the ensemble average because it commutes with temporal and spatial derivatives. The above notation of explicit indexes is burdensome. Because the tensors are symmetric, it suffices to show only the number of indexes. Define the following statistical tensors, which are symmetric under interchange of any pair of indexes, excluding the summation index n in the definition of $\mathbf{F}_{[N+1]}$:

$$\mathbf{D}_{[N]} \equiv \langle v_j \cdots v_i \rangle, \mathbf{F}_{[N+1]} \equiv \langle U_n v_j \cdots v_i \rangle, \mathbf{T}_{[N]} \equiv \langle \{v_j \cdots v_l P_i\} \rangle, \mathbf{E}_{[N]} \equiv \langle \{v_k \cdots v_l e_{ij}\} \rangle, \tag{16}$$

where angle brackets $\langle\rangle$ denote the ensemble average, and the subscripts N and $N+1$ within square brackets denote the number of indexes. The argument list $(\mathbf{X}, \mathbf{r}, t)$ is understood for each tensor. The left-hand sides of each definition in (16) are in implicit-index notation for which only the number of indexes is given; the right-hand sides in (16) are in explicit-index notation. The ensemble average of (12) is

$$\partial_t \mathbf{D}_{[N]} + \nabla_{\mathbf{X}} \cdot \mathbf{F}_{[N+1]} + \nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]} = -\mathbf{T}_{[N]} + 2\nu \left[\left(\nabla_{\mathbf{r}}^2 + \frac{1}{4} \nabla_{\mathbf{X}}^2 \right) \mathbf{D}_{[N]} - \mathbf{E}_{[N]} \right], \qquad (17)$$

where, $\nabla_{\mathbf{X}} \cdot \mathbf{F}_{[N+1]} \equiv \partial_{X_n} \langle U_n v_j \cdots v_i \rangle$, $\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]} \equiv \partial_{r_n} \langle v_n v_j \cdots v_i \rangle$, $\nabla_{\mathbf{r}}^2 \equiv \partial_{r_n} \partial_{r_n}$, $\nabla_{\mathbf{X}}^2 \equiv \partial_{X_n} \partial_{X_n}$. The notations $\nabla_{\mathbf{X}} \cdot$, $\nabla_{\mathbf{X}}^2$, $\nabla_{\mathbf{r}} \cdot$, and $\nabla_{\mathbf{r}}^2$ are the divergence and Laplacian operators in $\mathbf{X}$-space and $\mathbf{r}$-space, respectively.

4. HOMOGENEOUS AND LOCALLY HOMOGENEOUS TURBULENCE

Consider homogeneous turbulence and locally homogeneous turbulence; the latter applies for small r and large Reynolds number. The variation of the statistics with the location of measurement or of evaluation is neglected for these cases. That location being $\mathbf{X}$, the result of $\nabla_{\mathbf{X}} \cdot$ operating on a statistic is neglected. Thus the terms $\nabla_{\mathbf{X}} \cdot \mathbf{F}_{[N+1]}$ and $\frac{1}{4} \nabla_{\mathbf{X}}^2 \mathbf{D}_{[N]}$ are neglected in (17); then (17) becomes

$$\partial_t \mathbf{D}_{[N]} + \nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]} = -\mathbf{T}_{[N]} + 2\nu \left[\nabla_{\mathbf{r}}^2 \mathbf{D}_{[N]} - \mathbf{E}_{[N]} \right]. \qquad (18)$$

Because the $\mathbf{X}$-dependence is neglected, the argument list $(\mathbf{r}, t)$ is understood for each tensor. The ensemble average of (14) contains $\langle \partial_{X_i} [\{ v_j v_k \cdots v_l (p - p') \}] \rangle$, which can be written as the sum of $N-1$ statistics of the form $\langle \{ v_j v_k \cdots v_l (p - p') \} \rangle$ operated upon by the $\mathbf{X}$-space gradient. Since such $\mathbf{X}$-space derivative terms are neglected, (14) gives the alternative that

$$\mathbf{T}_{[N]} = -(N-1) \left\langle (p - p') \left\{ \left(s_{ij} - s'_{ij} \right) v_k \cdots v_l \right\} \right\rangle. \qquad (19)$$

Locally homogeneous turbulence is also locally stationary such that the term $\partial_t \mathbf{D}_{[N]}$ in (18) may be neglected. However, $\partial_t \mathbf{D}_{[N]}$ is not necessarily negligible for homogeneous turbulence.

5. ISOTROPIC AND LOCALLY ISOTROPIC TURBULENCE

Consider isotropic turbulence and locally isotropic turbulence; the latter applies for small r and large Reynolds number. The tensors $\mathbf{D}_{[N]}$, $\mathbf{T}_{[N]}$, and $\mathbf{E}_{[N]}$ in (16) obey the isotropic formula. The Kronecker delta $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Let $\delta_{[2P]}$ denote the product of P Kronecker deltas having $2P$ distinct indexes, and let $\mathbf{W}_{[N]}(\mathbf{r})$ denote the product of N factors $\frac{r_i}{r}$ each with a distinct index; the argument $\mathbf{r}$ is omitted when clarity does not suffer. Because each tensor in (16) is symmetric under interchange of any two indexes, their isotropic formulas are particularly simple. Each formula is a the sum of $M+1$ terms where

$$M = N/2 \text{ if } N \text{ is even, and } M = (N-1)/2 \text{ if } N \text{ is odd.} \tag{20}$$

Each term is the product of a distinct scalar function with a $\mathbf{W}_{[N]}$ and a $\delta_{[2P]}$. From one term to the next, a pair of indexes is transferred from a $\mathbf{W}_{[N]}$ to a $\delta_{[2P]}$; examples are given in (64) through (66) of Appendix E. For the tensor $\mathbf{D}_{[N]}$, denote the Pth scalar function by $D_{N,P}(r,t)$. Thus the scalar functions belonging to the isotropic formulas for $\mathbf{T}_{[N]}$, $\mathbf{E}_{[N]}$, and $\mathbf{D}_{[N+1]}$ are denoted by $T_{N,P}(r,t)$, $E_{N,P}(r,t)$, and $D_{N+1,P}(r,t)$, respectively. The scalar functions depend on the magnitude of the spacing r rather than on the vector spacing $\mathbf{r}$. The isotropic formula for $\mathbf{D}_{[N]}$ is

$$\mathbf{D}_{[N]}(\mathbf{r},t) = \sum_{P=0}^{M} D_{N,P}(r,t) \left\{ \mathbf{W}_{[N-2P]}(\mathbf{r}) \, \delta_{[2P]} \right\}, \tag{21}$$

and the isotropic formulas for $\mathbf{T}_{[N]}$ and $\mathbf{E}_{[N]}$ have the analogous notation. Recall that $\{\circ\}$ denotes the sum of all terms of a given type that produce symmetry under interchange of each pair of indexes. Henceforth, the argument list (r,t) will be deleted.

A special Cartesian coordinate system is typically used because it simplifies the isotropic formula. This coordinate system has the positive 1-axis parallel to the direction of $\mathbf{r}$, and the 2- and 3-axes are therefore perpendicular to $\mathbf{r}$. Let N_1, N_2, and N_3 be the number of indexes of a component of $\mathbf{D}_{[N]}$ that are 1, 2, and 3, respectively; such that

$N = N_1 + N_2 + N_3$. Because of symmetry, the order of indexes is immaterial such that a component of $\mathbf{D}_{[N]}$ can be identified by N_1, N_2, and N_3. Thus, denote a component of $\mathbf{D}_{[N]}$ by $D_{[N:N_1,N_2,N_3]}$, which is a function of $\mathbf{r}$ and t. The projection of (21) using N_1, N_2, and N_3 unit vectors in the directions of the 1-, 2-, and 3-axes, respectively, results in the component $D_{[N:N_1,N_2,N_3]}$ on the left-hand side of (21), and numerical values of the projection of $\left\{\mathbf{W}_{[N-2P]}\left(\mathbf{r}\right)\delta_{[2P]}\right\}$ appear on the right-hand side. Henceforth the word "projection" will be omitted for brevity. Those values of the coefficients $\left\{\mathbf{W}_{[N-2P]}\left(\mathbf{r}\right)\delta_{[2P]}\right\}$ in (21) are needed; the values obtained for the special coordinate system are determined in Appendix C; they are, from (43)-(44),

$$\text{if } 2P < N_2 + N_3 \text{ then } \left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} = 0, \text{ otherwise,} \tag{22}$$

$$\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} = N_1!N_2!N_3!/\left[(N-2P)!2^P\left(\frac{N_2}{2}\right)!\left(\frac{N_3}{2}\right)!\left(P-\frac{N_2}{2}-\frac{N_3}{2}\right)!\right]. \tag{23}$$

By applying (21) to (23) for all combinations of indexes, one can determine which components $D_{[N:N_1,N_2,N_3]}$ are zero and which are nonzero, identify $M+1$ linearly independent equations that determine the $D_{N,P}$ in terms of $M+1$ of the $D_{[N:N_1,N_2,N_3]}$, and find algebraic relationships between the remaining nonzero $D_{[N:N_1,N_2,N_3]}$. The derivations are in Appendix D; a summary follows.

A component $D_{[N:N_1,N_2,N_3]}$ is nonzero only if both N_2 and N_3 are even and when N_1 is odd if N is odd and when N_1 is even if N is even. Thereby, $(M+1)(M+2)/2$ components are nonzero. There are 3^N components of $\mathbf{D}_{[N]}$; thus the other $3^N - (M+1)(M+2)/2$ components are zero.

There exists exactly $(M+1)M/2$ kinematic relationships among the nonzero components of $\mathbf{D}_{[N]}$. For each of the $M+1$ cases of N_1, these relationships are expressed by the proportionality

$$D_{[N:N_1,2L,0]} : D_{[N:N_1,2L-2,2]} : D_{[N:N_1,2L-4,4]} : \cdots : D_{[N:N_1,0,2L]} =$$
$$[(2L)!0!/L!0!] : [(2L-2)!2!/(L-1)!1!] : [(2L-4)!4!/(L-2)!2!] : \cdots : [0!(2L)!/0!L!]. \tag{24}$$

Previously, only one such kinematic relationship was known(Millionshtchikov 1941). For

$N=4$, (24) gives $D_{[4:0,4,0]} : D_{[4:0,2,2]} : D_{[4:0,0,4]} = 12 : 4 : 12$. In explicit-index notation this can be written as $D_{2222} = 3D_{2233} = D_{3333}$, which was discovered by Millionshtchikov (1941). Now, all such relationships are known.

There remain $M+1$ linearly independent nonzero components of $\mathbf{D}_{[N]}$. This must be so because there are $M+1$ terms in (21), and the $M+1$ scalar functions $D_{N,P}$ therein must be related to $M+1$ components. Consider the $M+1$ linearly independent equations that determine the $D_{N,P}$ in terms of $M+1$ of the $D_{[N:N_1,N_2,N_3]}$. For simplicity, the chosen components can all have $N_3 = 0$, i.e., the choice of linearly independent components can be $D_{[N:N,0,0]}$, $D_{[N:N-2,2,0]}$, $D_{[N:N-4,4,0]}$, $\cdots$, $D_{[N:N-2M,2M,0]}$. As described above, projections of (21) result in the chosen components on the left-hand side and algebraic equations on the right-hand side. These equations can be expressed in matrix form and solved by matrix inversion methods; the result is given in (87) of Appendix F. Given experimental or direct numerical simulation (DNS) data or a theoretical formula for the chosen components, the solution in (87) determines the functions $D_{N,P}$ in (21); then (21) completely specifies the tensor $\mathbf{D}_{[N]}$. The matrix algorithm is an efficient means of determining isotropic expressions for the terms $\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}$ and $\nabla_{\mathbf{r}}^2 \mathbf{D}_{[N]}$ in (18). Those algorithms are given in Appendix F. From the example for $N=2$ in Appendix F, use of the matrix algorithm and the isotropic formulas in (18) gives the two scalar equations

$$\partial_t D_{11} + \left(\partial_r + \frac{2}{r}\right) D_{111} - \frac{4}{r} D_{122} = -T_{11} + 2\nu \left[\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{4}{r^2}\right) D_{11} + \frac{4}{r^2} D_{22} - E_{11}\right]$$
$$= 2\nu \left[\partial_r^2 D_{11} + \frac{2}{r}\partial_r D_{11} + \frac{4}{r^2}\left(D_{22} - D_{11}\right)\right] - 4\varepsilon/3, \quad (25)$$
$$\partial_t D_{22} + \left(\partial_r + \frac{4}{r}\right) D_{122} = -T_{22} + 2\nu \left[\frac{2}{r^2} D_{11} + \left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{2}{r^2}\right) D_{22} - E_{22}\right]$$
$$= 2\nu \left[\partial_r^2 D_{22} + \frac{2}{r}\partial_r D_{22} - \frac{2}{r^2}\left(D_{22} - D_{11}\right)\right] - 4\varepsilon/3, \quad (26)$$

where use was made of the fact (Hill, 1997) that local isotropy gives $T_{11} = T_{22} = 0$ and $2\nu E_{11} = 2\nu E_{22} = 4\varepsilon/3$ where ε is the average energy dissipation rate per unit mass of fluid. Now, (25)-(26) are the same as equations (43)-(44) of Hill (1997), and Hill (1997) shows how these equations lead to Kolmogorov's equation and his 4/5 law. From the example for

$N = 3$ in Appendix F,

$$\partial_t D_{111} + \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} = -T_{111} + 2\nu \left[C - E_{111}\right], \tag{27}$$

$$\partial_t D_{122} + \left(\partial_r + \frac{4}{r}\right) D_{1122} - \frac{4}{3r} D_{2222} = -T_{122} + 2\nu \left[B - E_{122}\right], \tag{28}$$

where

$$C \equiv \left(-\frac{4}{r^2} + \frac{4}{r}\partial_r + \partial_r^2\right) D_{111}, \text{ and } B \equiv \frac{1}{6}\left(\frac{4}{r^2} - \frac{4}{r}\partial_r + 5\partial_r^2 + r\partial_r^3\right) D_{111}. \tag{29}$$

The incompressibility condition, $D_{122} = \frac{1}{6}\left(D_{111} + r\partial_r D_{111}\right)$, was substituted in (104) to obtain (29). The matrix algorithm is checked by the fact that (27)-(29) are the same as given by Hill and Boratav (2001).

The equations for $N = 4$ are

$$\partial_t D_{1111} + \left(\partial_r + \frac{2}{r}\right) D_{11\,111} - \frac{8}{r} D_{11\,122} =$$

$$= -T_{1111} + 2\nu \left[\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{8}{r^2}\right) D_{1111} + \frac{14}{r^2} D_{1122} + \frac{10}{3r^2} D_{2222}\right] - 2\nu E_{1111}, \tag{30}$$

$$\partial_t D_{1122} + \left(\partial_r + \frac{4}{r}\right) D_{11\,122} - \frac{8}{3r} D_{12\,222} =$$

$$= -T_{1122} + 2\nu \left[\frac{2}{r^2} D_{1111} + \left(-\frac{52}{3r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{1122} + \frac{34}{9r^2} D_{2222}\right] - 2\nu E_{1122}, \tag{31}$$

$$\partial_t D_{2222} + \left(\partial_r + \frac{6}{r}\right) D_{12\,222} = -T_{2222} + 2\nu \left[\frac{2}{r^2} D_{1122} + \left(-\frac{2}{3r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{2222}\right] - 2\nu E_{2222}. \tag{32}$$

Since these equations have a repetitive structure, it suffices to give the divergence and Laplacian terms. For $N = 5$ to 7 these are, respectively:

$$\begin{pmatrix} \left(\partial_r + \frac{2}{r}\right) D_{111\,111} - \frac{10}{r} D_{111\,122} \\ \left(\partial_r + \frac{4}{r}\right) D_{111\,122} - \frac{4}{r} D_{112\,222} \\ \left(\partial_r + \frac{6}{r}\right) D_{112\,222} - \frac{6}{5r} D_{222\,222} \end{pmatrix}, \; 2\nu \begin{pmatrix} \left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{10}{r^2}\right) D_{11\,111} - \frac{14}{r^2} D_{11\,122} + \frac{54}{r^2} D_{12\,222} \\ \frac{2}{r^2} D_{11\,111} + \left(-\frac{154}{5r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{11\,122} + \frac{94}{5r^2} D_{12\,222} \\ \frac{6}{5r^2} D_{11\,122} + \left(-\frac{16}{5r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{12\,222} \end{pmatrix}$$

$$
\begin{pmatrix}
\left(\partial_r + \frac{2}{r}\right) D_{1111\,111} - \frac{12}{r} D_{1111\,122} \\
\left(\partial_r + \frac{4}{r}\right) D_{1111\,122} - \frac{16}{3r} D_{1112\,222} \\
\left(\partial_r + \frac{6}{r}\right) D_{1112\,222} - \frac{12}{5r} D_{1222\,222} \\
\left(\partial_r + \frac{8}{r}\right) D_{1222\,222}
\end{pmatrix},
$$

$$
2\nu \begin{pmatrix}
\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{12}{r^2}\right) D_{111\,111} - \frac{108}{r^2} D_{111\,122} + \frac{920}{3r^2} D_{112\,222} - \frac{416}{15r^2} D_{222\,222} \\
\frac{2}{r^2} D_{111\,111} + \left(-\frac{242}{5r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{111\,122} + \frac{824}{15r^2} D_{112\,222} - \frac{248}{75r^2} D_{222\,222} \\
\frac{4}{5r^2} D_{111\,122} + \left(-\frac{112}{15r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{112\,222} + \frac{4}{3r^2} D_{222\,222} \\
\frac{2}{3r^2} D_{112\,222} + \left(-\frac{2}{15r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{222\,222}
\end{pmatrix};
$$

$$
\begin{pmatrix}
\left(\partial_r + \frac{2}{r}\right) D_{11\,111\,111} - \frac{14}{r} D_{11\,111\,122} \\
\left(\partial_r + \frac{4}{r}\right) D_{11\,111\,122} - \frac{20}{3r} D_{11\,112\,222} \\
\left(\partial_r + \frac{6}{r}\right) D_{11\,112\,222} - \frac{18}{5r} D_{11\,222\,222} \\
\left(\partial_r + \frac{8}{r}\right) D_{11\,222\,222} - \frac{8}{7r} D_{22\,222\,222}
\end{pmatrix},
$$

$$
2\nu \begin{pmatrix}
\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{14}{r^2}\right) D_{1111\,111} - \frac{316}{r^2} D_{1111\,122} + \frac{3376}{3r^2} D_{1112\,222} - \frac{1376}{5r^2} D_{1222\,222} \\
\frac{2}{r^2} D_{1111\,111} + \left(-\frac{1472}{21r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{1111\,122} + \frac{7808}{63r^2} D_{1112\,222} - \frac{304}{15r^2} D_{1222\,222} \\
\frac{4}{7r^2} D_{1111\,122} + \left(-\frac{206}{15r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{1112\,222} + \frac{1132}{175r^2} D_{1222\,222} \\
\frac{2}{7r^2} D_{1112\,222} + \left(-\frac{76}{35r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{1222\,222}
\end{pmatrix}.
$$

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Appendix A: The Viscous Term

The quantity $\{v_j v_k \cdots v_l \partial_{x_n} \partial_{x_n} v_i\}$ requires special attention. Consider the repeated application of the identity

$$\partial_{x_n} \partial_{x_n} (fg) = f \partial_{x_n} \partial_{x_n} g + g \partial_{x_n} \partial_{x_n} f + 2 (\partial_{x_n} f) (\partial_{x_n} g) \tag{33}$$

to the quantity

$$\partial_{x_n} \partial_{x_n} (v_j v_k v_m \cdots v_i) \tag{34}$$

for N factors of velocity difference in $(v_j v_k v_m \cdots v_i)$. For the first application of (33) let $f = v_j$ and let g be the remaining factors $(v_k v_m \cdots v_i)$; this gives

$$\begin{aligned} \partial_{x_n} \partial_{x_n} (v_j v_k v_m \cdots v_i) &= v_j \partial_{x_n} \partial_{x_n} (v_k v_m \cdot \cdot v_i) \\ &\quad + (v_k v_m \cdot \cdot v_i) \partial_{x_n} \partial_{x_n} v_j + 2 [\partial_{x_n} v_j] [\partial_{x_n} (v_k v_m \cdot \cdot v_i)] . \end{aligned} \tag{35}$$

From the differentiation chain rule, $\partial_{x_n} (v_k v_m \cdot \cdot v_i)$ is the sum of $N-1$ terms of the form $(v_m \cdot \cdot v_p \partial_{x_n} v_i)$. Thus, the right-most term in (35) is $N-1$ terms of the form $2 v_m \cdots v_p (\partial_{x_n} v_i) (\partial_{x_n} v_j)$ each term containing N factors; two of those factors are distinguished by being derivatives of velocity differences. The second application of (33) is performed on $v_j \partial_{x_n} \partial_{x_n} (v_k v_m \cdot \cdot v_i)$ in (35), for which purpose $f = v_k$ and $g = (v_m \cdot \cdot v_i)$; this gives

$$v_j \partial_{x_n} \partial_{x_n} (v_k v_m \cdots v_i) = v_j v_k \partial_{x_n} \partial_{x_n} (v_m \cdot \cdot v_i) + v_j (v_m \cdot \cdot v_i) \partial_{x_n} \partial_{x_n} v_k + 2 v_j [\partial_{x_n} v_k] [\partial_{x_n} (v_m \cdot \cdot v_i)] .$$

The right-most term gives $N-2$ terms of the form $2v_j v_m \cdots v_p (\partial_{x_n} v_i)(\partial_{x_n} v_k)$ each term containing N factors.

There are $N-1$ steps to complete reduction of the formula. The number of terms of the form $2v_j v_m \cdots v_p (\partial_{x_n} v_i)(\partial_{x_n} v_k)$ is $(N-1)$ from the first step, $(N-2)$ from the second step, etc., such that the total number of terms is $(N-1)+(N-2)+\cdots+(N-(N-1)) = N(N-1)/2$. Now, $N(N-1)/2 = \binom{N}{2}$ is the binomial coefficient equal to the number of ways of choosing two indexes from a set of N indexes; the quantities $(\partial_{x_n} v_i)$ and $(\partial_{x_n} v_j)$ in $2v_m \cdots v_p (\partial_{x_n} v_i)(\partial_{x_n} v_j)$ contain the chosen two indexes i and j. The $\binom{N}{2}$ terms constitute $2\{v_m \cdots v_p (\partial_{x_n} v_i)(\partial_{x_n} v_j)\}$. Because two factors of the form $(v_j v_m \cdot \cdot v_i)\,\partial_{x_n}\partial_{x_n} v_k$ appear in the last step, the total number of terms of the form $(v_j \cdots v_n)\,\partial_{x_n}\partial_{x_n} v_i$ is N. Not surprisingly, these N terms constitute $\{(v_j \cdots v_n)\,\partial_{x_n}\partial_{x_n} v_i\}$, and $N = \binom{N}{1}$ is the binomial coefficient equal to the number of ways of choosing one index from a set of N indexes, the quantity $\partial_{x_n}\partial_{x_n} v_i$ contains the chosen one index i.

That is, for any N

$$\partial_{x_n}\partial_{x_n}(v_j \cdots v_i) = \{(v_j \cdots v_l)\,\partial_{x_n}\partial_{x_n} v_i\} + 2\{v_k \cdots v_l (\partial_{x_n} v_i)(\partial_{x_n} v_j)\}. \tag{36}$$

The left-hand side is symmetric under interchange of any pair of indexes (not including n because summation is implied over n), and the French brackets make the right-hand side likewise symmetric.

Use of (36) within the viscous term $\nu\left\{v_j v_k \cdots v_l \left(\partial_{x_n}\partial_{x_n} v_i + \partial_{x'_n}\partial_{x'_n} v_i\right)\right\}$ that arrises from (6), gives

$$\left\{v_j v_k \cdots v_l \left(\partial_{x_n}\partial_{x_n} v_i + \partial_{x'_n}\partial_{x'_n} v_i\right)\right\} =$$

$$\left(\partial_{x_n}\partial_{x_n} + \partial_{x'_n}\partial_{x'_n}\right)(v_j \cdots v_i) - 2\left\{v_k \cdots v_l \left[(\partial_{x_n} v_i)(\partial_{x_n} v_j) + \left(\partial_{x'_n} v_i\right)\left(\partial_{x'_n} v_j\right)\right]\right\}, \tag{37}$$

where the right-most term in (36) has been subtracted from both sides of (36) to obtain (37). Note that $(\partial_{x_n} u_i)(\partial_{x_n} u_j) = (\partial_{x_n} v_i)(\partial_{x_n} v_j)$ and $\left(\partial_{x'_n} u'_i\right)\left(\partial_{x'_n} u'_j\right) = \left(\partial_{x'_n} v_i\right)\left(\partial_{x'_n} v_j\right)$, and that use of (5) gives

$$\left(\partial_{x_n}\partial_{x_n}+\partial_{x_n'}\partial_{x_n'}\right)=2\left(\partial_{r_n}\partial_{r_n}+\frac{1}{4}\partial_{X_n}\partial_{X_n}\right).$$

Then, (37) can be written as

$$\begin{aligned}\left\{v_jv_k\cdots v_l\left(\partial_{x_n}\partial_{x_n}v_i+\partial_{x_n'}\partial_{x_n'}v_i\right)\right\}&=2\left(\partial_{r_n}\partial_{r_n}+\frac{1}{4}\partial_{X_n}\partial_{X_n}\right)(v_j\cdots v_i)\\&\quad-2\left\{v_k\cdots v_l\left[\left(\partial_{x_n}u_i\right)\left(\partial_{x_n}u_j\right)+\left(\partial_{x_n'}u_i'\right)\left(\partial_{x_n'}u_j'\right)\right]\right\}.\end{aligned}$$

Appendix B: Derivation of (14)

The purpose of this appendix is to derive (14). Since (5) allows P_i to be written as $P_i=\partial_{X_i}(p-p')$, the differentiation chain rule gives

$$\begin{aligned}v_jv_k\cdots v_lP_i&=v_jv_k\cdots v_l\partial_{X_i}(p-p')\\&=\partial_{X_i}\left[v_jv_k\cdots v_l(p-p')\right]-(p-p')\left\{\left(\partial_{X_i}v_j\right)v_k\cdots v_l\right\}_{/i/},\end{aligned}\tag{38}$$

where the notation $\{\circ\}_{/i/}$ denotes the sum of all terms of a given type that produce symmetry under interchange of each pair of indexes with the index i excluded. Recall that the product $v_jv_k\cdots v_l$ consists of $N-1$ factors. Sum the N equations of type (38) such that the sum is even under interchange of all pairs of indexes; then

$$\left\{v_jv_k\cdots v_lP_i\right\}=\left\{\partial_{X_i}\left[v_jv_k\cdots v_l(p-p')\right]\right\}-(N-1)(p-p')\left\{\left(\partial_{X_i}v_j\right)v_k\cdots v_l\right\},\tag{39}$$

where use was made of the fact that the $N-1$ terms in the sum $\left\{\left(\partial_{X_i}v_j\right)v_k\cdots v_l\right\}_{/i/}$ each give the same result, namely, $\left\{\left(\partial_{X_i}v_j\right)v_k\cdots v_l\right\}$. From (5), $\partial_{X_i}v_j=\partial_{x_i}u_j-\partial_{x_j'}u_i'$; such that the definition of strain rate (15) gives

$$\left(\partial_{X_i}v_j+\partial_{X_j}v_i\right)/2=s_{ij}-s_{ij}'.\tag{40}$$

Use of (40) gives $\left\{\left(\partial_{X_i}v_j\right)v_k\cdots v_l\right\}=\left(\left\{\left(\partial_{X_i}v_j\right)v_k\cdots v_l\right\}+\left\{\left(\partial_{X_j}v_i\right)v_k\cdots v_l\right\}\right)/2=\left\{\left(s_{ij}-s_{ij}'\right)v_k\cdots v_l\right\}$, substitution of which into (39) gives

$$\left\{v_jv_k\cdots v_lP_i\right\}=\left\{\partial_{X_i}\left[v_jv_k\cdots v_l(p-p')\right]\right\}-(N-1)(p-p')\left\{\left(s_{ij}-s_{ij}'\right)v_k\cdots v_l\right\}.\tag{41}$$

Alternatively, consider that (5) allows P_i to be written as $P_i = 2\partial_{r_i}(p+p')$. Analogous to (39), it follows that

$$\{v_j v_k \cdots v_l P_i\} = 2\left\{\partial_{r_i}\left[v_j v_k \cdots v_l (p+p')\right]\right\} - (N-1)(p+p')\left\{(2\partial_{r_i} v_j) v_k \cdots v_l\right\}.$$

The $2\partial_{r_i} v_j$ can be replaced by, $2\partial_{r_i} v_j = \partial_{x_i} u_j + \partial_{x'_j} u'_i$, such that $\{(2\partial_{r_i} v_j) v_k \cdots v_l\} = \left(\{(2\partial_{r_i} v_j) v_k \cdots v_l\} + \left\{\left(2\partial_{r_j} v_i\right) v_k \cdots v_l\right\}\right)/2 = \left\{\left(s_{ij} + s'_{ij}\right) v_k \cdots v_l\right\}$. Then, analogous to (41), it follows that

$$\{v_j v_k \cdots v_l P_i\} = 2\left\{\partial_{r_i}\left[v_j v_k \cdots v_l (p+p')\right]\right\} - (N-1)(p+p')\left\{\left(s_{ij} + s'_{ij}\right) v_k \cdots v_l\right\}.$$

Appendix C: The Coefficient in (21)

The purpose of this appendix is to obtain a formula for the coefficient $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ in the special Cartesian coordinate system. In this coordinate system, $\frac{r_i}{r} = \delta_{1i}$ such that $\mathbf{W}_{[N]}$ is the product of N Kronecker deltas of the form δ_{1i}. Consider setting N_1 of the indexes equal to 1, an even number N_2 to 2, and an even number N_3 to 3 such that $N = N_1 + N_2 + N_3$. Then $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ becomes a sum of zeros and ones. What is that sum? If all indexes are set to 1, i.e., $N_1 = N$, then all terms in the sum $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ are unity such that $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ equals its number of terms; from (48) in Appendix E, that number is $\binom{N}{N-2P}(2P-1)!!$. The notation $\binom{N}{N-2P}$ is a binomial coefficient. To interpret $(2P-1)!!$, recall that $q!! \equiv q(q-2)(q-4)\cdots q_L$, where q_L is 2 or 1 for q even or odd, respectively, and $(-1)!! \equiv 1$. Now consider setting two indexes to 2, thus $N_1 = N-2$, and $N_2 = 2$. Name the two indexes $i = 2$ and $j = 2$. The only term in $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ that is nonzero is that which has i and j together in a single Kronecker delta, δ_{ij}, within $\delta_{[2P]}$. For $P = 0$ there is no such δ_{ij}, in which case $\left\{\mathbf{W}_{[N]}\delta_{[0]}\right\} = 0$. For $P \geq 1$, there is one such δ_{ij}, which is set to 1 and it multiplies the quantity $\left\{\mathbf{W}_{[N-2P]}\delta_{[2(P-1)]}\right\}$; since this quantity is evaluated with all 1s, it is equal to its number of terms, namely $\binom{(N-2P)+2(P-1)}{(N-2P)}(2(P-1)-1)!!$. Now consider setting four indexes to 2; thus, $N_1 = N-4$ and $N_2 = 4$. Name the four indexes

$i = 2$, $j = 2$, $k = 2$, $l = 2$. The only terms in $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ that are nonzero are those that have factors $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$, or $\delta_{il}\delta_{jk}$ within $\delta_{[2P]}$. For $P \leq 1$ there is no such pair of Kronecker deltas such that $\left\{\mathbf{W}_{[N]}\delta_{[0]}\right\} = 0$ and $\left\{\mathbf{W}_{[N-2]}\delta_{[2]}\right\} = 0$. For $P \geq 2$, there are the above $3 = (N_2 - 1)!!$ nonzero factors and each multiplies the quantity $\left\{\mathbf{W}_{[N-2P]}\delta_{[2(P-2)]}\right\}$; since this latter quantity is subsequently evaluated with all 1s, it is equal to its number of terms, namely $\binom{(N-2P)+2(P-2)}{(N-2P)}\left(2\left(P-2\right)-1\right)!!$. Continuation of this study reveals the pattern that when $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ is evaluated with N_2 2s and N_1 1s such that $N = N_1 + N_2$ then

$$\begin{aligned}
&\text{if } 2P < N_2 \text{ then } \left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} = 0, \text{ otherwise} \\
\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} &= (N_2-1)!!\binom{(N-2P)+2\left(P-\frac{N_2}{2}\right)}{(N-2P)}\left(2\left(P-\frac{N_2}{2}\right)-1\right)!!. \\
&= (N_2-1)!!\left[\frac{N_1!}{\left(N_1-2\left(P-\frac{N_2}{2}\right)\right)!\left(2\left(P-\frac{N_2}{2}\right)\right)!}\right]\left(2\left(P-\frac{N_2}{2}\right)-1\right)!!
\end{aligned}$$

If one ceases increasing the number of 2s and commences increasing the number of 3s in pairs such that $N = N_1 + N_2 + N_3$, then $(N_2-1)!!\binom{(N-2P)+2\left(P-\frac{N_2}{2}\right)}{(N-2P)}\left(2\left(P-\frac{N_2}{2}\right)-1\right)!!$ is replaced by $(N_2-1)!!\,(N_3-1)!!\binom{(N-2P)+2\left(P-\frac{N_2}{2}-\frac{N_3}{2}\right)}{(N-2P)}\left(2\left(P-\frac{N_2}{2}-\frac{N_3}{2}\right)-1\right)!!$. Of course, the binomial coefficient can be expressed as follows: $\binom{(N-2P)+2\left(P-\frac{N_2}{2}-\frac{N_3}{2}\right)}{(N-2P)} = (N-N_2-N_3)!/\left[(N-2P)!\,(2P-N_2-N_3)!\right]$; also, $(N-N_2-N_3)! = N_1!$. The double factorial can be eliminated by means of the following identities:

$$(2Q-1)!!/\,(2Q)! = 1/\,(2Q)!! = 1/\left(2^Q Q!\right). \tag{42}$$

That is, $(2P-N_2-N_3-1)!!/\,(2P-N_2-N_3)! = 1/\,(2P-N_2-N_3)!! = 1/\left[2^{\left(P-\frac{N_2}{2}-\frac{N_3}{2}\right)}\left(P-\frac{N_2}{2}-\frac{N_3}{2}\right)!\right]$; also, $(N_2-1)!! = N_2!/\left[2^{N_2/2}\,(N_2/2)!\right]$. Finally,

$$\text{if } 2P < N_2 + N_3 \text{ then } \left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} = 0, \text{ otherwise} \tag{43}$$

$$\begin{aligned}
\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} &= (N_2-1)!!\,(N_3-1)!!N_1!/\left[(N-2P)!2^{\left(P-\frac{N_2}{2}-\frac{N_3}{2}\right)}\left(P-\frac{N_2}{2}-\frac{N_3}{2}\right)!\right] \\
&= N_1!N_2!N_3!/\left[(N-2P)!2^P\left(\frac{N_2}{2}\right)!\left(\frac{N_3}{2}\right)!\left(P-\frac{N_2}{2}-\frac{N_3}{2}\right)!\right].
\end{aligned} \tag{44}$$

Appendix D: Identities for Isotropic Symmetric Tensors

The purpose of this appendix is to determine (1) which components of a symmetric, isotropic tensor are zero; (2) how many components are zero and how many are nonzero; and (3) the relationships between the nonzero components.

Consider which components $D_{[N:N_1,N_2,N_3]}$ are nonzero and which are zero. In (21), $\mathbf{W}_{[N-2P]}$ vanishes if any of its indexes is 2 or 3, $\delta_{[2P]}$ vanishes unless it contains an even number of indexes equal to 2 and a likewise even number of 3s (and of 1s). Thus, a component $D_{[N:N_1,N_2,N_3]}$ is nonzero only if both N_2 and N_3 are even. Because $N = N_1 + N_2 + N_3$, $D_{[N:N_1,N_2,N_3]}$ is nonzero only when N_1 is odd if N is odd and only when N_1 is even if N is even. The values of N_1 that can give nonzero values of $D_{[N:N_1,N_2,N_3]}$ are N, $N-2$, $\cdots$, 0 or 1; i.e., $M+1$ cases of N_1. Given N_1, the values of $[N_2, N_3]$ that give nonzero values of $D_{[N:N_1,N_2,N_3]}$ are $[N-N_1, 0]$, $[N-N_1-2, 2]$, $\cdots$,$[0, N-N_1]$; i.e., $(N-N_1+2)/2$ cases (note that $N-N_1 = N_2+N_3$ is necessarily even). Counting the number of cases of $[N_2, N_3]$ as N_1 varies from N to 0 or 1, (i.e., substituting $N_1 = N$, then $N_1 = N-2$, $\cdots$ into $(N-N_1+2)/2$ and adding the resultant numbers) shows that there are $1+2+3+\cdots+(M+1) = (M+1)(M+2)/2$ components $D_{[N:N_1,N_2,N_3]}$ that are nonzero. Since there are 3^N components of $\mathbf{D}_{[N]}$, the remaining $3^N - (M+1)(M+2)/2$ components are zero. Since there are $M+1$ linearly independent components (that are related to the $D_{N,P}$), there are $(M+1)(M+2)/2-(M+1) = M(M+1)/2$ relationships among the nonzero components $D_{[N:N_1,N_2,N_3]}$. For instance, interchange of the values of N_2 and N_3 produces components that are equal.

Consider the $M+1$ linearly independent equations that determine the $D_{N,P}$ in terms of $M+1$ of the $D_{[N:N_1,N_2,N_3]}$. $D_{N,0}$ is related by (21) to only the component $D_{[N:N,0,0]}$ because (22) shows that the coefficient of $D_{N,0}$, namely $\left\{\mathbf{W}_{[N]}\delta_{[0]}\right\}$, vanishes unless $N_1 = N$. That is, if all indexes in (21) are 1, then $D_{[N:N,0,0]}$ appears on the left-hand side and the $D_{N,P}$ for all P appear in the equation. This equation is essential for determining $D_{N,0}$ and is called "the equation for $D_{N,0}$"; similar terminology "the equation for $D_{N,P}$" is used below. With the equation for $D_{N,0}$ in hand, consider $D_{N,1}$. $D_{N,1}$ is related by (21) to $D_{[N:N-2,2,0]}$ or $D_{[N:N-2,0,2]}$. When $D_{[N:N-2,2,0]}$ or $D_{[N:N-2,0,2]}$ is on the left-hand side of (21) the equation for $D_{N,1}$ results because the coefficients $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ of $D_{N,P}$ for $P \geq 1$ do not vanish, but the coefficient of $D_{N,0}$ does vanish. Now consider an equation for $D_{N,2}$. $D_{N,2}$ is related by (21) to $D_{[N:N-4,4,0]}$ or $D_{[N:N-4,0,4]}$ or $D_{[N:N-4,2,2]}$; these components also involve $D_{N,P}$ for $P \geq 3$ but not for $P \leq 1$. This procedure repeats until the last equation is produced; only $D_{N,M}$ appears in the last equation. If N is even then $D_{N,M}$ is related to $D_{[N:0,N_2,N_3]}$ with N_2 and N_3 equal to any positive even numbers such that $N = N_2 + N_3$. If N is odd then $D_{N,M}$ is related to $D_{[N:1,N_2,N_3]}$ with N_2 and N_3 equal to any positive even numbers such that $N = 1 + N_2 + N_3$. This procedure results in a set of $M+1$ linearly independent equations that can be solved to obtain the $D_{N,P}$ in terms of the $D_{[N:N_1,N_2,N_3]}$. Note that $M+1$ components must be chosen for use in the $M+1$ equations. For instance, from the above example, $D_{[N:N,0,0]}$ must be used; either $D_{[N:N-2,2,0]}$ or $D_{[N:N-2,0,2]}$ must be chosen, and one of $D_{[N:N-4,4,0]}$ or $D_{[N:N-4,0,4]}$ or $D_{[N:N-4,2,2]}$ must be chosen, etc. For simplicity, the chosen components can all have $N_3 = 0$, i.e., the choice can be $D_{[N:N,0,0]}$, $D_{[N:N-2,2,0]}$, $D_{[N:N-4,4,0]}$, $\cdots$, $D_{[N:N-2M,2M,0]}$.

The above procedure also reveals algebraic relationships between the nonzero $D_{[N:N_1,N_2,N_3]}$. The equation for $D_{N,1}$ can be expressed in terms of either $D_{[N:N-2,2,0]}$ or $D_{[N:N-2,0,2]}$; the left-hand side is the same in either case because the coefficients (23) are the same; hence $D_{[N:N-2,2,0]} = D_{[N:N-2,0,2]}$. The equation for $D_{N,2}$ can be expressed in terms of $D_{[N:N-4,2,2]}$ or $D_{[N:N-4,4,0]}$ or $D_{[N:N-4,0,4]}$ such that (23) gives $D_{[N:N-4,4,0]} = D_{[N:N-4,0,4]}$; but what is the relationship of $D_{[N:N-4,2,2]}$ to $D_{[N:N-4,4,0]}$ and $D_{[N:N-4,0,4]}$? When $D_{[N:N-4,0,4]}$ or $D_{[N:N-4,4,0]}$ is on the left-hand side of (21), the nonzero coefficients are, from (23), $(N-4)!4!0!/\left[(N-2P)!2^P 2!0!\,(P-2-0)!\right]$, but when $D_{[N:N-4,2,2]}$ is on the left-hand side, the nonzero coefficients are $(N-4)!2!2!/\left[(N-2P)!2^P 1!1!\,(P-1-1)!\right]$. The ratio of these coefficients is 3. Since this ratio is independent of P, the entire right-hand side of (21) is three times greater when $D_{[N:N-4,0,4]}$ is on the left-hand side as compared to when $D_{[N:N-4,2,2]}$ is on the left-hand side. Therefore, the proportionality $D_{[N:N-4,0,4]} : D_{[N:N-4,2,2]}$ (and also

$D_{[N:N-4,4,0]}$: $D_{[N:N-4,2,2]}$) is 3 : 1. In general, for given N and N_1, and hence given $N_2 + N_3 = N - N_1$, and another choice of N_2 and N_3, call them N_2' and N_3', such that $N_2' + N_3' = N_2 + N_3 = N - N_1$, the proportionality obtained from (23) is $D_{[N:N_1,N_2,N_3]}$: $D_{[N:N_1,N_2',N_3']} = [N_2!N_3!/(N_2/2)!(N_3/2)!] : [N_2'!N_3'!/(N_2'/2)!(N_3'/2)!]$. Parameterized in terms of an integer L such that $N = N_1 + 2L$, for given N_1, the proportionalities are $D_{[N:N_1,2L,0]} : D_{[N:N_1,2L-2,2]} : D_{[N:N_1,2L-4,4]} : \cdots : D_{[N:N_1,0,2L]} = [(2L)!0!/L!0!] : [(2L-2)!2!/(L-1)!1!] : [(2L-4)!4!/(L-2)!2!] : \cdots : [0!(2L)!/0!L!]$. This constitutes $(N - N_1)/2$ relationships among the nonzero components $D_{[N:N_1,N_2,N_3]}$ for given N_1. Substituting the $M+1$ cases of N_1 (i.e., $N_1 = N$, $N_1 = N-1$, $\cdots$) into $(N - N_1)/2$ the number of relationships thus identified among the components of $\mathbf{D}_{[N]}$ is $0+1+2+\cdots+M = M(M+1)/2$. In the paragraph above, it was determined that the total number of relationships among the nonzero components of $\mathbf{D}_{[N]}$ is $M(M+1)/2$. Consequently, all such relationships have now been found.

Appendix E: Derivatives of Isotropic Tensors

The objective of this appendix is to develop succinct notation for isotropic tensors and their derivatives with specific attention to their first-order divergence and their Laplacian. Those derivatives appear in (18). A derivation of those derivatives operating on an isotropic tensor that is symmetric under interchange of any pair of indexes is given.

First, notation is developed: $\delta_{[2P]}$ is the product of P Kronecker deltas having $2P$ distinct indexes. For example, $\delta_{[6]} = \delta_{ij}\delta_{kl}\delta_{mn}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. $\mathbf{W}_{[N]}$ is the product of N factors $\frac{r_i}{r}$ each with an index distinct from the other indexes. For example, $\mathbf{W}_{[4]} = \frac{r_i}{r}\frac{r_j}{r}\frac{r_k}{r}\frac{r_l}{r}$. For convenience, define

$$\delta_{[0]} \equiv 1, \delta_{[-2]} \equiv 0, \mathbf{W}_{[0]} \equiv 1, \mathbf{W}_{[-1]} \equiv 0, \mathbf{W}_{[-2]} \equiv 0. \tag{45}$$

The plural of δ is δs and that of $\mathbf{W}$ is $\mathbf{W}$s. It is understood that products of $\mathbf{W}$s (e.g., $\mathbf{W}_{[N]}\mathbf{W}_{[K]}$) and of δs (e.g., $\delta_{[2N]}\delta_{[2K]}$) and of $\mathbf{W}$s with δs (e.g., $\mathbf{W}_{[N]}\delta_{[2K]}$) have all distinct indexes. Then, the $\mathbf{W}$s factor, e.g., $\mathbf{W}_{[4]} = \mathbf{W}_{[1]}\mathbf{W}_{[3]} = \mathbf{W}_{[2]}\mathbf{W}_{[2]} = \mathbf{W}_{[1]}\mathbf{W}_{[1]}\mathbf{W}_{[2]}$, etc., and the δs likewise factor. The operation "contraction" means to set two indexes equal and sum over their range of values; the summation convention over repeated Roman indexes is used, e.g., $\delta_{ii} \equiv \sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$. A contraction of two indexes of $\delta_{[2P]}$ produces either $3\delta_{[2(P-1)]}$ or $\delta_{[2(P-1)]}$ depending on whether the two indexes are on the same Kronecker delta or different ones, respectively. The contraction of $\mathbf{W}_{[N]}$ on two indexes produces $\mathbf{W}_{[N-1]}$ because $\frac{r_i}{r}\frac{r_i}{r} = \frac{r^2}{r^2} = 1$. Consider the contraction of $\frac{r_i}{r}$ with $\mathbf{W}_{[N]}\delta_{[2P]}$. If the index i is in $\mathbf{W}_{[N]}$ then the contraction $\frac{r_i}{r}\mathbf{W}_{[N]}\delta_{[2P]}$ is $\mathbf{W}_{[N-1]}\delta_{[2P]}$ because $\frac{r_i}{r}\frac{r_i}{r} = 1$. If the index i is in $\delta_{[2P]}$ then the contraction $\frac{r_i}{r}\mathbf{W}_{[N]}\delta_{[2P]}$ is $\mathbf{W}_{[N+1]}\delta_{[2(P-1)]}$ because $\frac{r_i}{r}\delta_{ij} = \frac{r_j}{r}$. The notation $\Updownarrow \circ \Updownarrow_j^N$ means the sum of N terms where each term contains a distinct index j, and the index j is interchanged with all implied indexes, but j is not interchanged with any explicit index. For example,

$$\Updownarrow \mathbf{W}_{[2]}\delta_{ij} \Updownarrow_j^3 = \frac{r_k}{r}\frac{r_l}{r}\delta_{ij} + \frac{r_k}{r}\frac{r_j}{r}\delta_{il} + \frac{r_j}{r}\frac{r_l}{r}\delta_{ik}, \tag{46}$$

$$\text{and } \Updownarrow \mathbf{W}_{[1]}\delta_{[2]}\frac{r_i}{r}\frac{r_j}{r} \Updownarrow_j^4 = \frac{r_k}{r}\delta_{nm}\frac{r_i}{r}\frac{r_j}{r} + \frac{r_j}{r}\delta_{nm}\frac{r_i}{r}\frac{r_k}{r} + \frac{r_k}{r}\delta_{jm}\frac{r_i}{r}\frac{r_n}{r} + \frac{r_k}{r}\delta_{nj}\frac{r_i}{r}\frac{r_m}{r}, \tag{47}$$

where, i is an explicit index and is therefore not interchanged with j.

French braces (i.e., $\{\circ\}$) means: add all such distinct terms required to make the tensor symmetric under interchange of any pair of indexes. For example,

$$\left\{\mathbf{W}_{[2]}\delta_{[2]}\right\} = \left\{\frac{r_i}{r}\frac{r_j}{r}\delta_{kl}\right\} = \frac{r_i}{r}\frac{r_j}{r}\delta_{kl} + \frac{r_i}{r}\frac{r_l}{r}\delta_{kj} + \frac{r_l}{r}\frac{r_j}{r}\delta_{ki} + \frac{r_k}{r}\frac{r_l}{r}\delta_{ij} + \frac{r_k}{r}\frac{r_j}{r}\delta_{il} + \frac{r_i}{r}\frac{r_k}{r}\delta_{jl}.$$

Note that terms that are necessarily equal do not appear; i.e., since $\frac{r_i}{r}\frac{r_j}{r}\delta_{kl}$ appears, neither $\frac{r_j}{r}\frac{r_i}{r}\delta_{kl}$ nor $\frac{r_i}{r}\frac{r_j}{r}\delta_{lk}$ appear. Because of the commutative law of addition, $\{\circ\}$ commutes with addition; e.g., $\left\{\mathbf{W}_{[N]}\right\} + \left\{\mathbf{W}_{[Q]}\delta_{[2P]}\right\} = \left\{\mathbf{W}_{[N]} + \mathbf{W}_{[Q]}\delta_{[2P]}\right\}$. Because of the distributive law of multiplication, multiplication by a scalar function commutes with the $\{\circ\}$ notation; i.e., $A(r)\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\} = \left\{A(r)\,\mathbf{W}_{[N]}\delta_{[2P]}\right\}$.

The number of terms in various sums, $\{\circ\}$, is required repeatedly; $\left\{\delta_{[2P]}\right\}$ has $(2P-1)!! = (2P-1)(2P-3)(2P-5)\cdots(1)$ terms. Since $\mathbf{W}_{[N]}\delta_{[2P]}$ has $2P+N$ indexes, $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}$ has $\binom{2P+N}{N}(2P-1)!!$ terms, where the binomial coefficient $\binom{2P+N}{N}$ is the number of ways of selecting the N indexes in $\mathbf{W}_{[N]}$ from the total $2P+N$ indexes. If i is an index in $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}$, then $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}$ has $\binom{2P+N-1}{N}(2P-1)!!$ terms in which i appears in a Kronecker delta because there are N indexes to select for $\mathbf{W}_{[N]}$ from the remaining $2P+N-1$ indexes. Similarly, $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}$ has $\binom{2P+N-1}{N-1}(2P-1)!!$ terms in which i appears in a factor (r_i/r) because there are $N-1$ indexes remaining to select for $\mathbf{W}_{[N]}$ from the remaining $2P+N-1$ indexes. Note that $\left\{\mathbf{W}_{[N]}\right\}$ has only 1 term. Hence, $\left\{\mathbf{W}_{[N]}\right\} = \mathbf{W}_{[N]}$. In summary,

$$\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\} \text{ has } \binom{2P+N}{N}(2P-1)!! \text{ terms;} \tag{48}$$

$$\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\} \text{ has } \binom{2P+N-1}{N}(2P-1)!! \text{ terms with } i \text{ in } \delta_{[2P]}; \tag{49}$$

$$\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\} \text{ has } \binom{2P+N-1}{N-1}(2P-1)!! \text{ terms with } i \text{ in } \mathbf{W}_{[N]}. \tag{50}$$

The sum of the number of terms in (49)-(50), namely $\binom{2P+N-1}{N}(2P-1)!! + \binom{2P+N-1}{N-1}(2P-1)!! = (2P+N-1)!\left[\frac{2P}{N!(2P)!}+\frac{N}{N!(2P)!}\right](2P-1)!! = \binom{2P+N}{N}(2P-1)!!$, agrees with the total number of terms in (48).

Now, rules for differentiation of symmetric, isotropic tensors are developed. Note the identity

$$\partial_{r_i}(r_j/r) = \left[\delta_{ij} - \left(r_i r_j/r^2\right)\right]/r = \left(\delta_{[2]} - \mathbf{W}_{[2]}\right)/r, \tag{51}$$

from which it follows that

$$\partial_{r_i}(r_i/r) = (3-1)/r = 2/r, \text{ and } r_i\partial_{r_i}(r_j/r) = \left[r_j - \left(r^2 r_j/r^2\right)\right]/r = 0. \tag{52}$$

The latter formula greatly simplifies the divergence of $\mathbf{W}_{[N]}$ because ∂_{r_i} operating on $\mathbf{W}_{[N]}$ vanishes when it operates on any factor other than the factor $\frac{r_i}{r}$ within $\mathbf{W}_{[N]}$.

The divergence and gradient of $\mathbf{W}_{[N]}$ are needed. If i is an index in $\mathbf{W}_{[N]}$ then the divergence of $\mathbf{W}_{[N]}$ is denoted by $\nabla_{\mathbf{r}}\cdot\mathbf{W}_{[N]} = \partial_{r_i}\mathbf{W}_{[N]}$. Application of (52) gives $\partial_{r_i}\mathbf{W}_{[1]} = \partial_{r_i}(r_i/r) = 2/r$, and $\partial_{r_i}\mathbf{W}_{[N]} = \mathbf{W}_{[N-1]}\partial_{r_i}(r_i/r) = \frac{2}{r}\mathbf{W}_{[N-1]}$. The gradient of $\mathbf{W}_{[N]}$ is denoted by $\partial_{r_i}\mathbf{W}_{[N]}$ where i is not an index in $\mathbf{W}_{[N]}$. From the differentiation chain rule, $\partial_{r_i}\mathbf{W}_{[N]}$ is the sum of N terms, each of which has the form $\mathbf{W}_{[N-1]}\partial_{r_i}(r_j/r)$. Therefore, by use of (51), $\partial_{r_i}\mathbf{W}_{[N]} = \llbracket \mathbf{W}_{[N-1]}\left[\delta_{ij} - (r_i r_j/r^2)\right]/r \rrbracket_j^N = \frac{1}{r}\llbracket \mathbf{W}_{[N-1]}\delta_{ij}\rrbracket_j^N - \frac{N}{r}\mathbf{W}_{[N+1]}$, where use was made of $\llbracket \mathbf{W}_{[N-1]}\left[-(r_i r_j/r^2)\right]/r \rrbracket_j^N = \left[-(r_i/r)/r\right]\llbracket \mathbf{W}_{[N-1]}(r_j/r)\rrbracket_j^N = -(r_i/r)/r\left(N\mathbf{W}_{[N]}\right) = -\frac{N}{r}\mathbf{W}_{[N+1]}$, because $\llbracket \mathbf{W}_{[N-1]}(r_j/r)\rrbracket_j^N$ is the sum of N identical terms each equal to $\mathbf{W}_{[N]}$. In summary,

$$\text{If } i \text{ is in } \mathbf{W}_{[N]} \text{ then } \nabla_{\mathbf{r}}\cdot\mathbf{W}_{[N]} = \partial_{r_i}\mathbf{W}_{[N]} = \frac{2}{r}\mathbf{W}_{[N-1]}. \tag{53}$$

$$\text{If } i \text{ is not in } \mathbf{W}_{[N]} \text{ then } \partial_{r_i}\mathbf{W}_{[N]} = \frac{1}{r}\llbracket \mathbf{W}_{[N-1]}\delta_{ij}\rrbracket_j^N - \frac{N}{r}\mathbf{W}_{[N+1]}, \tag{54}$$

Consider the divergence of $\mathbf{W}_{[N]}\delta_{[2P]}$. If the index i is in $\mathbf{W}_{[N]}$ then, from (53), $\partial_{r_i}\left(\mathbf{W}_{[N]}\delta_{[2P]}\right) = \delta_{[2P]}\mathbf{W}_{[N|1]} = \frac{2}{r}\delta_{[2P]}\mathbf{W}_{[N-1]}$. If the index i is in $\delta_{[2P]}$, then (given that index k is not in $\mathbf{W}_{[N]}$) $\partial_{r_i}\left(\mathbf{W}_{[N]}\delta_{[2P]}\right) = \delta_{[2(P-1)]}\delta_{ik}\partial_{r_i}\mathbf{W}_{[N]} = \delta_{[2(P-1)]}\partial_{r_k}\mathbf{W}_{[N]} = \delta_{[2(P-1)]}\frac{1}{r}\llbracket \mathbf{W}_{[N-1]}\delta_{kj}\rrbracket_j^N - \delta_{[2(P-1)]}\frac{N}{r}\mathbf{W}_{[N+1]}$, where the last expression follows from (54). In summary,

$$\text{If } i \text{ is in } \mathbf{W}_{[N]} \text{ then } \partial_{r_i}\left(\mathbf{W}_{[N]}\delta_{[2P]}\right) = \frac{2}{r}\delta_{[2P]}\mathbf{W}_{[N-1]}, \tag{55}$$

$$\text{If } i \text{ is in } \delta_{[2P]} \text{ then } \partial_{r_i}\left(\mathbf{W}_{[N]}\delta_{[2P]}\right) = \delta_{[2(P-1)]}\frac{1}{r}\llbracket \mathbf{W}_{[N-1]}\delta_{kj}\rrbracket_j^N - \delta_{[2(P-1)]}\frac{N}{r}\mathbf{W}_{[N+1]}. \tag{56}$$

The above results allow evaluation of the divergence $\partial_{r_i}\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\} \equiv \nabla_{\mathbf{r}}\cdot\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}$. It follows from use of (55) and the distributive law of multiplication and the fact that the number of terms in $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}$ in which i appears in the factor $\frac{r_i}{r}$ is the same as the number of terms in $\left\{\mathbf{W}_{[N-1]}\delta_{[2P]}\right\}$ [see (48)-(50)], that for those terms in which i is in $\mathbf{W}_{[N]}$, the divergence of $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}$ yields $\frac{2}{r}\left\{\mathbf{W}_{[N-1]}\delta_{[2P]}\right\}$. Similar use of (56) gives that for those terms in which i is in $\delta_{[2P]}$, the divergence of $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}$ yields $\frac{2P}{r}\left\{\mathbf{W}_{[N-1]}\delta_{[2P]}\right\} - \frac{N(N+1)}{r}\left\{\mathbf{W}_{[N+1]}\delta_{[2(P-1)]}\right\}$. Thus,

$$\begin{aligned}\nabla_{\mathbf{r}}\cdot\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\} &= \frac{2}{r}\left\{\mathbf{W}_{[N-1]}\delta_{[2P]}\right\} + \left[\frac{2P}{r}\left\{\mathbf{W}_{[N-1]}\delta_{[2P]}\right\} - \frac{N(N+1)}{r}\left\{\mathbf{W}_{[N+1]}\delta_{[2(P-1)]}\right\}\right] \\ &= \frac{2}{r}(P+1)\left\{\mathbf{W}_{[N-1]}\delta_{[2P]}\right\} - \frac{N(N+1)}{r}\left\{\mathbf{W}_{[N+1]}\delta_{[2(P-1)]}\right\}. \end{aligned} \tag{57}$$

Because of the definitions in (45), (57) remains valid if N is 0 or 1 or if P is 0 or 1.

Derivation of the formula for the divergence of an isotropic tensor requires evaluation of the contraction $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}\frac{r_i}{r}$. From (49), in $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}$ there are $\binom{2P+N-1}{N}(2P-1)!!$ occurrences of the index i within $\delta_{[2P]}$ and each gives the contraction $\delta_{ij}\frac{r_i}{r} = \frac{r_j}{r}$, which decreases P by unity and increases N by unity thereby producing several $\left\{\mathbf{W}_{[N+1]}\delta_{[2(P-1)]}\right\}$. From (48), there are $\binom{2(P-1)+(N+1)}{(N+1)}(2(P-1)-1)!!$ terms in a $\left\{\mathbf{W}_{[N+1]}\delta_{[2(P-1)]}\right\}$; thus the number of $\left\{\mathbf{W}_{[N+1]}\delta_{[2(P-1)]}\right\}$ so produced is $\left[\binom{2P+N-1}{N}(2P-1)!!\right] / \left[\binom{2(P-1)+(N+1)}{(N+1)}(2(P-1)-1)!!\right] = (N+1)$. From (50), the contraction $\left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}\frac{r_i}{r}$ contains $\binom{2P+N-1}{N-1}(2P-1)!!$ terms in which i appears within $\mathbf{W}_{[N]}$ and each results in the contraction $\frac{r_i}{r}\frac{r_i}{r} = 1$, which decreases N by unity. The number of $\left\{\mathbf{W}_{[N-1]}\delta_{[2P]}\right\}$ so produced is $\left[\binom{2P+N-1}{N-1}(2P-1)!!\right] / \left[\binom{2P+(N-1)}{(N-1)}(2P-1)!!\right] = 1$ because $\left\{\mathbf{W}_{[N-1]}\delta_{[2P]}\right\}$ has $\binom{2P+(N-1)}{(N-1)}(2P-1)!!$ terms, which is also the number of terms given in (50). Thus,

$$\text{contraction on } i\text{: } \left\{\mathbf{W}_{[N]}\delta_{[2P]}\right\}\frac{r_i}{r} = (N+1)\left\{\mathbf{W}_{[N+1]}\delta_{[2(P-1)]}\right\} + \left\{\mathbf{W}_{[N-1]}\delta_{[2P]}\right\}. \quad (58)$$

The general isotropic formula for a tensor $\mathbf{A}_{[N]}(\mathbf{r})$ of order N that is symmetric under interchange of any pair of indexes is

$$\begin{aligned}\mathbf{A}_{[N]}(\mathbf{r}) &= A_0(r)\left\{\mathbf{W}_{[N]}\right\} + A_1(r)\left\{\mathbf{W}_{[N-2]}\delta_{[2]}\right\} + A_2(r)\left\{\mathbf{W}_{[N-4]}\delta_{[4]}\right\} + \cdots + T_{last} \\ &= \sum_{P=0}^{M} A_P(r)\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}, \end{aligned} \quad (59)$$

where the $A_0(r)$, $A_1(r)$, etc. are scalar functions of r, and T_{last} is the last term. Note that for brevity in this appendix, the subscript N has been omitted from $A_{N,0}(r)$, $A_{N,1}(r)$, etc. If N is even, then $T_{last} = A_{N/2}(r)\left\{\delta_{[N]}\right\}$ and $M = N/2$. If N is odd, then $T_{last} = A_{(N-1)/2}(r)\left\{\mathbf{W}_{[1]}\delta_{[N-1]}\right\}$ and $M = (N-1)/2$.

All of the foregoing has set the stage for efficient derivation of a formula for the divergence $\nabla_{\mathbf{r}} \cdot \mathbf{A}_{[N]}(\mathbf{r})$. Also needed is the fact that the gradient of a scalar function of $r \equiv \sqrt{r_i r_i}$ is $\partial_{r_i} A(r) = (\partial_{r_i} r)\,\partial_r A(r) = \frac{r_i}{r}\partial_r A(r) = \mathbf{W}_{[1]}\partial_r A(r)$. Consider the divergence of a term in (59). By use of the differentiation chain rule, and substitution of (58) and (57),

$$\nabla_{\mathbf{r}} \cdot \left[A_P(r)\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}\right] =$$

$$
\begin{aligned}
&\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}\frac{r_i}{r}\partial_r A_P(r) + A_P(r)\nabla_{\mathbf{r}}\cdot\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} \\
&= \left[(N-2P+1)\left\{\mathbf{W}_{[N-2P+1]}\delta_{[2(P-1)]}\right\} + \left\{\mathbf{W}_{[N-2P-1]}\delta_{[2P]}\right\}\right]\partial_r A_P(r) \\
&+ A_P(r)\left[\frac{2}{r}(P+1)\left\{\mathbf{W}_{[N-2P-1]}\delta_{[2P]}\right\}\right] \\
&- A_P(r)\left[\frac{(N-2P)(N-2P+1)}{r}\left\{\mathbf{W}_{[N-2P+1]}\delta_{[2(P-1)]}\right\}\right] \\
&= B_{N,P}(r)\left\{\mathbf{W}_{[N-2P+1]}\delta_{[2(P-1)]}\right\} + C_P(r)\left\{\mathbf{W}_{[N-2P-1]}\delta_{[2P]}\right\}, \qquad (60)
\end{aligned}
$$

where $B_{N,P}(r)$ and $C_P(r)$ are defined by the following operators, $O_B(N,P)$ and $O_C(P)$, operating on $A_P(r)$:

$$
B_{N,P}(r) \equiv O_B(N,P)\,A_P(r), \text{ where } O_B(N,P) \equiv (N-2P+1)\left[\partial_r - \frac{N-2P}{r}\right], \qquad (61)
$$

$$
C_P(r) \equiv O_C(P)\,A_P(r) \text{ where } O_C(P) \equiv \left[\partial_r + \frac{2}{r}(P+1)\right]. \qquad (62)
$$

Thereby, the divergence of (59) is

$$
\nabla_{\mathbf{r}}\cdot\mathbf{A}_{[N]}(\mathbf{r}) = \sum_{P=0}^{M} B_{N,P}(r)\left\{\mathbf{W}_{[N-2P+1]}\delta_{[2(P-1)]}\right\} + \sum_{P=0}^{M} C_P(r)\left\{\mathbf{W}_{[N-2P-1]}\delta_{[2P]}\right\}, \qquad (63)
$$

where (60) and (45) were used.

Now, (63) can be checked by comparison with the divergence performed on the explicit-index formulas for symmetric, isotropic tensors of rank 1 to 4. The lowest-order tensor for which the divergence is defined is a vector (i.e., $N=1$), in which case (63) gives

$$
\partial_{r_i} A_i(\mathbf{r}) = \partial_r A_0(r) + \frac{2}{r}A_0(r),
$$

which is easily verified by evaluating the divergence of a isotropic vector, namely $\partial_{r_i}\left[A_0(r)\frac{r_i}{r}\right]$. Expressed with explicit indexes as well as in the implicit-index form of (59), isotropic tensors of rank 2 to 4 that are symmetric under interchange of any pair of indexes are:

$$
A_{ij}(\mathbf{r}) = A_0(r)\frac{r_i}{r}\frac{r_j}{r} + A_1(r)\,\delta_{ij} = A_0(r)\,\mathbf{W}_{[2]} + A_1(r)\,\delta_{[2]}. \qquad (64)
$$

$$A_{ijk}(\mathbf{r}) = A_0(r)\frac{r_i}{r}\frac{r_j}{r}\frac{r_k}{r} + A_1(r)\left(\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik} + \frac{r_k}{r}\delta_{ij}\right) = A_0(r)\mathbf{W}_{[3]} + A_1(r)\left\{\mathbf{W}_{[1]}\delta_{[2]}\right\}. \tag{65}$$

$$\begin{aligned} A_{ijkl}(\mathbf{r}) &= A_0(r)\frac{r_ir_jr_kr_l}{r^4} + A_1(r)\left(\frac{r_ir_j}{r^2}\delta_{kl} + \frac{r_ir_k}{r^2}\delta_{jl} + \frac{r_jr_k}{r^2}\delta_{il} + \frac{r_ir_l}{r^2}\delta_{jk} + \frac{r_jr_l}{r^2}\delta_{ik} + \frac{r_kr_l}{r^2}\delta_{ij}\right) \\ &\quad + A_2(r)\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\right) \\ &= A_0(r)\mathbf{W}_{[4]} + A_1(r)\left\{\mathbf{W}_{[2]}\delta_{[2]}\right\} + A_2(r)\left\{\delta_{[4]}\right\}. \end{aligned} \tag{66}$$

One can see the brevity of the implicit-index formula as the rank of the tensor increases. The first-order divergences obtained by differentiating the above explicit-index formulas as well as from (63) are

$$\nabla_{\mathbf{r}} \cdot \mathbf{A}_{[2]}(\mathbf{r}) = \left[\left(\partial_r + \frac{2}{r}\right)A_0(r) + \partial_r A_1(r)\right]\frac{r_j}{r} \tag{67}$$

$$= \left[B_{2,1}(r) + C_0(r)\right]\left\{\mathbf{W}_{[1]}\delta_{[0]}\right\}. \tag{68}$$

$$\nabla_{\mathbf{r}} \cdot \mathbf{A}_{[3]}(\mathbf{r}) = \left[\left(\partial_r + \frac{2}{r}\right)A_0(r) + \left(2\partial_r - \frac{2}{r}\right)A_1(r)\right]\frac{r_jr_k}{r^2} + \left[\left(\partial_r + \frac{4}{r}\right)A_1(r)\right]\delta_{jk} \tag{69}$$

$$= \left[B_{3,1}(r) + C_0(r)\right]\left\{\mathbf{W}_{[2]}\delta_{[0]}\right\} + C_1(r)\left\{\mathbf{W}_{[0]}\delta_{[2]}\right\}. \tag{70}$$

$$\begin{aligned} \nabla_{\mathbf{r}} \cdot \mathbf{A}_{[4]}(\mathbf{r}) &= \left[\left(\partial_r + \frac{2}{r}\right)A_0(r) + \left(3\partial_r - \frac{6}{r}\right)A_1(r)\right]\frac{r_jr_kr_l}{r^3} \\ &\quad + \left[\left(\partial_r + \frac{4}{r}\right)A_1(r) + \partial_r A_2(r)\right]\left(\frac{r_i}{r}\delta_{jk} + \frac{r_j}{r}\delta_{ik} + \frac{r_k}{r}\delta_{ij}\right) \end{aligned} \tag{71}$$

$$= \left[B_{4,1}(r) + C_0(r)\right]\left\{\mathbf{W}_{[3]}\delta_{[0]}\right\} + \left(B_{4,2}(r) + C_1(r)\right)\left\{\mathbf{W}_{[1]}\delta_{[2]}\right\}. \tag{72}$$

In the implicit-index formulas in (68), (70), and (72), terms from (63) that are zero because of (45) have been omitted. Equation (63) has been checked by using the implicit-index formulas in (68), (70), and (72) to obtain the explicit-index formulas in (67), (69), and (71), respectively.

The Laplacian of a symmetric, isotropic tensor is also needed for the term $2\nu\nabla_{\mathbf{r}}^2\mathbf{D}_{[N]}$ in (18). Application of (36) to $\nabla^2\mathbf{W}_{[N-2P]}$ and use of (51)-(52) gives

$$\nabla^2\mathbf{W}_{[N-2P]} = \left\{\mathbf{W}_{[N-2P-1]}\partial_{r_n}\partial_{r_n}\frac{r_j}{r}\right\} + 2\left\{\mathbf{W}_{[N-2P-2]}\left(\partial_{r_n}\frac{r_k}{r}\right)\left(\partial_{r_n}\frac{r_j}{r}\right)\right\}. \tag{73}$$

Now, $\left\{\mathbf{W}_{[N-2P-1]}\partial_{r_n}\partial_{r_n}\frac{r_j}{r}\right\}$ is $\binom{N-2P}{1}$ terms, each one is of the form $\mathbf{W}_{[N-2P-1]}\partial_{r_n}\partial_{r_n}\frac{r_j}{r} = \mathbf{W}_{[N-2P-1]}\left(\frac{-2}{r^2}\frac{r_j}{r}\right) = -\frac{2}{r^2}\mathbf{W}_{[N-2P]}$. Also, $\left\{\mathbf{W}_{[N-2P-2]}\left(\partial_{r_n}\frac{r_k}{r}\right)\left(\partial_{r_n}\frac{r_j}{r}\right)\right\}$ is $\binom{N-2P}{2}$ terms, each one is of the form $\mathbf{W}_{[N-2P-2]}\left(\partial_{r_n}\frac{r_k}{r}\right)\left(\partial_{r_n}\frac{r_j}{r}\right) = \mathbf{W}_{[N-2P-2]}\frac{1}{r^2}\left(\delta_{kj} - \frac{r_k r_j}{r^2}\right) = \frac{1}{r^2}\mathbf{W}_{[N-2P-2]}\delta_{ij} - \frac{1}{r^2}\mathbf{W}_{[N-2P]}$. From (48), $\left\{\mathbf{W}_{[N-2P-2]}\delta_{[2]}\right\}$ has $\binom{N-2P}{2}$ terms; thus, (73) gives

$$\nabla^2\mathbf{W}_{[N-2P]} = \frac{2}{r^2}\left\{\mathbf{W}_{[N-2P-2]}\delta_{[2]}\right\} - \frac{2}{r^2}\left[\binom{N-2P}{2} + \binom{N-2P}{1}\right]\mathbf{W}_{[N-2P]}. \tag{74}$$

The binomial coefficients prevent a nonzero term in (74) when $\mathbf{W}_{[N-2P-1]}$ or $\mathbf{W}_{[N-2P-2]}$ vanish in (73) as required by definition (45) provided that we define

$$\binom{N-2P}{1} \equiv 0 \text{ if } N-2P<1, \text{ and } \binom{N-2P}{2} \equiv 0 \quad \text{if } N-2P<2. \tag{75}$$

Of course, (75) is consistent with $1/K! = 0$ for $K < 0$ (Abramowitz and Stegun, 1964, equation 6.1.7). Given (75), we can define, for brevity

$$S_{N-2P} \equiv 2\binom{N-2P}{2} + 2\binom{N-2P}{1}. \tag{76}$$

Now (74) and (76) give

$$\nabla^2\mathbf{W}_{[N-2P]} = \frac{2}{r^2}\left\{\mathbf{W}_{[N-2P-2]}\delta_{[2]}\right\} - \frac{S_{N-2P}}{r^2}\mathbf{W}_{[N-2P]}. \tag{77}$$

Now, use of (77) gives

$$\begin{aligned}
\nabla^2\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} &= \left\{\left[\nabla^2\mathbf{W}_{[N-2P]}\right]\delta_{[2P]}\right\} \\
&= \left\{\left[\frac{2}{r^2}\left\{\mathbf{W}_{[N-2P-2]}\delta_{[2]}\right\} - \frac{S_{N-2P}}{r^2}\mathbf{W}_{[N-2P]}\right]\delta_{[2P]}\right\} \\
&= \frac{2}{r^2}\left\{\left\{\mathbf{W}_{[N-2P-2]}\delta_{[2]}\right\}\delta_{[2P]}\right\} - \frac{S_{N-2P}}{r^2}\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} \qquad (78) \\
&= \frac{R(N,P)}{r^2}\left\{\mathbf{W}_{[N-2(P+1)]}\delta_{[2(P+1)]}\right\} - \frac{S_{N-2P}}{r^2}\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} \qquad (79)
\end{aligned}$$

Noting the appearance of $\mathbf{W}_{[N-2P-2]}$ in (78) and recalling that $\mathbf{W}_{[N-2P-2]} = 0$ if $N-2P-2 < 0$, the coefficient $R(N,P)$ is defined by

$$R(N,P) \equiv 0 \text{ if } N-2P-2<0, \tag{80}$$

$$\text{otherwise, } R(N,P) \equiv 2\binom{N-2P}{2} \Big/ \left[\binom{N}{2P+2}(2P+1)!!\right]. \tag{81}$$

The coefficient $\binom{N-2P}{2}/\left[\binom{N}{2P+2}(2P+1)!!\right]$ is the number of terms in $\left\{\mathbf{W}_{[N-2P-2]}\delta_{[2]}\right\}$ divided by the number in $\left\{\mathbf{W}_{[N-2(P+1)]}\delta_{[2(P+1)]}\right\}$ as obtained from (48). Because of (42), (80)-(81) can be simplified to

$$R(N,P) \equiv \left[2^{P+1}(N-2P)!\,(P+1)!/N!\right]\Theta(N-2P-2), \tag{82}$$

where $\Theta(x) = 1$ for $x \geq 0$ and $\Theta(x) = 0$ for $x < 0$.

The Laplacian of the product of two functions f and g is (33). When applied to (59), the case $f = A_P(r)$ and $g = \left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ are needed. Recall that $\partial_{r_i}A(r) = \frac{r_i}{r}\partial_r A(r)$. The last term in (33) vanishes as follows: $(\partial_{r_i}f)(\partial_{r_i}g) = \left[\frac{1}{r}\partial_r A_P(r)\right]r_i\partial_{r_i}\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} = \left[\frac{1}{r}\partial_r A(r)\right]\left\{\left[r_i\partial_{r_i}\mathbf{W}_{[N-2P]}\right]\delta_{[2P]}\right\} = 0$; this vanishes because (52) shows that $r_i\partial_{r_i}\mathbf{W}_{[N-2P]} = 0$. Then (79) used in (33) combined with $\nabla^2 A(r) = \left(\partial_r^2 + \frac{2}{r}\partial_r\right)A(r)$ give

$$\begin{aligned}\nabla^2\left[A_P(r)\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}\right] &= \left[\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{S_{N-2P}}{r^2}\right)A_P(r)\right]\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} \\ &\quad + A_P(r)\frac{R(N,P)}{r^2}\left\{\mathbf{W}_{[N-2(P+1)]}\delta_{[2(P+1)]}\right\}.\end{aligned} \tag{83}$$

The Laplacian operation on (59) is simply the sum, $\sum_{P=0}^{M}$, of terms (83).

Appendix F: Matrix Algorithms

For computations, it is useful to write (21) as a matrix equation. Let the column index be $J \equiv P+1$, and the row index be $I \equiv (N_2/2)+1$, such that both J and I range from 1 to $M+1$ in (21). Use $N_3 = 0$ in (22)-(23) to define the following matrix elements

$$M_N(I,J) = 0, \text{ for } J < I, \text{ i.e., } M_N(I,J) = 0 \text{ below the main diagonal;} \tag{84}$$

$$M_N(I,J) = (N-2I+2)!\,(2I-2)!/\left[(N-2(J-1))!2^{J-1}(I-1)!\,(J-I)!\right], \text{ for } J \geq I. \tag{85}$$

The chosen $M+1$ linearly independent components of $\mathbf{D}_{[N]}$ are arranged in a column vector having $D_{[N:N-2I+2,2I-2,0]}$ in its I-th row, and the $M+1$ scalar functions $D_{N,P}$ are likewise arranged in a column vector having $D_{N,I-1}$ in its I-th row. Then (21) is written as the matrix equation

$$\begin{pmatrix} D_{[N:N,0,0]} \\ D_{[N:N-2,2,0]} \\ \vdots \\ D_{[N:N-2M,2M,0]} \end{pmatrix} = \begin{pmatrix} M_N(1,1) & M_N(1,2) & \cdots & M_N(1,M+1) \\ 0 & M_N(2,2) & \cdots & M_N(2,M+1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_N(M+1,M+1) \end{pmatrix} \begin{pmatrix} D_{N,0} \\ D_{N,1} \\ \vdots \\ D_{N,M} \end{pmatrix}. \quad (86)$$

Denote a matrix having matrix elements $A(I,J)$ by $\boxed{A(I,J)}$ Then (86) and its solution are (respectively)

$$\boxed{D_{[N:N-2J+2,2J-2,0]}} = \boxed{M_N(I,J)}\,\boxed{D_{N,I-1}}, \text{ and } \boxed{D_{N,I-1}} = \boxed{M_N(I,J)}^{-1}\boxed{D_{[N:N-2J+2,2J-2,0]}}, \quad (87)$$

where $\boxed{M_N(I,J)}^{-1}$ is the inverse of $\boxed{M_N(I,J)}$. The determinant of $\boxed{M_N(I,J)}$ is the product of its diagonal elements; from (85) that product is nonzero, hence $\boxed{M_N(I,J)}^{-1}$ exists. This inverse matrix is to be calculated numerically. In effect, evaluation of the components $D_{[N:N-2J+2,2J-2,0]}$ by means of experimental data or DNS data and use of the solution in (87) produces the $D_{N,P}$ for use in (21) to completely specify $\mathbf{D}_{[N]}$.

A matrix algorithm is useful for determining the isotropic formula for the first-order divergence $\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}$. By replacing N by $N+1$ and the symbol A by D in the divergence formula (63), we have

$$\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]} = \sum_{P=0}^{M'} \left\{\mathbf{W}_{[N-2(P-1)]}\delta_{[2(P-1)]}\right\} O_B(N+1,P)\, D_{N+1,P} + \sum_{P=0}^{M'} \left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\} O_C(P)\, D_{N+1,P}, \quad (88)$$

$$O_B(N+1,P) \equiv ((N+1)-2P+1)\left(\partial_r - \frac{(N+1)-2P}{r}\right), \quad (89)$$

$$O_C(P) \equiv \left[\partial_r + \frac{2(P+1)}{r}\right], \quad (90)$$

$$M' = N/2 \text{ if } N \text{ is even, and } M' = 1 + (N-1)/2 \text{ if } N \text{ is odd.} \quad (91)$$

The differential operators, i.e., $\partial_r \equiv \partial/\partial r$, in (89)-(90) are obtained from (61)-(62), and (91) is obtained by replacing N by $N+1$ in (20) and simplifying and rearranging the terms. Comparison of (91) with (20) shows that if N is even then $M' = M$; thus the matrix representation of $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ within (88) is the same as in (86), which representation was abbreviated by $\boxed{M_N(I,J)}$ above. On the other hand, if N is odd, then $M' = M + 1$, and the last column of the matrix representation of $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ within (88) corresponds to $P = M' = 1 + (N-1)/2$, in which case $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ contains $\mathbf{W}_{[-1]} = 0$ such that the last column of the matrix is zero. Thus, the matrix representation of $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$ within (88) is

$$\boxed{M_N^*(I,J)} = \begin{pmatrix} & 0 \\ \boxed{M_N(I,J)} & \vdots \\ & 0 \end{pmatrix} \text{ if } N \text{ is odd;}$$

$$\boxed{M_N^*(I,J)} = \boxed{M_N(I,J)} \text{ if } N \text{ is even.}$$

In addition to the coefficient $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$, (88) contains the coefficient $\left\{\mathbf{W}_{[N-2(P-1)]}\delta_{[2(P-1)]}\right\}$. From the matrix representation of $\left\{\mathbf{W}_{[N-2P]}\delta_{[2P]}\right\}$, namely (84)-(85), the matrix representation of $\left\{\mathbf{W}_{[N-2(P-1)]}\delta_{[2(P-1)]}\right\}$ is (recall that $J \equiv P + 1$)

for $J - 1 < I$, $M'_N(I,J) = 0$, i.e., $M'_N(I,J) = 0$ on and below the main diagonal;

whereas for $J \geq I$:

$$M_N(I,J) = (N - 2I + 2)!\,(2I-2)!/\left[(N - 2(J-1))!2^{J-1}(I-1)!\,(J-I)!\right].$$

The matrix having these elements is denoted by $\boxed{M'_N(I,J)}$. Because of (91), if N is odd, then the matrix $\boxed{M'_N(I,J)}$ contains the matrix $\boxed{M_N(I,J)}$ shifted to the right by one column and a first column of zeros is included; that is,

$$\boxed{M'_N(I,J)} = \begin{pmatrix} 0 & \\ \vdots & \boxed{M_N(I,J-1)} \\ 0 & \end{pmatrix} \text{ if } N \text{ is odd.}$$

Because of (91), the same is true if N is even except that the right-most column of $\boxed{M_N(I,J)}$ is discarded. Thus,

$$\boxed{M'_N(I,J)} = \begin{pmatrix} 0 & \\ \vdots & \boxed{M_N(I,J-1)} \\ 0 & \end{pmatrix} \quad \text{if } N \text{ is even (discard the right-most column).}$$

Define operator matrices that are of dimension $M'+1$ by $M'+1$, that have zeros off of the diagonal, and that have the operators (89)-(90) on the diagonals. Thus, recall that $J \equiv P+1$, and that $\partial_r \equiv \partial/\partial_r$, and define matrix elements

$$B(I,J) \equiv \delta_{IJ}(N-2J+4)\left(\partial_r - \frac{N-2J+3}{r}\right) \text{ and } C(I,J) \equiv \delta_{IJ}\left(\partial_r + \frac{2J}{r}\right). \tag{92}$$

The matrices corresponding to $O_B(N+1,P)$ and $O_C(P)$ in (89)-(90) are denoted by $\boxed{B(I,J)}$, and $\boxed{C(I,J)}$, respectively.

Let the components of $\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}$ be denoted by $\left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}\right)_{[N:N_1,N_2,N_3]}$, which denotes the fact that $\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}$ is a tensor of order N. In matrix notation, (88) gives

$$\begin{pmatrix} \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}\right)_{[N:N,0,0]} \\ \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}\right)_{[N:N-2,2,0]} \\ \vdots \\ \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}\right)_{[N:N-2M,2M,0]} \end{pmatrix} = \left[\boxed{M'_N(I,J)}\,\boxed{B(I,J)} + \boxed{M^*_N(I,J)}\,\boxed{C(I,J)}\right] \begin{pmatrix} D_{N+1,0} \\ D_{N+1,1} \\ \vdots \\ D_{N+1,M'} \end{pmatrix}. \tag{93}$$

The solution of (86) is $\boxed{D_{N+1,I-1}} = \boxed{M_{N+1}(I,J)}^{-1}\boxed{D_{[N+1:N+1-2J+2,2J-2,0]}}$ (when it is applied to $\mathbf{D}_{[N+1]}$), substitution of which into (93) gives

$$\begin{pmatrix} \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}\right)_{[N:N,0,0]} \\ \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}\right)_{[N:N-2,2,0]} \\ \vdots \\ \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}\right)_{[N:N-2M,2M,0]} \end{pmatrix} = \boxed{Y(I,J)} \begin{pmatrix} D_{[N+1:N+1,0,0]} \\ D_{[N+1:N+1-2,2,0]} \\ \vdots \\ D_{[N+1:N+1-2M',2M',0]} \end{pmatrix}, \tag{94}$$

$$\text{where, } \boxed{Y(I,J)} \equiv \left[\boxed{M'_N(I,J)}\,\boxed{B(I,J)} + \boxed{M^*_N(I,J)}\,\boxed{C(I,J)}\right]\boxed{M_{N+1}(I,J)}^{-1}. \tag{95}$$

We see that $\boxed{Y(I,J)}$ is the operator matrix that operates on the column matrix representation of $\mathbf{D}_{[N+1]}$ to produce $\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[N+1]}$; this is true for any completely symmetric isotropic tensor, not just true for $\mathbf{D}_{[N+1]}$.

It is helpful to illustrate this algorithm for $N = 2$ and $N = 3$. Two examples are needed because the algorithm differs for even N as compared to odd N. For $N = 2$, (95) is

$$\boxed{Y(I,J)} = \left[\begin{array}{c} \begin{pmatrix} 0 & M_2(1,1) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} B(1,1) & 0 \\ 0 & B(2,2) \end{pmatrix} + \\ + \begin{pmatrix} M_2(1,1) & M_2(1,2) \\ 0 & M_2(2,2) \end{pmatrix} \begin{pmatrix} C(1,1) & 0 \\ 0 & C(2,2) \end{pmatrix} \end{array} \right] \cdot \begin{pmatrix} M_3(1,1) & M_3(1,2) \\ 0 & M_3(2,2) \end{pmatrix}^{-1} \tag{96}$$

$$= \begin{pmatrix} \partial_r + \frac{2}{r} & -\frac{4}{r} \\ 0 & \partial_r + \frac{4}{r} \end{pmatrix}. \tag{97}$$

Computer evaluation of (96) produced (97). Consequently, (94) is

$$\begin{pmatrix} \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[3]}\right)_{[2:2,0,0]} \\ \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[3]}\right)_{[2:0,2,0]} \end{pmatrix} = \begin{pmatrix} \partial_r + \frac{2}{r} & -\frac{4}{r} \\ 0 & \partial_r + \frac{4}{r} \end{pmatrix} \begin{pmatrix} D_{[3:3,0,0]} \\ D_{[3:1,2,0]} \end{pmatrix} = \begin{pmatrix} \left(\partial_r + \frac{2}{r}\right) D_{111} - \frac{4}{r} D_{122} \\ \left(\partial_r + \frac{4}{r}\right) D_{122} \end{pmatrix},$$

where explicit-index notation is given at far right by use of $D_{[3:3,0,0]} \equiv D_{111}$ and $D_{[3:1,2,0]} \equiv D_{122}$.

For $N = 3$, $\boxed{Y(I,J)}$ from (95) is

$$\left[\begin{array}{c} \begin{pmatrix} 0 & M_3(1,1) & M_3(1,2) \\ 0 & 0 & M_3(2,2) \end{pmatrix} \begin{pmatrix} B(1,1) & 0 & 0 \\ 0 & B(2,2) & 0 \\ 0 & 0 & B(3,3) \end{pmatrix} + \\ \begin{pmatrix} M_3(1,1) & M_3(1,2) & 0 \\ 0 & M_3(2,2) & 0 \end{pmatrix} \begin{pmatrix} C(1,1) & 0 & 0 \\ 0 & C(2,2) & 0 \\ 0 & 0 & C(3,3) \end{pmatrix} \end{array} \right] \cdot$$

$$
\cdot \begin{pmatrix} M_4(1,1) & M_4(1,2) & M_4(1,3) \\ 0 & M_4(2,2) & M_4(2,3) \\ 0 & 0 & M_4(3,3) \end{pmatrix}^{-1}
$$

$$
= \begin{pmatrix} \partial_r + \frac{2}{r} & -\frac{6}{r} & 0 \\ 0 & \partial_r + \frac{4}{r} & -\frac{4}{3r} \end{pmatrix}
$$

As with (97), the matrix was evaluated using a computer program. Consequently, (94) is

$$
\begin{pmatrix} \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[4]}\right)_{[3:3,0,0]} \\ \left(\nabla_{\mathbf{r}} \cdot \mathbf{D}_{[4]}\right)_{[3:1,2,0]} \end{pmatrix} = \begin{pmatrix} \partial_r + \frac{2}{r} & -\frac{6}{r} & 0 \\ 0 & \partial_r + \frac{4}{r} & -\frac{4}{3r} \end{pmatrix} \begin{pmatrix} D_{[4:4,0,0]} \\ D_{[4:2,2,0]} \\ D_{[4:0,4,0]} \end{pmatrix} \tag{98}
$$

$$
= \begin{pmatrix} \left(\partial_r + \frac{2}{r}\right) D_{1111} - \frac{6}{r} D_{1122} \\ \left(\partial_r + \frac{4}{r}\right) D_{1122} - \frac{4}{3r} D_{2222} \end{pmatrix}, \tag{99}
$$

where explicit-index notation is used in (99).

A matrix algorithm is also needed for the Laplacian of a symmetric tensor. Performing the Laplacian of (59) and use of (83) gives

$$
\nabla_{\mathbf{r}}^2 \mathbf{D}_{[N]}(\mathbf{r}) = \sum_{P=0}^{M} \begin{pmatrix} \left\{\mathbf{W}_{[N-2P]} \delta_{[2P]}\right\} \left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{S_{N-2P}}{r^2}\right) D_{N,P} \\ + \left\{\mathbf{W}_{[N-2(P+1)]} \delta_{[2(P+1)]}\right\} \frac{R(N,P)}{r^2} D_{N,P} \end{pmatrix}. \tag{100}
$$

It is necessary to recall the definitions (76) and (82). The matrix representation of $\left\{\mathbf{W}_{[N-2P]} \delta_{[2P]}\right\}$ within (100) is the same as in (84)-(85), namely $\boxed{M_N(I,J)}$. The matrix representation of $\left\{\mathbf{W}_{[N-2(P+1)]} \delta_{[2(P+1)]}\right\}$ within (100) is obtained from (84)-(85) by replacing J by $J+1$, i.e.,

$$
\text{for } J+1 < I, \; M_N^{\#}(I,J) = 0,
$$

whereas for $J+1 \geq I$,

$$
M_N^{\#}(I,J) = (N-2I+2)!\,(2I-2)!/\left[(N-2J)!2^J\,(I-1)!\,(J+1-I)!\right]. \tag{101}
$$

This is just the square matrix that appears in (86) except that the left-most column in (86) is discarded and the matrix is then shifted leftward by one column and the right-most column is zeros. Those zeros appear because in the right-most column $J = P + 1 = M + 1$ such that $M_N^{\#}(I, M+1)$ contains the factor $1/(N - 2(M+1))!$ which is $1/(-2)! = 0$ if N is even and is $1/(-1)! = 0$ if N is odd(see Abramowitz and Stegun, 1964, equation 6.1.7). Thus,

$$\boxed{M_N^{\#}(I,J)} = \begin{pmatrix} & 0 \\ M_N(I, J+1) & \vdots \\ & 0 \end{pmatrix} \text{ (discard the left-most column).}$$

Define two operator matrices that are zero off the main diagonal and contain $\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{S_{N-2P}}{r^2}\right)$ and $\frac{R(N,P)}{r^2}$ on the main diagonal; i.e., their matrix elements are $E(I,J) = \delta_{IJ}\left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{S_{N-2(J-1)}}{r^2}\right)$ and $F(I,J) = \delta_{IJ} R(N, J-1)/r^2$. Analogous to the derivation of (94), the matrix representation of (100) is

$$\begin{pmatrix} \left(\nabla_{\mathbf{r}}^2 \mathbf{D}_{[N]}\right)_{[N:N,0,0]} \\ \left(\nabla_{\mathbf{r}}^2 \mathbf{D}_{[N]}\right)_{[N:N-2,2,0]} \\ \vdots \\ \left(\nabla_{\mathbf{r}}^2 \mathbf{D}_{[N]}\right)_{[N:N-2M,2M,0]} \end{pmatrix} = \boxed{X(I,J)} \begin{pmatrix} D_{[N:N,0,0]} \\ D_{[N:N-2,2,0]} \\ \vdots \\ D_{[N:N-2M,2M,0]} \end{pmatrix}, \tag{102}$$

where, $\boxed{X(I,J)} \equiv \left[\boxed{M_N(I,J)}\,\boxed{E(I,J)} + \boxed{M_N^{\#}(I,J)}\,\boxed{F(I,J)}\right] \boxed{M_N(I,J)}^{-1}$.

For both $N = 2$ and $N = 3$, the matrix representation of $\boxed{X(I,J)}$ is

$$\boxed{X(I,J)} \equiv \left[\begin{matrix} \begin{pmatrix} M_N(1,1) & M_N(1,2) \\ 0 & M_N(2,2) \end{pmatrix} \begin{pmatrix} E(1,1) & 0 \\ 0 & E(2,2) \end{pmatrix} \\ + \begin{pmatrix} M_N(1,2) & 0 \\ M_N(2,2) & 0 \end{pmatrix} \begin{pmatrix} F(1,1) & 0 \\ 0 & F(2,2) \end{pmatrix} \end{matrix} \right] \begin{pmatrix} M_N(1,1) & M_N(1,2) \\ 0 & M_N(2,2) \end{pmatrix}^{-1}$$

For $N = 2$ (102) is

$$\begin{pmatrix} \left(\nabla_{\mathbf{r}}^2 \mathbf{D}_{[2]}\right)_{[2:2,0,0]} \\ \left(\nabla_{\mathbf{r}}^2 \mathbf{D}_{[2]}\right)_{[2:0,2,0]} \end{pmatrix} = \boxed{X(I,J)} \begin{pmatrix} D_{[2:2,0,0]} \\ D_{[2:0,2,0]} \end{pmatrix} = \begin{pmatrix} \left(\nabla_{\mathbf{r}}^2 \mathbf{D}_{[2]}\right)_{11} \\ \left(\nabla_{\mathbf{r}}^2 \mathbf{D}_{[2]}\right)_{22} \end{pmatrix} =$$

$$\begin{pmatrix} \partial_r^2 + \frac{2}{r}\partial_r - \frac{4}{r^2} & \frac{4}{r^2} \\ \frac{2}{r^2} & \partial_r^2 + \frac{2}{r}\partial_r - \frac{2}{r^2} \end{pmatrix} \begin{pmatrix} D_{11} \\ D_{22} \end{pmatrix} = \begin{pmatrix} \left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{4}{r^2}\right) D_{11} + \frac{4}{r^2} D_{22} \\ \frac{2}{r^2} D_{11} + \left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{2}{r^2}\right) D_{22} \end{pmatrix}, \tag{103}$$

where the matrix was evaluated using a computer program. For $N = 3$ the matrix algorithm is

$$\begin{pmatrix} \left(\nabla_{\mathbf{r}}^2 \mathbf{D}_{[3]}\right)_{111} \\ \left(\nabla_{\mathbf{r}}^2 \mathbf{D}_{[3]}\right)_{122} \end{pmatrix} = \boxed{X(I,J)} \begin{pmatrix} D_{111} \\ D_{122} \end{pmatrix} = \begin{pmatrix} \partial_r^2 + \frac{2}{r}\partial_r - \frac{6}{r^2} & \frac{12}{r^2} \\ \frac{2}{r^2} & -\frac{8}{r^2} + \partial_r^2 + \frac{2}{r}\partial_r \end{pmatrix} \begin{pmatrix} D_{111} \\ D_{122} \end{pmatrix}$$
$$= \begin{pmatrix} \left(\partial_r^2 + \frac{2}{r}\partial_r - \frac{6}{r^2}\right) D_{111} + \frac{12}{r^2} D_{122} \\ \frac{2}{r^2} D_{111} + \left(-\frac{8}{r^2} + \partial_r^2 + \frac{2}{r}\partial_r\right) D_{122} \end{pmatrix}. \qquad (104)$$