

Internet Appendix for “Fast Filtering with Large Option Panels”

A. Option Valuation

We use a strike-optimized method based on the formula of [Carr and Madan \(1998\)](#). The time- t price of a call option with strike K and maturity τ can be written as:

$$(A.1) \quad C_t = \int_k^\infty e^{-r\tau} (e^{s_{t+\tau}} - e^k) f_s(s_{t+\tau}) ds_{t+\tau},$$

where $s_t = \log(S_t)$ and $k = \log(K)$. In the above formula, as k approaches $-\infty$, the call option price converges to S_t rather than zero and thus is not square-integrable. Carr and Madan introduce a dampening factor α to solve the problem and let:

$$(A.2) \quad c = e^{\alpha k} C.$$

Now the Fourier transform can be applied to c :

$$(A.3) \quad \begin{aligned} \psi(u) &= \int_{-\infty}^{\infty} e^{iuk} c dk, \\ &= \int_{-\infty}^{\infty} e^{iuk} e^{\alpha k} \int_k^\infty e^{-r\tau} (e^{s_{t+\tau}} - e^k) f_s(s_{t+\tau}) ds_{t+\tau} dk, \\ &= \int_{-\infty}^{\infty} e^{-r\tau} f_s(s_{t+\tau}) \int_{-\infty}^{s_{t+\tau}} [e^{s_{t+\tau} + (\alpha + iu)k} - e^{(\alpha + 1 + iu)k}] dk ds_{t+\tau}, \\ &= \int_{-\infty}^{\infty} e^{-r\tau} f_s(s_{t+\tau}) \left[\frac{e^{s_{t+\tau}(\alpha + 1 + iu)}}{(\alpha + iu)(\alpha + 1 + iu)} \right] ds_{t+\tau}, \\ &= \frac{e^{-r\tau}}{(\alpha + iu)(\alpha + 1 + iu)} \int_{-\infty}^{\infty} e^{(\alpha + 1 + iu)s_{t+\tau}} f_s(s_{t+\tau}) ds_{t+\tau}, \\ &= \frac{e^{-r\tau} \phi_{s_{t+\tau}}(\alpha + 1 + iu)}{(\alpha + iu)(\alpha + 1 + iu)}, \end{aligned}$$

where ϕ_{s_t} denotes the risk neutral characteristic function of the log-price. Dropping the notation for dependence on t , the call option value is given by:

$$(A.4) \quad C = e^{\alpha k} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuk} \psi(u) du.$$

Since the imaginary part of $\psi(u)$ is odd and the real part is even, Equation (A.4) can be further simplified as

$$(A.5) \quad C = \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} \text{Re}[e^{-iuk} \psi(u)] du.$$

Carr and Madan (1998) evaluate Equation (A.5) with the fast Fourier transform algorithm.

However, given the availability of improved computing power, we instead directly solve equation (A.5) with a numerical integration method (See Crisóstomo (2018) for a comparison of the two methods in terms of speed and accuracy). We vectorize equation (A.5) with respect to the strikes and we apply Simpson's rule for computing the call prices.

B. The Orthogonal MCMC Algorithm

In this section, we present the implementational details of the Orthogonal MCMC (O-MCMC) algorithm. In a nutshell, the algorithm consists in running Z MCMC algorithms in parallel and makes the Markov chains interacting at some iterations, called the horizontal moves. It starts from a population of Z initial values $\{\Theta_1^1, \dots, \Theta_Z^1\}$, which are the initial states of the Z Markov chains. Then the algorithm applies M iterations in which it alternates between vertical and horizontal moves. For each of the Z chains, the

vertical moves update the current state independently of the other chains by using a standard adaptive random-walk (RW) sampler. The horizontal moves update the states of all the chains at once using the simple Metropolis-Hasting sampler, a method that implies interaction between the Markov chains. The O-MCMC algorithm is detailed in the algorithm below.

The O-MCMC algorithm

For $z = 1, \dots, Z$, initialize the state Θ_z of the chain, the RW covariance matrix Σ_z , the number α_z of accepted draws to zero ($\alpha_z = 0$), the type of proposal distribution of the chain $\text{AF}_z = \text{mod}(z, 2)$ and set $\text{RW}_{\text{AF},z} = 2.38^2/d$ in which d is the number of model parameters.

for iter=1 to M **do**

If $\text{mod}(\text{iter},10)=0$, then do a horizontal move at this iteration. Else, do a vertical move.

Vertical move

for $z=1$ to Z **do**

Set $\Theta_z^{\text{iter}} = \Theta_z^{\text{iter}-1}$ and sample $\eta_z \sim N(0, I_d)$.

Compute a proposal $\tilde{\Theta}_z = \Theta_z^{\text{iter}-1} + C_z \eta_z$ in which $C_z C_z' = (1 + \text{AF}_z(\text{RW}_{\text{AF},z} - 1))\Sigma_z$.

Accept the proposal draw $\tilde{\Theta}_z$ according to the Metropolis acceptance ratio:

$$(B.1) \quad \min \left[\left(\frac{f(R_{1:T}, C_{1:T} | \tilde{\Theta}_z) f(\tilde{\Theta}_z)}{f(R_{1:T}, C_{1:T} | \Theta_z^{\text{iter}-1}) f(\Theta_z^{\text{iter}-1})} \right), 1 \right]$$

if the proposal draw has been accepted, **then** $\Theta_z^{\text{iter}} = \tilde{\Theta}_z$ and set $\alpha_z = \alpha_z + 1$, .

if $\text{AF}_z=1$, **then** update the covariance matrix Σ_z according to [Atchadé and Fort \(2010\)](#):

Set Σ_z to the empirical covariance of $\{\Theta_z^1, \dots, \Theta_z^{\text{iter}}\}$ and $\text{RW}_{\text{AF},z} = \max \left[\text{RW}_{\text{AF},z} + \frac{(\frac{\alpha_z}{\text{iter}} - 0.33)}{\text{iter}^{0.6}}, 0.0001 \right]$

else Update the covariance matrix Σ_z according to Robust Adaptive Metropolis (RAM):

Set $\eta_{\text{RAM}} = \min[1, \frac{d}{\text{iter}^{3/2}}]$ and $\Sigma_z = C_z(I_d + \frac{\eta_{\text{RAM}}(\frac{\alpha_z}{\text{iter}} - 0.33)}{\eta_z' \eta_z} \eta_z \eta_z') C_z'$

end if

end for

Horizontal move

Compute the mean $\bar{\mu}$ and the empirical covariance matrix $\bar{\Sigma}$ of $\{\Theta_1^1, \dots, \Theta_Z^{\text{iter}}\}$.

for $z=1$ to Z **do**

Draw a proposal candidate $\Theta_0^{\text{iter}} \sim N(\bar{\mu}, \bar{\Sigma})$.

Select $k \in \{1, \dots, Z\}$ with probability proportional to $\gamma_k \propto \frac{f_N(\Theta_z^{\text{iter}} | \bar{\mu}, \bar{\Sigma})}{f(R_{1:T}, C_{1:T} | \Theta_z^{\text{iter}}) f(\Theta_z^{\text{iter}})}$ where $f_N(\cdot | \bar{\mu}, \bar{\Sigma})$ is the normal density with expectation and covariance $\bar{\mu}, \bar{\Sigma}$, respectively.

Set $\Theta_k^{\text{iter}} = \Theta_0^{\text{iter}}$ if the draw is accepted according to the probability:

$$(B.2) \quad \min \left[\left(\frac{\sum_{z=1}^Z \frac{f_N(\Theta_z^{\text{iter}} | \bar{\mu}, \bar{\Sigma})}{f(R_{1:T}, C_{1:T} | \Theta_z^{\text{iter}}) f(\Theta_z^{\text{iter}})}}{\sum_{z=0}^Z \frac{f_N(\Theta_z^{\text{iter}} | \bar{\mu}, \bar{\Sigma})}{f(R_{1:T}, C_{1:T} | \Theta_z^{\text{iter}}) f(\Theta_z^{\text{iter}})}} - \min_{i \in 0, \dots, Z} \frac{f_N(\Theta_i^{\text{iter}} | \bar{\mu}, \bar{\Sigma})}{f(R_{1:T}, C_{1:T} | \Theta_i^{\text{iter}}) f(\Theta_i^{\text{iter}})} \right), 1 \right]$$

end for

Burn-in period

If $\text{mod}(\text{iter},10)=0$ and $\text{iter} \leq 0.4M$, then replace the current state Θ_k^{iter} and the chain attributes ($\text{AF}_k, \Sigma_k, \dots$) of the least performing chain (i.e., $k = \text{argmin}_{i \in 1, \dots, Z} f(R_{1:T}, C_{1:T} | \Theta_i) f(\Theta_i)$) with the state and the attributes of the best performing chain.

end for

C. Likelihood Evaluation Using The Particle Filter

This section details the particle filter used for evaluating the integrated log-likelihood function of the SVCJ model with parameter $\Theta = \{\kappa, \theta, \sigma, \rho, \eta_s, \eta_v, \lambda, \mu_s, \sigma_s, \eta_{J^s}, \mu_v, \eta_{J^v}, \rho_J, \sigma_c\}$. Let us denote the latent variables at time t by $L_t = (V_t, B_t, J_t^v)$ ²⁹. The particle filter is based on the following decomposition of the posterior distribution of the latent states:

$$\begin{aligned} \pi(L_{0:T}|\Theta, R_{1:T}, C_{1:T}) &\propto f(R_{1:T}, C_{1:T}, L_{0:T}|\Theta), \\ \text{(C.1)} \qquad \qquad \qquad &= f(R_1|L_0, \Theta) \prod_{t=1}^T f(R_{t+1}, C_t|L_t, \Theta) f(L_t|L_{t-1}, R_t, \Theta). \end{aligned}$$

Relying on this decomposition, a standard particle filter samples the latent variables from $L_t|L_{t-1}, R_t, \Theta$ and then evaluates the likelihood function $f(R_{t+1}, C_t|L_t, \Theta)$ which can be decomposed into $f(R_{t+1}|L_t, \Theta)$ and $f(C_t|L_t, \Theta)$ as explained in Section C.

A. Particle filter

The section details the particle filter of the SVCJ model which deals with both options and returns. To estimate the SVCJ model with returns only, it suffices to skip the option likelihood evaluation (i.e., Step 3 (b)) and the step 5 (b) on the particle rejuvenation. In this case, the risk-neutral parameters $\{\eta_v, \eta_{J^s}, \eta_{J^v}\}$ are set to zero which implies that $\bar{\mu}_s^Q = \bar{\mu}_s$.

The particle filter operates as follows. For $i = 1, \dots, N$, sample the latent variable $L_0^i = (V_0^i, B_0^i, (J_0^v)^i)$ from the prior distribution. Set $\pi_0^i = \frac{1}{N}$ and $\Delta t = \frac{1}{252}$. Then, for each time t from $t = 1, \dots, T$, do for $i = 1, \dots, N$:

²⁹Note that in contrast to the previous exposition of the latent state $L_t = (V_t, B_t, J_t^s, J_t^v)$ we drop here the jump in mean state J_t^s because it can be directly integrated out rather than simulated.

1. Step 1: Resampling

The set $\{\pi_{t-1}^i\}_{i=1}^N$ constitute a discrete probability distribution of

$L_{t-1} = (V_{t-1}, J_{t-1}^v, B_{t-1})$ from which we resample using a stratified method. The

resampled $\{L_{t-1}^i\}_{i=1}^N$ as well as its ancestors are stored for the next period.

2. Step 2: Simulating the State Forward

(a) Simulate the jumps at time t :

$$(C.2) \quad f(B_t = 1 | R_t, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta) = \frac{\lambda \Delta t f(R_t | B_t = 1, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta)}{\lambda \Delta t f(R_t | B_t = 1, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta) + (1 - \lambda \Delta t) f(R_t | B_t = 0, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta)}$$

in which the function $f(R_t | B_t = 1, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta)$ is an exponentially

modified gaussian density function as shown in Section B while

$f(R_t | B_t = 0, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta)$ stands for a standard Normal density. In

particular, $R_t | B_t = 0, V_{t-1}^i, B_{t-1}, (J_{t-1}^v)^i, \Theta \sim$

$$N\left((r_{t-1} - \delta_{t-1} + (\eta_s - \frac{1}{2})V_{t-1}^i - \lambda \bar{\mu}_s^Q) \Delta t, \Delta t V_{t-1}^i\right).$$

(b) If $B_t^i = 1$, simulate the jumps in volatility at time t :

$$(C.3) \quad f((J_t^v)^i | V_{t-1}^i, B_t^i = 1, R_t, \Theta) \propto f(R_t | (J_t^v)^i, V_{t-1}^i, B_t = 1, \Theta) f((J_t^v)^i | \Theta),$$

in which $R_t | (J_t^v)^i, V_{t-1}^i, B_t^i = 1, \Theta \sim$

$N((r_{t-1} - \delta_{t-1} + (\eta_s - \frac{1}{2})V_{t-1}^i - \lambda \bar{\mu}_s^Q) \Delta t + (\mu_s + \rho_J (J_t^v)^i), \Delta t V_{t-1}^i + \sigma_s^2)$ and

$$f((J_t^v)^i | \Theta) = \frac{\exp(-(J_t^v)^i / \mu_v)}{\mu_v}.$$

(c) Compute the error term w_t^i given by

$$(C.4) \quad w_t^i = R_t - (r_{t-1} - \delta_{t-1} + (\eta_s - \frac{1}{2})V_{t-1}^i - \lambda \bar{\mu}_s^Q) \Delta t - (\mu_s + \rho_J (J_t^v)^i) B_t^i$$

(d) Simulate the spot variances at time t :

(C.5)

$$V_t^i \sim N\left(V_{t-1}^i + \kappa(\theta - V_{t-1}^i)\Delta t + \frac{\sigma\rho\Delta t V_{t-1}^i}{\sigma_s^2 B_t^i + \Delta t V_{t-1}^i} w_t^i + (J_t^v)^i B_t^i, \sigma^2 \Delta t V_{t-1}^i \left(1 - \frac{\rho^2(\Delta t V_{t-1}^i)}{\sigma_s^2 B_t^i + \Delta t V_{t-1}^i}\right)\right)$$

3. Step 3: Computing the unnormalized weights

For each particle $i = 1, \dots, N$, compute the unnormalized weights according to the likelihood:

$$\tilde{\omega}^i = f(R_{t+1}|L_t^i, \Theta) f(C_t|L_t^i, \Theta).$$

(a) The likelihood related to the asset returns, i.e. $f(R_{t+1}|L_t^i, \Theta)$, is given by

(C.6)

$$f(R_{t+1}|L_t, \Theta) = (1 - \lambda\Delta t)f(R_{t+1}|B_{t+1} = 0, V_t, B_t, J_t^v, \Theta) + (\lambda\Delta t)f(R_{t+1}|B_{t+1} = 1, V_t, B_t, J_t^v, \Theta),$$

with

(C.7)

$$f(R_{t+1}|B_{t+1} = 0, V_t, B_t, J_t^v, \Theta) = \frac{\exp\left(\frac{(R_{t+1} - (r_t - \delta_t + (\eta_s - \frac{1}{2})V_t - \lambda_J \bar{\mu}_s^Q)\Delta t)^2}{2V_t \Delta t}\right)}{\sqrt{2\pi(V_t \Delta t)}},$$

$$f(R_{t+1}|B_{t+1} = 1, V_t, B_t, J_t^v, \Theta) = \frac{A_\lambda \exp\left(\frac{A_\lambda}{2}(A_\lambda A_\sigma^2 - 2\text{sgn}(\rho_J)(R_{t+1} - A_\mu))\right)}{2} \text{erfc}\left(\frac{\text{sgn}(\rho_J)(A_\mu - R_{t+1}) + A_\sigma^2 A_\lambda}{\sqrt{2A_\sigma^2}}\right),$$

in which $A_\lambda = |\rho_J| \mu_v$, $A_\mu = (r_t - \delta_t + (\eta_s - \frac{1}{2})V_t - \lambda \bar{\mu}_s^Q)\Delta t + \mu_s$ and $A_\sigma^2 = \Delta t V_t + \sigma_s^2$.

(b) The likelihood related to the option prices, i.e. $f(C_t|L_t^i, \Theta)$, is given by

$$(C.8) \quad f(C_t|L_t^i, \Theta) = \left(\prod_{h=1}^{H_t} \frac{1}{\sqrt{2\pi\sigma_c^2}} \exp\left\{-\frac{1}{2} \frac{[C_{t,h} - C_{t,h}^M(V_t^i|\Theta)]^2}{\sigma_c^2}\right\}\right)^{\frac{1}{H_t}}.$$

Note that this step is computationally intensive. In particular, for each particle $i = 1, \dots, N$ and for each option $h = 1, \dots, H_t$, the option price $C_{t,h}^M(V_t^i|\Theta)$ must be computed. To reduce the computational burden, we apply the quantile spot variance method. It consists of evaluating the option prices for Q spot variances (that are evenly-spaced quantiles of the spot variance particles) and then interpolating the

option prices of the other spot variances using a polynomial interpolation function, see Section D for more details.

4. Step 4: Assessing the particle approximation quality

Check the effective sample size (ESS):

$$(C.9) \quad \text{ESS} = \frac{\left(\sum_{j=1}^N \tilde{\omega}^j\right)^2}{\sum_{i=1}^N (\tilde{\omega}^i)^2}.$$

If ESS is larger than a targeted ESS (set to $0.5N$ in our implementation), then go to step 5 (a). Otherwise, the particles do not correctly approximate the current distribution. In such a case, skip step 5(a) and go to step 5 (b) to address the problem.

5. Step 5 (a): Computing the likelihood and Normalizing the Weights

Compute the unbiased approximation of the integrated likelihood function as:

$$(C.10) \quad f(R_{t+1}, C_t | \Theta, R_{1:t}, C_{1:t-1}) \approx \frac{1}{N} \sum_{i=1}^N \tilde{\omega}^i.$$

Then, compute the normalized weights π_t^i as follows:

$$(C.11) \quad \pi_t^i = \frac{\tilde{\omega}_i}{\sum_{j=1}^N \tilde{\omega}^j}$$

Skip Step 5(b).

6. Step 5 (b): Rejuvenating the particles using several MCMC algorithms

Build a sequence of $k = 1, \dots, K$ bridging distributions,

$$(C.12) \quad \pi_{\phi_k}(L_t | R_{1:t+1}, C_{1:t}, \Theta) = \frac{[f(R_{t+1}, C_t | L_t, \Theta)]^{\phi_k} f(L_t | R_{1:t}, C_{1:t-1}, \Theta)}{f_{\phi_k}(R_{t+1}, C_t | R_{1:t}, C_{1:t-1}, \Theta)},$$

in which $f_{\phi_k}(R_{t+1}, C_t | R_{1:t}, C_{1:t-1}, \Theta) = \int [f(R_{t+1}, C_t | L_t, \Theta)]^{\phi_k} f(L_t | R_{1:t}, C_{1:t-1}, \Theta) dL_t$, i.e. the normalizing constant, and ϕ_k stands for an increasing function with $\phi_1 = 0$ and $\phi_K = 1$ such that, following [Jasra, Stephens, Doucet, and Tsagaris \(2011\)](#), ESS is targeted at $0.5N$. Note that $f_{\phi_1}(R_{t+1}, C_t | R_{1:t}, C_{1:t-1}, \Theta) = 1$ when $\phi_1 = 0$.

For each distribution of the sequence, i.e. for $k = 2, \dots, K$,

(a) Compute the unnormalized weights:

$$\tilde{\omega}^{i, \phi_k} = (f(R_{t+1} | L_t^i, \Theta) f(C_t | L_t^i, \Theta))^{\phi_k - \phi_{k-1}}.$$

(b) Compute the ratio of normalizing constants:

$$\frac{f_{\phi_k}(R_{t+1}, C_t | R_{1:t}, C_{1:t-1}, \Theta)}{f_{\phi_{k-1}}(R_{t+1}, C_t | R_{1:t}, C_{1:t-1}, \Theta)} \approx \frac{1}{N} \sum_{i=1}^N \tilde{\omega}^{i, \phi_k}.$$

(c) Compute the normalized weights π_t^{i, ϕ_k} as follows:

$$(C.13) \quad \pi_t^{i, \phi_k} = \frac{\tilde{\omega}^{i, \phi_k}}{\sum_{j=1}^N \tilde{\omega}^{j, \phi_k}}.$$

(d) Re-sample the particles according to the normalized weights $\{\pi_t^{i, \phi_k}\}_{i=1}^N$ and keep track of the particle ancestors that remain active.

(e) Apply M iterations of an MCMC algorithm exhibiting $\pi_{\phi_k}(L_t | \Theta, R_{1:t+1}, C_{1:t})$ as invariant distribution. First, set $\sigma_Q^2 = \hat{\text{Var}}(V_t^i)$ in which $\hat{\text{Var}}(V_t^i)$ stands for the empirical variance of the particles. Then, For $m = 1, \dots, M$, apply the following MCMC steps:

i. For $i = 1, \dots, N$, draw proposal candidates $\tilde{V}_t^i \sim N(V_t^i, \sigma_Q^2)$,

$\tilde{B}_t^i \sim \text{Bern}(\lambda \Delta t)$ and $(\tilde{J}_t^v)^i \sim \exp(\mu_v)$.

ii. For $i = 1, \dots, N$, accept/reject the proposal draws according to the

Metropolis acceptance probability:

$$(C.14) \quad \min \left[\left(\frac{f(R_{t+1}, C_t | \tilde{V}_t^i, \tilde{B}_t^i, (\tilde{J}_t^v)^i, \Theta)}{f(R_{t+1}, C_t | V_t^i, B_t^i, (J_t^v)^i, \Theta)} \right)^{\phi_k} \frac{f(\tilde{V}_t^i | V_{t-1}^i, \tilde{B}_t^i, (\tilde{J}_t^v)^i, R_t, \Theta)}{f(V_t^i | V_{t-1}^i, B_t^i, (J_t^v)^i, R_t, \Theta)}, 1 \right].$$

When we have approximated all the distributions in the sequence, we end up with a set of particles that approximates the distribution $f(L_t | R_{1:t+1}, C_{1:t}, \Theta)$. The integrated likelihood at time t is computed as

$$(C.15) \quad \begin{aligned} f(R_{t+1}, C_t | \Theta, R_{1:t}, C_{1:t-1}) &= \prod_{k=1}^K \frac{f_{\phi_k}(R_{t+1}, C_t | R_{1:t}, C_{1:t-1}, \Theta)}{f_{\phi_{k-1}}(R_{t+1}, C_t | R_{1:t}, C_{1:t-1}, \Theta)}, \\ &\approx \left(\prod_{k=1}^K \frac{1}{N} \sum_{i=1}^N \tilde{\omega}^{i, \phi_k} \right), \end{aligned}$$

where the approximation is unbiased as shown in [Herbst and Schorfheide \(2019\)](#). To end this step of the particle filter, set the normalized weights to $\pi_t^i = \frac{1}{N}$ for each particle $i = 1, \dots, N$.

B. Analytical expressions for the distributions used in the particle filter

Transition distribution $L_t | L_{t-1}, R_t, \Theta$ used in Step 2.

The SIR algorithm requires to sample the state variable L_t from the transition distribution $L_t | L_{t-1}, R_t, \Theta$. The transition density can be expressed as,

$$(C.16) \quad \begin{aligned} f(L_t | L_{t-1}, R_t, \Theta) &= f(V_t, B_t, J_t^v | V_{t-1}, B_{t-1}, J_{t-1}^v, R_t, \Theta), \\ &= f(V_t | B_t, J_t^v, V_{t-1}, B_{t-1}, J_{t-1}^v, R_t, \Theta) f(J_t^v | R_t, B_t, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta) f(B_t | R_t, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta). \end{aligned}$$

Therefore, we first sample B_t , then J_t^v and finally V_t . The sampling of each latent variable is detailed below.

Sampling B_t :

The posterior density of B_t is given by

$$(C.17) \quad f(B_t = 1 | R_t, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta) = \frac{\lambda \Delta t f(R_t | B_t = 1, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta)}{\lambda \Delta t f(R_t | B_t = 1, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta) + (1 - \lambda \Delta t) f(R_t | B_t = 0, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta)}$$

in which we use that fact that B_t is independent of L_{t-1} . The function

$f(R_t | B_t = 1, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta)$ is a exponentially modified gaussian density function as shown below while $f(R_t | B_t = 0, V_{t-1}, B_{t-1}, J_{t-1}^v, \Theta)$ stands for a standard Normal density.

Sampling J_t^v :

The variance jump distribution when $B_t = 1$ is given by

$$(C.18) \quad f(J_t^v | V_{t-1}, B_t = 1, R_t, \Theta) = \frac{f(R_t | J_t^v, V_{t-1}, B_t = 1, \Theta) f(J_t^v | \Theta)}{\int f(R_t | J_t^v, V_{t-1}, B_t = 1, \Theta) f(J_t^v | \Theta) dJ_t^v},$$

in which $\int f(R_t | J_t^v, V_{t-1}, B_t = 1, \Theta) f(J_t^v | \Theta) dJ_t^v$ is known as an exponentially modified Gaussian density and stands for the normalizing constant of the distribution. We sample the jump size using a Griddy-Gibbs approach using a grid of 200 equally-spaced values.

Sampling V_t :

The spot variance is sampled from the distribution conditional on the

contemporaneous return:

$$(C.19) \quad V_t | V_{t-1}, B_t, J_t^v, R_t, \Theta \sim N(V_{t-1} + \kappa(\theta - V_{t-1})\Delta t + \rho\sigma\sqrt{V_{t-1}\Delta t} w_t + J_t^v B_t, \sigma^2 \Delta t V_{t-1} (1 - \rho^2)),$$

in which $w_t = \frac{R_t - (r_{t-1} - \delta_{t-1} + (\eta_s - \frac{1}{2})V_{t-1} - \lambda_J \bar{\mu}_s^Q)\Delta t - (\mu_s + \rho_J J_t^v)B_t}{\sqrt{\Delta t V_{t-1} + \sigma_s^2 B_t}}$. Note that we do not sample the spot variance from its prior distribution (i.e., $V_t | V_{t-1}, J_t^v, B_t, \Theta$) as it is the case in standard SIR algorithms. Due to the timing convention adopted in this paper, the spot variance is sampled from a distribution that accounts for the contemporaneous asset return. In that regard, this particle filter does not blindly sample the latent variables and is similar to an auxiliary particle filter (without the need of augmenting the posterior space with an auxiliary variable).

Likelihood evaluation given the latent variables used in Step 3.

The likelihood function $f(R_{t+1}, C_t | L_t, \Theta)$ is decomposed into $f(R_{t+1} | L_t, \Theta)$ and $f(C_t | L_t, \Theta)$. Since the options in $C_t = (C_{t,1}, \dots, C_{t,H_t})$ are conditionally independent, $f(C_t | L_t, \Theta)$ is simply a product of Gaussian distributions. The evaluation of $f(R_{t+1} | L_t, \Theta)$, however, requires integrating out the latent variable L_{t+1} . In particular, we have

$$\begin{aligned}
(C.20) \quad f(R_{t+1} | L_t, \Theta) &= \int f(R_{t+1}, L_{t+1} | L_t, \Theta) dL_{t+1}, \\
&= \int f(R_{t+1} | L_{t+1}, L_t, \Theta) f(L_{t+1} | L_t, \Theta) dL_{t+1}, \\
&= \int f(R_{t+1} | B_{t+1}, J_{t+1}^v, V_t, B_t, J_t^v, \Theta) f(B_{t+1}, J_{t+1}^v | V_t, B_t, J_t^v, \Theta) dB_{t+1} dJ_{t+1}^v, \\
&= \int f(R_{t+1} | B_{t+1}, J_{t+1}^v, V_t, B_t, J_t^v, \Theta) f(B_{t+1} | \Theta) dB_{t+1} f(J_{t+1}^v | \Theta) dJ_{t+1}^v, \\
&= (1 - \lambda \Delta t) f(R_{t+1} | B_{t+1} = 0, V_t, B_t, J_t^v, \Theta) \\
&\quad + (\lambda \Delta t) \int f(R_{t+1} | B_{t+1} = 1, J_{t+1}^v, V_t, B_t, J_t^v, \Theta) f(J_{t+1}^v | \Theta) dJ_{t+1}^v,
\end{aligned}$$

where we use the fact that the density of R_{t+1} does not depend on J_{t+1}^v when there is no jump at time $t + 1$. Analytical expressions for $f(R_{t+1} | B_{t+1} = 0, V_t, B_t, J_t^v, \Theta)$ and $\int f(R_{t+1} | B_{t+1} = 1, J_{t+1}^v, V_t, B_t, J_t^v, \Theta) f(J_{t+1}^v | \Theta) dJ_{t+1}^v$ are obtained as follows.

1. Analytical expression of $f(R_{t+1} | B_{t+1} = 0, V_t, B_t, J_t^v, \Theta)$.

$$(C.21) \quad f(R_{t+1} | B_{t+1} = 0, V_t, B_t, J_t^v, \Theta) = \frac{\exp\left(\frac{(R_{t+1} - (r_t - \delta_t + (\eta_s - \frac{1}{2})V_t - \lambda \bar{\mu}_s^Q)\Delta t)^2}{2V_t \Delta t}\right)}{\sqrt{2\pi(V_t \Delta t)}}.$$

2. Analytical expression of $\int f(R_{t+1} | B_{t+1} = 1, J_{t+1}^v, V_t, B_t, J_t^v, \Theta) f(J_{t+1}^v | \Theta) dJ_{t+1}^v$.

The distribution of the first term in the integral is a standard normal distribution:

$$(C.22) \quad R_{t+1} | B_{t+1} = 1, J_{t+1}^v, V_t, B_t, J_t^v, \Theta \sim N\left((r_t - \delta_t + (\eta_s - \frac{1}{2})V_t - \lambda \bar{\mu}_s^Q)\Delta t + \mu_{J_s} + \text{sgn}(\rho_J) |\rho_J| J_{t+1}^v, V_t \Delta t + \sigma_s^2\right),$$

where $\text{sgn}(\rho_J)$ denotes the sign of ρ_J and $|\cdot|$ denotes the absolute value operator. To integrate out the variance jump, note that $|\rho_J| J_{t+1}^v \sim \exp(|\rho_J| \mu_v)$. Therefore, when

we integrate out the exponential distribution, we end up with an exponentially modified Gaussian density function. To facilitate the notation, we derive the density function for two random variables given by $X \sim N(\mu + \text{sgn}(\rho_J)J, \sigma^2)$ and $J \sim \exp(\alpha)$.

In our setting, those parameters are equal to $\alpha = |\rho_J|\mu_v$,

$\mu = (r_t - \delta_t + (\eta_s - \frac{1}{2})V_t - \lambda\bar{\mu}_s^Q)\Delta t + \mu_s$ and $\sigma^2 = \Delta tV_t + \sigma_s^2$. The exponentially

modified Gaussian density is given by

$$\begin{aligned}
f(x|\mu, \sigma^2, \alpha) &= \int_0^\infty \frac{\exp(-\frac{(x-\mu-\text{sgn}(\rho_J)J)^2}{2\sigma^2})}{\sqrt{2\pi\sigma^2}} \alpha \exp(-\alpha J) dJ, \\
&= \frac{\alpha}{\sqrt{2\pi\sigma^2}} \int_0^\infty \exp(-\frac{(x-\mu)^2 + J^2 - 2(\text{sgn}(\rho_J)(x-\mu) - \sigma^2\alpha)J}{2\sigma^2}) dJ, \\
&= \frac{\alpha \exp(-\frac{(x-\mu)^2}{2\sigma^2})}{\sqrt{2\pi\sigma^2}} \int_0^\infty \exp(-\frac{J^2 - 2(\text{sgn}(\rho_J)(x-\mu) - \sigma^2\alpha)J}{2\sigma^2}) dJ, \\
&= \frac{\alpha \exp(-\frac{(x-\mu)^2}{2\sigma^2})}{\sqrt{2\pi\sigma^2}} \int_0^\infty \exp(-\frac{(J - (\text{sgn}(\rho_J)(x-\mu) - \sigma^2\alpha))^2 + (\text{sgn}(\rho_J)(x-\mu) - \sigma^2\alpha)^2}{2\sigma^2}) dJ, \\
\text{(C.23)} \quad &= \frac{\alpha \exp(-\frac{(x-\mu)^2 + (\text{sgn}(\rho_J)(x-\mu) - \sigma^2\alpha)^2}{2\sigma^2})}{\sqrt{2\pi\sigma^2}} \int_0^\infty \exp(-(\frac{J - (\text{sgn}(\rho_J)(x-\mu) - \sigma^2\alpha)}{\sqrt{2}\sigma})^2) dJ, \\
&= \frac{\alpha \exp(-\frac{(x-\mu)^2 + (\text{sgn}(\rho_J)(x-\mu) - \sigma^2\alpha)^2}{2\sigma^2})}{\sqrt{2\pi\sigma^2}} \int_{-\frac{(\text{sgn}(\rho_J)(x-\mu) - \sigma^2\alpha)}{\sqrt{2}\sigma}}^\infty \exp(-t^2) \sqrt{2}\sigma dt, \\
&= \frac{\alpha \exp(-\frac{(x-\mu)^2 + (\text{sgn}(\rho_J)(x-\mu) - \sigma^2\alpha)^2}{2\sigma^2})}{2} \text{erfc}(\frac{\text{sgn}(\rho_J)(\mu - x) + \sigma^2\alpha}{\sqrt{2}\sigma}), \\
&= \frac{\alpha \exp(\frac{(\sigma^2\alpha^2 - 2\text{sgn}(\rho_J)(x-\mu)\alpha)}{2})}{2} \text{erfc}(\frac{\text{sgn}(\rho_J)(\mu - x) + \sigma^2\alpha}{\sqrt{2\sigma^2}}), \\
&= \frac{\alpha \exp(\frac{\alpha}{2}(\alpha\sigma^2 - 2\text{sgn}(\rho_J)(x-\mu)))}{2} \text{erfc}(\frac{\text{sgn}(\rho_J)(\mu - x) + \sigma^2\alpha}{\sqrt{2\sigma^2}}).
\end{aligned}$$

D. A Monte Carlo Experiment

In contrast to standard particle filter methods, the quantile spot variance method makes it possible to estimate the models we consider in this paper with both returns and large panels of option data. Because of the improvement in computational efficiency, it is also possible to study the sampling properties of the estimator in a (small-scale) Monte Carlo exercise.

Table [A1](#) presents the results of the Monte Carlo study. We simulate fifty samples

of return time series and option panels for two parameterizations of the SV model. The length of the samples is one year and the sample frequency is daily. Column 2 in Table [A1](#) shows the two sets of parameters. They are chosen based on the empirical results below. The main difference between them is that the ρ parameters are set at -0.90 and -0.70 respectively, and we modify the σ to keep the unconditional return kurtosis constant (see [Das and Sundaram \(1999\)](#) for the model-implied moments). The simulated option samples are similar to the sample used in the empirical analysis and summarized in Table [1](#). For a one-year sample, we thus end up with 6731 options.

Table [A1](#) shows that for both parameterizations, the medians of the parameter estimates based on the 50 replications are close to the true values, with good precision as indicated by the first and third quartiles. The estimates of the ρ and σ parameters are slightly biased downward. The price of risk parameters η_s and η_v are precisely estimated.

Table A1: Monte Carlo Results

Parameter	True value	Median	Q25	Q75
κ	2.0000	2.0144	1.9089	2.1613
θ	0.0350	0.0361	0.0333	0.0378
σ	0.3800	0.3669	0.3647	0.3693
ρ	-0.9000	-0.9277	-0.9324	-0.9226
η_s	2.5000	2.5302	2.3948	2.6702
η_v	1.0000	0.9956	0.8855	1.1384
Parameter	True value	Median	Q25	Q75
κ	2.0000	2.0537	1.9047	2.1757
θ	0.0350	0.0358	0.0341	0.0385
σ	0.4900	0.4642	0.4551	0.4682
ρ	-0.7000	-0.7288	-0.7397	-0.7233
η_s	2.5000	2.5074	2.3946	2.5709
η_v	1.0000	0.9911	0.8603	1.0759

Notes: We report Monte Carlo results based on two configurations of the SV model. For each of the two parameter settings, we simulate 50 samples of returns and option panels samples for a one-year period. The maturity and moneyness structure of the simulated option panels are based on the 2015 option sample, implying 6731 options used for estimation of each SV model.

Table A1: Parameter Estimates in Existing Studies: The SVJR Model

Panel A: Based on Returns								
Author	Period	κ	θ	σ	ρ	λ	μ_s	σ_s
ABL	1953-1996	3.704	0.013	0.184	-0.620	5.040	0.000	0.012
CV	1980-2000	12.187	0.019	4.274	-0.271	0.372	0.051	$exp(+)$
						5.379	0.026	$exp(-)$
CGGT	1953-1999	3.276	0.015	0.151	-0.279			
EJP	1980-2000	3.226	0.021	0.240	-0.467	1.512	-0.025	0.041
Eraker	1987-1990	3.024	0.021	0.202	-0.468	0.756	-0.037	0.066
Bates ₁	1953-1996	4.380	0.014	0.244	-0.612	0.744	-0.010	0.052
CJM	1996-2004	6.589	0.032	0.450	-0.777	2.790	-0.013	0.013

Panel B: Based on Options								
Author	Period	κ	θ	σ	ρ	λ	μ_s	σ_s
BCC	1988-1991	2.030	0.040	0.380	-0.570	0.590	-0.050	0.070
BCJ	1987-2003	5.796	0.012	0.240	-0.467	1.512	-0.100	0.041
CJM	1996-2004	2.638	0.063	0.448	-0.782	2.832	-0.015	0.006

Panel C: Based on Returns and Options								
Author	Period	κ	θ	σ	ρ	λ	μ_s	σ_s
Pan	1989-1996	6.400	0.015	0.300	-0.530	12.300	-0.008	0.039
		3.300*	0.030*				-0.192*	
Eraker	1987-1990	4.788	0.042	0.512	-0.586	0.504	-0.010	0.167
		2.772*	0.072*				-0.050*	
HLM	1990-2007	1.711	0.049	0.653	-0.740	2.332	-0.021	0.019
		0.415*	0.203*			2.831*		

Notes: We report parameter estimates for the SVJR model from existing studies. Estimates in Panel A are physical values. Estimates in Panel B are risk-neutral values. In Panel C, estimates with a star (*) indicate risk-neutral values and the rest are physical values. All parameters are annualized. BCC: [Bakshi et al. \(1997\)](#), based on the S&P 500; ABL: [Andersen, Benzoni, and Lund \(2002\)](#), based on the S&P 500; CV: [Chacko and Viceira \(2003\)](#), based on the S&P 500; CGGT: [Chernov, Gallant, Ghysels, and Tauchen \(2003\)](#), based on the DJIA; EJP: [Eraker et al. \(2003\)](#), based on S&P 500; Eraker: [Eraker \(2004\)](#), based on S&P 500; Bates₁: [Bates \(2006\)](#), based on S&P 500; CJM: [Christoffersen et al. \(2010\)](#), based on the S&P 500; BCJ: [Broadie et al. \(2007\)](#), based on the S&P 500; Pan: [Pan \(2002\)](#), based on the S&P 500; HLM: [Hurn et al. \(2015\)](#), based on the S&P 500.

Table A2: Parameter Estimates in Existing Studies: The SVCJ Model

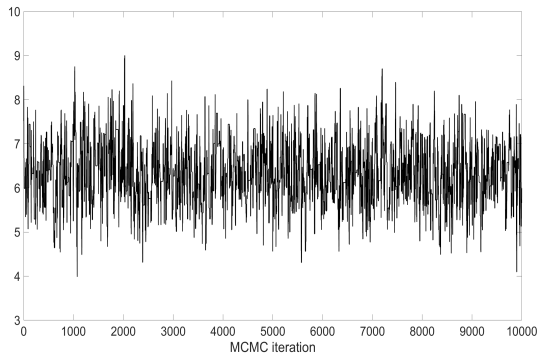
Panel A: Based on Returns										
Author	Period	κ	θ	σ	ρ	λ	μ_s	σ_s	μ_v	ρ_J
EJP	1980-2000	6.552	0.014	0.199	-0.484	1.663	-0.018	0.029	0.037	-0.601
Eraker	1987-1990	4.032	0.014	0.146	-0.461	1.008	-0.032	0.049	0.032	0.312

Panel B: Based on Options										
Author	Period	κ	θ	σ	ρ	λ	μ_s	σ_s	μ_v	ρ_J
BCJ	1987-2003	14.112	0.006	0.199	-0.484	1.663	-0.066	0.029	0.108	-0.601
AFT	1996-2010	2.049	0.033	0.354	-0.934	4.435	0.005	0.004	0.052	-0.502

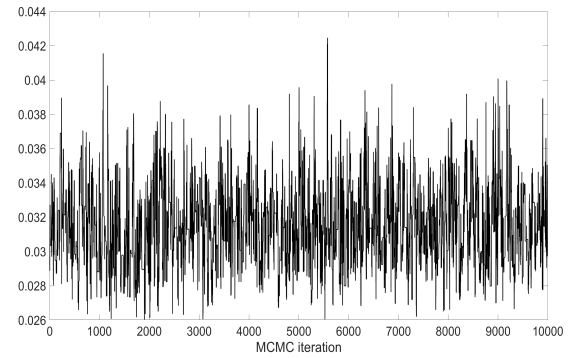
Panel C: Based on Returns and Options										
Author	Period	κ	θ	σ	ρ	λ	μ_s	σ_s	μ_v	ρ_J
Eraker	1987-1990	5.796	0.034	0.411	-0.582	0.504	-0.061	0.036	0.041	-0.693
		2.772*	0.071*				-0.075*			

Notes: We report parameter estimates for the SVCJ model from existing studies. Estimates in Panel A are physical values. Estimates in Panel B are risk-neutral values. In Panel C, estimates with a star (*) indicate risk-neutral values and the rest are physical values. All parameters are annualized. EJP: [Eraker et al. \(2003\)](#), based on the S&P 500; Eraker: [Eraker \(2004\)](#), based on the S&P 500; BCJ: [Broadie et al. \(2007\)](#), based on the S&P 500; AFT: [Andersen et al. \(2015a\)](#), based on the S&P 500.

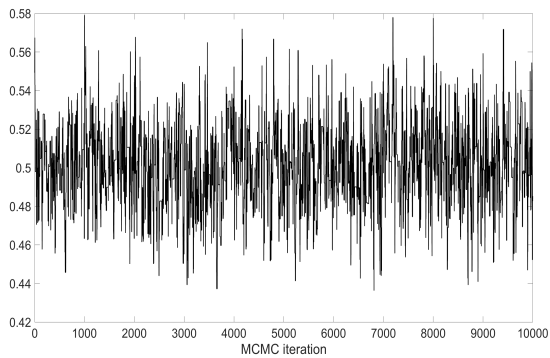
Figure A1: Parameter Trace for SV Model Parameters. Return-Based Estimation



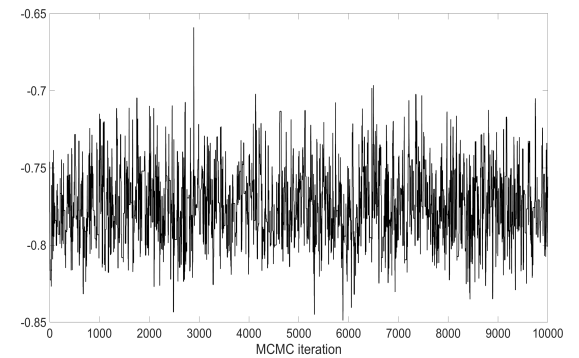
(a) κ



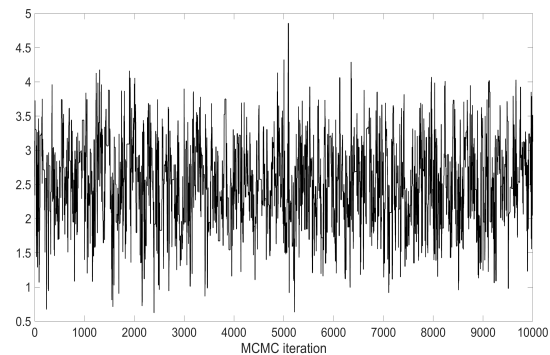
(b) θ



(c) σ



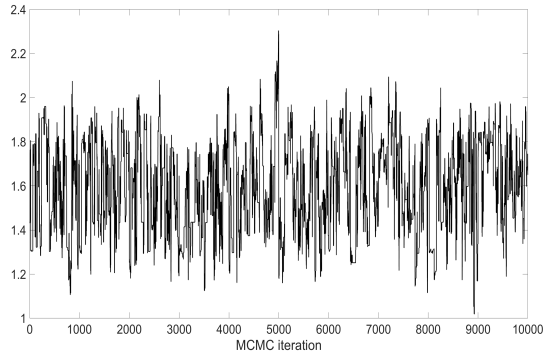
(d) ρ



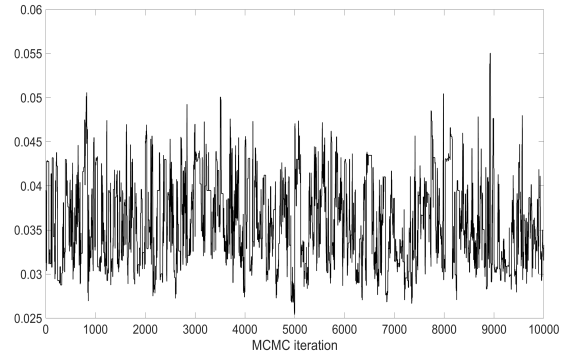
(e) η_s

Notes: We plot the traces for each parameter in the SV model. We use 10,000 iterations. The first 4,000 of the iterations are treated as burn-in.

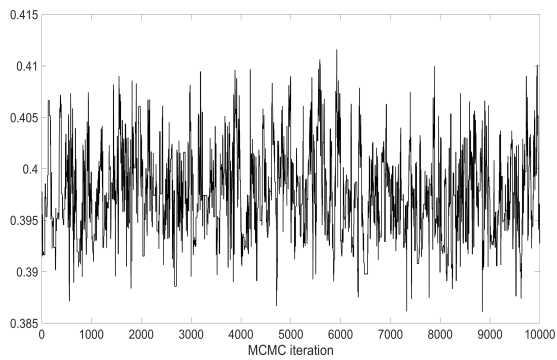
Figure A2: Parameter Trace for SV Model Parameters. Joint Estimation



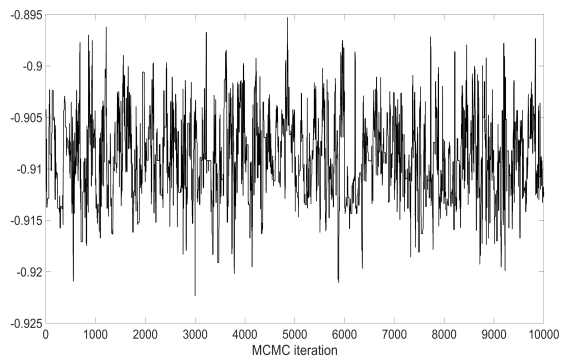
(a) κ



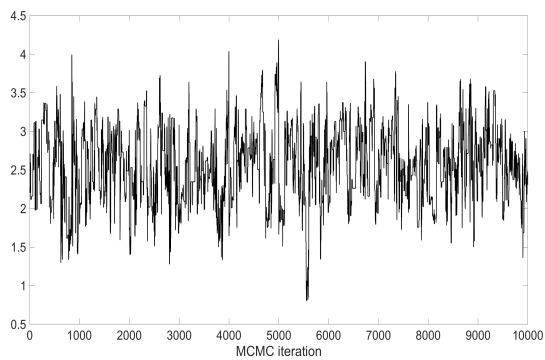
(b) θ



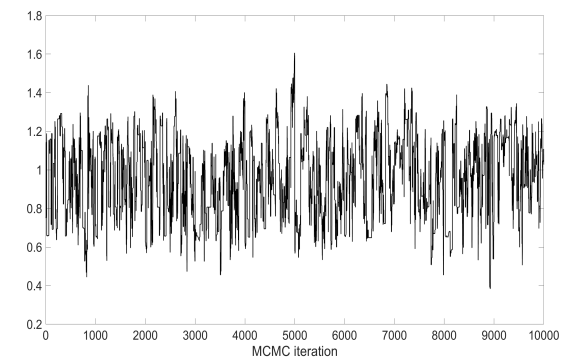
(c) σ



(d) ρ



(e) η_s



(f) η_v

Notes: We plot the traces for each parameter in the SV model. We use 10,000 iterations. The first 4,000 of the iterations are treated as burn-in.