Counterparty Risk in Over-the-Counter Markets

Internet Appendix

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A Results and their Proofs

This section contains the proofs of our results, which consider the more general case of arbitrary sizes s_i for the banks. When the formulation of the statement is different from that in the case $s_i = 1$, we restate the result.

A.A Proof of Lemma IV.1

Using $P[D=1]=q$, we compute

$$\Gamma^i(y_1,\dots,y_M) = \frac{1}{\eta}\log E\Big[\exp\Big(\eta D\Big(\omega_i + \sum_{n\neq i} y_n\big(\mathbb{1}_{A_i^\complement}\mathbb{1}_{y_n>0} + \mathbb{1}_{A_n^\complement}\mathbb{1}_{y_n<0}\big)\Big)\Big)\Big]$$

$$= \frac{1}{\eta}\log\Big(1-q+qE\Big[\exp\Big(\eta\omega_i + \eta\sum_{n\neq i} y_n\big(\mathbb{1}_{A_i^\complement|D=1}\mathbb{1}_{y_n>0} + \mathbb{1}_{A_n^\complement|D=1}\mathbb{1}_{y_n<0}\big)\Big)\Big]\Big)$$

$$= \frac{1}{\eta}\log\Big(1-q+q\mathrm{e}^{\eta\omega_i}E\Big[\exp\Big(\eta\mathbb{1}_{A_i^\complement|D=1}\sum_{n\neq i:\, y_n>0} y_n\Big)\Big]\prod_{n\neq i:\, y_n<0} E\big[\exp\big(\eta\mathbb{1}_{A_n^\complement|D=1}y_n\big)\big]\Big).$$

Using that $p_i = P[A_i|D=1]$, we obtain

$$\Gamma^i(y_1,\dots,y_M) = \frac{1}{\eta}\log\left(1-q+q\mathrm{e}^{\eta\omega_i}\Big((1-p_i)\mathrm{e}^{\eta\sum_{n:\, y_n>0} y_n} + p_i\Big)\prod_{n\neq i:\, y_n<0}\big((1-p_n)\mathrm{e}^{\eta y_n}+p_n\big)\right),$$

which can be brought into the form $\Gamma^i(y_1,\dots,y_M)$ written in the statement of Lemma IV.1.

To show the additional properties of $\Gamma^i(y_1,\dots,y_M)$, we first note that the function Ξ given by

$$\Xi(y) = \frac{1}{\eta}\log\big(1-q+q\mathrm{e}^{\eta y}\big)$$

is strictly increasing and strictly convex. Indeed, we can calculate

$$\Xi'(y) = \frac{q\mathrm{e}^{\eta y}}{1-q+q\mathrm{e}^{\eta y}} > 0, \qquad \Xi''(y) = \frac{(1-q)q\eta\mathrm{e}^{\eta y}}{(1-q+q\mathrm{e}^{\eta y})^2} > 0.$$

Next, we consider

$$f(y,p) = \frac{1}{\eta}\log\big((1-p)\mathrm{e}^{\eta y}+p\big)$$

for $p > 0$ and calculate

$$\begin{aligned} f_y(y,p) &= \frac{(1-p)\mathrm{e}^{\eta y}}{(1-p)\mathrm{e}^{\eta y}+p} > 0, \\ f_{yy}(y,p) &= \eta\frac{((1-p)\mathrm{e}^{\eta y}+p)((1-p)\mathrm{e}^{\eta y})-((1-p)\mathrm{e}^{\eta y})^2}{((1-p)\mathrm{e}^{\eta y}+p)^2} = \eta\frac{p(1-p)\mathrm{e}^{\eta y}}{((1-p)\mathrm{e}^{\eta y}+p)^2} > 0. \end{aligned} \tag{13}$$

These inequalities show that the function $y \mapsto f(y,p)$ is strictly increasing and strictly convex for $p > 0$. Because $f(y,p)$ either equals y (if $p = 0$) or is strictly increasing and strictly convex (if $p > 0$), we see that $\Gamma^i(y_1,\ldots,y_M)$ is strictly increasing, and the statements on convexity of $\Gamma^i(y_1,\ldots,y_M)$ now follow from the fact that convexity is maintained under sums and compositions with a convex, nondecreasing function.

Finally, to prove (3), let $y_1 < y_2$, $y_3 \in \big(0, \frac{y_2-y_1}{2}\big]$ and $p_1 \geq p_2$. We first note that (3) is equivalent to

$$\big((1-p_1)\mathrm{e}^{\eta y_1}+p_1\big)\big((1-p_2)\mathrm{e}^{\eta y_2}+p_2\big) > \big((1-p_1)\mathrm{e}^{\eta(y_1+y_3)}+p_1\big)\big((1-p_2)\mathrm{e}^{\eta(y_2-y_3)}+p_2\big),$$

which can be further simplified to

$$(1-p_1)p_2\mathrm{e}^{\eta y_1} + (1-p_2)p_1\mathrm{e}^{\eta y_2} > (1-p_1)p_2\mathrm{e}^{\eta(y_1+y_3)} + (1-p_2)p_1\mathrm{e}^{\eta(y_2-y_3)}.$$

This inequality follows from

$$a\mathrm{e}^{x_1} + b\mathrm{e}^{x_2} > a\mathrm{e}^{x_1+x_3} + b\mathrm{e}^{x_2-x_3} \tag{14}$$

for all $a \le b$, $x_1 < x_2$ and $x_3 \in \left(0, \frac{x_2-x_1}{2}\right]$ by choosing

$$a = (1-p_1)p_2, \quad b = (1-p_2)p_1, \quad x_1 = \eta y_1, \quad x_2 = \eta y_2, \quad x_3 = \eta y_3,$$

where we note that $p_1 \ge p_2$, $y_1 < y_2$, and $y_3 \in \left(0, \frac{y_2-y_1}{2}\right]$ imply $a \le b$, $x_1 < x_2$, and $x_3 \in \left(0, \frac{x_2-x_1}{2}\right]$. The inequality (14) can be seen from the convexity of the exponential function or checked directly by calculating the partial derivative

$$\frac{\partial}{\partial z}(a\mathrm{e}^{x_1+z} + b\mathrm{e}^{x_2-z}) = a\mathrm{e}^{x_1+z} - b\mathrm{e}^{x_2-z} \le b\mathrm{e}^{x_1+z} - b\mathrm{e}^{x_2-z} < 0$$

for all $z \in \left[0, \frac{x_2-x_1}{2}\right)$. □

A.B Results of Section IV.B and their Proofs

Theorem A.1 (Theorem IV.3)**.** *Feasible contracts* $(\gamma_{i,n})_{i,n=1,\dots,M}$ *are a market equilibrium if and only if they solve the optimization problem*

$$\text{(15)} \qquad \textit{minimize} \sum_{i=1}^{M} s_i \Gamma^i(\gamma_i s) \quad \textit{over } \gamma \textit{ subject to } \gamma_{i,n} = -\gamma_{n,i} \textit{ and } -k \le \gamma_{i,n} \le k,$$

where $\gamma_i s := (\gamma_{i,1} s_1, \dots, \gamma_{i,M} s_M)$*.*

Proof. The Lagrangian function corresponding to (15) is

$$\sum_{i=1}^{M} s_i \Gamma^i(\gamma_i s) - \sum_{i,n=1}^{M} s_i s_n \alpha_{i,n}(\gamma_{i,n} + \gamma_{n,i}) - \sum_{i,n=1}^{M} s_i s_n \underline{\beta}_{i,n}(k - \gamma_{i,n}) - \sum_{i,n=1}^{M} s_i s_n \overline{\beta}_{i,n}(k + \gamma_{i,n}).$$

The optimality conditions are

$$\text{(16)} \qquad \begin{aligned} &\Gamma^i_{y_n}(\gamma_i s) = \alpha_{i,n} + \alpha_{n,i} - \underline{\beta}_{i,n} + \overline{\beta}_{i,n}, \qquad \underline{\beta}_{i,n} \ge 0, \qquad \overline{\beta}_{i,n} \ge 0, \\ &\underline{\beta}_{i,n}(k - \gamma_{i,n}) = 0, \qquad \overline{\beta}_{i,n}(k + \gamma_{i,n}) = 0. \end{aligned}$$

All of them are satisfied for

$$\underline{\beta}_{n,i} = \overline{\beta}_{i,n} = \frac{1}{2} \max\left\{\Gamma^i_{y_n}(\gamma_i s) - \Gamma^n_{y_i}(\gamma_n s), 0\right\}, \quad \alpha_{i,n} + \alpha_{n,i} = \frac{1}{2}\left(\Gamma^i_{y_n}(\gamma_i s) + \Gamma^n_{y_i}(\gamma_n s)\right)$$

if γ satisfies (4) and $\gamma_{i,n} = -\gamma_{n,i}$. This means that if γ is a market equilibrium, it is a solution to (15). Conversely, if γ is a solution to (15), then (16) implies

$$\Gamma^i_{y_n}(\gamma_i s)(k^2 - \gamma_{i,n}^2) = (\alpha_{i,n} + \alpha_{n,i})(k^2 - \gamma_{i,n}^2) = (\alpha_{n,i} + \alpha_{i,n})(k^2 - \gamma_{n,i}^2) = \Gamma^n_{y_i}(\gamma_n s)(k^2 - \gamma_{n,i}^2).$$

This equation shows that if $\gamma_{i,n} \neq \pm k$, we need $\Gamma^i_{y_n}(\gamma_i s) = \Gamma^n_{y_i}(\gamma_n s)$. In turn, $\Gamma^i_{y_n}(\gamma_i s) \neq \Gamma^n_{y_i}(\gamma_n s)$ implies $\gamma_{i,n} = \pm k$. Consider the case $\Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s)$ and assume $\gamma_{i,n} = -k$, then $\gamma_{n,i} = k$; it follows from (16) that $\underline{\beta}_{i,n} = 0$, $\overline{\beta}_{n,i} = 0$ and

$$\Gamma^i_{y_n}(\gamma_i s) = \alpha_{i,n} + \alpha_{n,i} + \overline{\beta}_{i,n} \geq \alpha_{i,n} + \alpha_{n,i} \geq \alpha_{n,i} + \alpha_{i,n} - \underline{\beta}_{i,n} = \Gamma^n_{y_i}(\gamma_n s),$$

which is a contradiction to $\Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s)$. Therefore, $\Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s)$ implies $\gamma_{i,n} = k$. By symmetry, $\Gamma^i_{y_n}(\gamma_i s) > \Gamma^n_{y_i}(\gamma_n s)$ implies $\gamma_{i,n} = -k$. This shows that a solution to (15) satisfies (4) and thus is a market equilibrium. □

Theorem A.2 (Theorem IV.4)**.** *There exists a market equilibrium $(\gamma_{i,n})_{i,n=1,\ldots,M}$. The $\gamma_{i,n}$ are unique for $p_n > 0$ and $\gamma_{i,n} < 0$, or $p_i > 0$ and $\gamma_{i,n} > 0$. For every i, the value is the same for $\sum \gamma_{i,n} s_n$ where the sum is over n such that $p_n = 0$ and $\gamma_{i,n} < 0$, or $p_i = 0$ and $\gamma_{i,n} > 0$. In particular, $\Gamma(\gamma_n s)$ are uniquely determined for a market equilibrium $(\gamma_{i,n})_{i,n=1,\ldots,M}$.*

Proof. We prove first the existence of a market equilibrium. To this end, we will apply Kakutani's fixed-point theorem (see, for example, Corollary 15.3 in Border (1985)). We fix k, set $S = [-k, k]^{M(M-1)/2}$, and define a mapping $\Phi : S \to 2^S$ as follows, where 2^S denotes the power set of S, i.e., the set of all subsets of S. Each element in S corresponds to the lower triangular matrix of $(\gamma_{i,n})_{i,n=1,\ldots,M}$, where we set the diagonal elements γ_{ii} equal to zero and the upper diagonal elements are defined by $\gamma_{i,n} = -\gamma_{n,i}$. Let $\Phi(\gamma)$ consist of all $(\tilde{\gamma}_{i,n})_{i,n=1,\ldots,M}$ that satisfy $\tilde{\gamma}_{i,n} = -\tilde{\gamma}_{n,i}$, $-k \leq \tilde{\gamma}_{i,n} \leq k$, with the further restriction in the following three cases

$$\tilde{\gamma}_{i,n} \begin{cases} = k & \text{if } \Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s), \\ = \gamma_{i,n} & \text{if } \Gamma^i_{y_n}(\gamma_i s) = \Gamma^n_{y_i}(\gamma_n s), \\ = 0 & \text{if } \Gamma^i_{y_n}(\gamma_i) \text{ or } \Gamma^n_{y_i}(\gamma_n) \text{ do not exist}, \\ = -k & \text{if } \Gamma^i_{y_n}(\gamma_i s) > \Gamma^n_{y_i}(\gamma_n s). \end{cases}$$

Note that these "if" conditions depend on γ and not on $\tilde{\gamma}$. We can see that $\Phi(\gamma)$ is nonempty, compact and convex. To show that Φ has a closed graph, consider a sequence $\big(\gamma^{(m)}, \tilde{\gamma}^{(m)}\big)$ converging to $(\gamma, \tilde{\gamma})$ with $\tilde{\gamma}^{(m)} \in \Phi\big(\gamma^{(m)}\big)$ for all m. Because $\tilde{\gamma}^{(m)} \to \tilde{\gamma}$ and $\tilde{\gamma}^{(m)} \in \Phi\big(\gamma^{(m)}\big)$, we have $\tilde{\gamma}_{i,n} = -\tilde{\gamma}_{n,i}$ and $-k \le \tilde{\gamma}_{i,n} \le k$. Moreover, if $\Gamma^i_{y_n}(\gamma_i s) < \Gamma^n_{y_i}(\gamma_n s)$, we have $\Gamma^i_{y_n}\big(\gamma_i^{(m)} s\big) < \Gamma^n_{y_i}\big(\gamma_n^{(m)} s\big)$ for all m big enough, as $\gamma^{(m)} \to \gamma$. This yields $\tilde{\gamma}_{i,n}^{(m)} = k$ for all m big enough; hence, $\tilde{\gamma}_{i,n} = k$. Similarly, $\Gamma^i_{y_n}(\gamma_i s) > \Gamma^n_{y_i}(\gamma_n s)$ implies $\tilde{\gamma}_{i,n} = -k$. The condition is also satisfied for the last case $\Gamma^i_{y_n}(\gamma_i s) = \Gamma^n_{y_i}(\gamma_n s)$, as we have already shown $-k \le \tilde{\gamma}_{i,n} \le k$. Therefore, there exists γ with $\Phi(\gamma) = \gamma$ by Kakutani's fixed-point theorem; hence, there is a market equilibrium.

To prove uniqueness, we first apply Theorem IV.3, which says that finding a market equilibrium is equivalent to solving (15). We then write the objective function in (15) as

$$\sum_{i=1}^{M} s_i \Gamma^i(\gamma_i s) = \sum_{i=1}^{M} s_i \Xi\left(\omega_i + f\left(\sum_{n:\gamma_{i,n}s_n \ge 0} \gamma_{i,n} s_n, p_i\right) + \sum_{n:\gamma_{i,n}s_n < 0} f(\gamma_{i,n} s_n, p_n)\right),$$

where the function Ξ is given in Lemma IV.1. The uniqueness statements now follow from the statements on convexity in Lemma IV.1. $\square$

A.C Results of Section IV.C and their Proofs

Proposition A.3 (Proposition IV.5). *Assume that at least one of the following conditions holds:*

(a) $\sum_{\ell:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}s_\ell \geq s_i \max_\ell \gamma_{i,\ell}$, *or*

(b) $\sum_{\ell:\gamma_{j,\ell}\geq 0}\gamma_{j,\ell}s_\ell \geq s_j \max_\ell \gamma_{j,\ell}$.

We then have the following relations between initial and post-trade exposures:

1. *If* $\omega_i \geq \omega_j$, $p_i \leq p_j$, *and* $s_i \leq s_j$, *then* $\Omega_i \geq \Omega_j$.
2. *If* $\omega_i > \omega_j$, $p_i \geq p_j$, *and* $s_i \geq s_j$, *then* $\omega_i - \omega_j > \Omega_i - \Omega_j$.

Proof. Under conditional independence and for general sizes, the post-trade exposure is given by

$$\Omega_i = \omega_i + f\left(\sum_{n:\gamma_{i,n}\geq 0}\gamma_{i,n}s_n, p_i\right) + \sum_{n:\gamma_{i,n}<0} f(\gamma_{i,n}s_n, p_n).$$

We split the proof in several steps, starting with some preparation.

Claim 1a. For two banks i and j, we have

$$\Omega_j > \Omega_i \Longrightarrow \gamma_{j,i} \leq 0. \tag{C1a}$$

Proof of Claim 1a. From Lemma IV.1, it follows that

$$\Gamma^j_{y_i}(\gamma_j s) = \begin{cases} \Xi'(\Omega_j)\eta f_y\left(\sum_{n:\gamma_{j,n}\geq 0}\gamma_{j,n}s_n, p_j\right) & \text{if } \gamma_{j,i} > 0, \\ \Xi'(\Omega_j)\eta f_y(\gamma_{j,i}s_i, p_i) & \text{if } \gamma_{j,i} < 0, \end{cases}$$

with an analogous expression for $\Gamma^i_{y_j}(\gamma_i s)$. If $\gamma_{j,i} > 0$ (and thus $\gamma_{i,j} < 0$), we obtain

$$\begin{aligned}
\Gamma^j_{y_i}(\gamma_j s) &= \Xi'(\Omega_j)\eta f_y\left(\sum_{n:\gamma_{j,n}\geq 0} \gamma_{j,n} s_n, p_j\right) \\
&> \Xi'(\Omega_i)\eta f_y\left(\sum_{n:\gamma_{j,n}\geq 0} \gamma_{j,n} s_n, p_j\right) \\
&\geq \Xi'(\Omega_i)\eta f_y(\gamma_{i,j} s_j, p_j) \\
&= \Gamma^i_{y_j}(\gamma_i s)
\end{aligned}$$

by strict convexity of Ξ and convexity of $f(., p_j)$ from Lemma IV.1. However, this implies $\gamma_{j,i} = -k$ by (4) in contradiction to the assumption $\gamma_{j,i} > 0$.

Claim 1b. For two banks i and j, we have

$$\Omega_j > \Omega_i \Longrightarrow \gamma_{j,n} \leq \gamma_{i,n} \text{ or } \gamma_{j,n} = -k \text{ for all } n \text{ with } \Omega_n < \Omega_j. \tag{C1b}$$

Proof of Claim 1b. We distinguish the following three cases:

- If $\Omega_n \in (\Omega_i, \Omega_j)$, we have $\gamma_{j,n} \leq 0$ and $\gamma_{i,n} \geq 0$ by (C1a) so that $\gamma_{j,n} \leq \gamma_{i,n}$ holds.

- If $\Omega_n < \Omega_i$, we have $\gamma_{j,n} \leq 0$ and $\gamma_{i,n} \leq 0$ by (C1a); thus,

$$\Gamma^j_{y_n}(\gamma_j s) = \Xi'(\Omega_j)\eta f_y(\gamma_{j,n} s_n, p_n), \tag{17}$$

$$\Gamma^i_{y_n}(\gamma_i s) = \Xi'(\Omega_i)\eta f_y(\gamma_{i,n} s_n, p_n), \tag{18}$$

$$\Gamma^n_{y_j}(\gamma_n s) = \Xi'(\Omega_n)\eta f_y\left(\sum_{\ell:\gamma_{n,\ell}\geq 0} \gamma_{n,\ell} s_\ell, p_n\right) = \Gamma^n_{y_i}(\gamma_n s). \tag{19}$$

Assume that $\gamma_{j,n} \neq -k$, which implies

$$\Gamma^j_{y_n}(\gamma_j s) = \Gamma^n_{y_j}(\gamma_n s) = \Gamma^n_{y_i}(\gamma_n s) \leq \Gamma^i_{y_n}(\gamma_i s)$$

by (4) and (19); thus,

$$1 < \frac{\Xi'(\Omega_j)}{\Xi'(\Omega_i)} \leq \frac{f_y(\gamma_{i,n} s_n, p_n)}{f_y(\gamma_{j,n} s_n, p_n)}$$

by (17) and (18). This is only possible if $\gamma_{j,n} < \gamma_{i,n}$.

- If $\Omega_n = \Omega_i$, we argue as in the first item if $\gamma_{i,n} \geq 0$, or as in the second item if $\gamma_{i,n} < 0$.

Note that (C1b) holds regardless of the default probabilities of banks i and j. This is because we are considering banks n with smaller post-trade exposures; thus, banks that are sellers of protection by (C1a) so that the same counterparty risk p_n applies to trades with i and j.

Claim 1c. For two banks i and j, we have

(C1c)

$$\Omega_j > \Omega_i \Longrightarrow \frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)} < \frac{f_y(\gamma_{n,j} s_j, p_j)}{f_y(\gamma_{n,i} s_i, p_i)} \text{ or } \gamma_{n,i} = -k \text{ for all } n \text{ with } \Omega_n > \Omega_j.$$

Proof of Claim 1c. $\Omega_n > \Omega_j$ implies $\gamma_{j,n} \geq 0$ by (C1a), and thus $\Gamma^j_{y_n}(\gamma_j s) \leq \Gamma^n_{y_j}(\gamma_n s)$. If $\gamma_{n,i} \neq -k$, it follows that $\Gamma^i_{y_n}(\gamma_i s) \geq \Gamma^n_{y_i}(\gamma_n s)$; hence,

$$\begin{aligned}\Gamma^n_{y_j}(\gamma_n s) &\geq \Gamma^j_{y_n}(\gamma_j s) = \Xi'(\Omega_j)\eta f_y\Bigg(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\Bigg) > \Xi'(\Omega_i)\eta f_y\Bigg(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\Bigg) \\ &= \Gamma^i_{y_n}(\gamma_i s)\frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)} \geq \Gamma^n_{y_i}(\gamma_n s)\frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)},\end{aligned}$$

which shows (C1c), as $\Gamma^n_{y_i}(\gamma_n s) = \Xi'(\Omega_n)\eta f_y(\gamma_{n,i}s_i, p_i)$ and $\Gamma^n_{y_j}(\gamma_n s) = \Xi'(\Omega_n)\eta f_y(\gamma_{n,j}s_j, p_j)$.

Claim 1d. For three banks i, j, and n, we have

$$\text{(C1d)} \qquad \Omega_i < \Omega_j = \Omega_n \Longrightarrow \gamma_{j,n} \leq \gamma_{i,n} \text{ or (C1c) holds.}$$

Proof of Claim 1d. If $\gamma_{j,n} \leq 0$, we obtain $\gamma_{j,n} \leq \gamma_{i,n}$, as $\gamma_{i,n} \geq 0$ by (C1a). If $\gamma_{j,n} > 0$, we can argue as (C1c).

We can summarize (C1a)–(C1d) as

$$\text{(C1)} \qquad \Omega_j > \Omega_i \Longrightarrow \begin{cases} \gamma_{j,n} \leq \gamma_{i,n} & \text{for all } \gamma_{j,n} \leq 0, \\ \text{(C1c) holds} & \text{for all } \gamma_{j,n} > 0. \end{cases}$$

Claim 2. For two banks i and j, we have

$$\text{(C2)} \quad \omega_i \geq \omega_j,\ p_j \geq p_i,\ s_j \geq s_i,\ \text{and (a), (b) or (c) of the proposition holds} \Longrightarrow \Omega_i \geq \Omega_j.$$

Proof of Claim 2. We prove the claim by contradiction and assume that $\Omega_i < \Omega_j$. This implies $\gamma_{j,n} \leq \gamma_{i,n}$ for all $\gamma_{j,n} \leq 0$ by (C1); hence,

$$\begin{aligned} f\left(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell}s_\ell, p_j\right) &= \Omega_j - \omega_j - \sum_{n:\gamma_{j,n}<0} f(\gamma_{j,n}s_n, p_n) \\ &> \Omega_i - \omega_i - \sum_{n:\gamma_{i,n}<0} f(\gamma_{i,n}s_n, p_n) \\ &= f\left(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell}s_\ell, p_i\right) \end{aligned}$$

$$\geq f\Bigg(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_j \Bigg),$$

using (8), $p_j \geq p_i$, and that $f(y,p)$ is decreasing in p for $y \geq 0$ because, using definition (2),

$$f_p(y,p) = \frac{\partial}{\partial p}\frac{1}{\eta} \log\big((1-p)\mathrm{e}^{\eta y} + p\big) = \frac{-\mathrm{e}^{\eta y}+1}{\eta((1-p)\mathrm{e}^{\eta y}+p)} < 0 \quad \text{for } y \geq 0. \tag{20}$$

This yields $\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell > \sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell$, as $y \mapsto f(y,p_j)$ is strictly increasing by Lemma IV.1. This implies that there exists n with $\gamma_{j,n} > \gamma_{i,n} \geq 0$; thus,

$$\gamma_{n,j} < \gamma_{n,i} \leq 0 \quad \text{and} \quad \gamma_{n,j} s_j < \gamma_{n,i} s_i \tag{21}$$

because $s_j \geq s_i$ by assumption. Moreover, $\gamma_{j,n} > 0$ implies $\Omega_n \geq \Omega_j$ by (C1a). On the other hand, $\Omega_i < \Omega_j$ implies by (C1c) and (C1d) that $\gamma_{n,i} = -k$ (which stands in contradiction to (21) because $\gamma_{n,j} \geq -k$) or $\gamma_{j,n} \leq \gamma_{i,n}$ (also a contradiction to (21)) or

$$\frac{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)}{f_y(\gamma_{n,i} s_i, p_i)} > \frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y(\gamma_{n,j} s_j, p_j)}. \tag{22}$$

We will show that (22) contradicts

$$p_j \geq p_i, \quad \sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell > \sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell \quad \text{and} \quad \gamma_{n,j} s_j < \gamma_{n,i} s_i \tag{23}$$

if one of the conditions (a)–(c) of the proposition holds.

As an auxiliary step, we next analyze the function $p \mapsto \frac{f_y(y_1,p)}{f_y(y_2,p)}$ and show that

$$\frac{\partial}{\partial p}\frac{f_y(y_1,p)}{f_y(y_2,p)} \geq 0 \quad \text{for all } p \in [0,1) \text{ and } y_1 \geq -y_2 \geq 0. \tag{24}$$

To show this, first note that if $p=0$, then $f_y(y_1,p) = f_y(y_2,p) = 1$ so that $\frac{\partial}{\partial p}\frac{f_y(y_1,p)}{f_y(y_2,p)} = 0$. Now assume that $p > 0$. We use (13) and

$$\begin{aligned} f_{yp}(y,p) = \frac{\partial}{\partial p}\frac{(1-p)\mathrm{e}^{\eta y}}{(1-p)\mathrm{e}^{\eta y}+p} &= \frac{((1-p)\mathrm{e}^{\eta y}+p)(-\mathrm{e}^{\eta y}) - ((1-p)\mathrm{e}^{\eta y})(-\mathrm{e}^{\eta y}+1)}{((1-p)\mathrm{e}^{\eta y}+p)^2} \\ &= \frac{-\mathrm{e}^{\eta y}}{((1-p)\mathrm{e}^{\eta y}+p)^2} \end{aligned}$$

to deduce that

$$\begin{aligned} \frac{\partial}{\partial p}\frac{f_y(y_1,p)}{f_y(y_2,p)} &= \frac{f_y(y_2,p)f_{yp}(y_1,p) - f_{yp}(y_2,p)f_y(y_1,p)}{(f_y(y_2,p))^2} \\ &= \frac{\frac{(1-p)\mathrm{e}^{\eta y_2}}{(1-p)\mathrm{e}^{\eta y_2}+p}\frac{-\mathrm{e}^{\eta y_1}}{((1-p)\mathrm{e}^{\eta y_1}+p)^2} - \frac{-\mathrm{e}^{\eta y_2}}{((1-p)\mathrm{e}^{\eta y_2}+p)^2}\frac{(1-p)\mathrm{e}^{\eta y_1}}{(1-p)\mathrm{e}^{\eta y_1}+p}}{(f_y(y_2,p))^2} \\ &= \frac{-\mathrm{e}^{\eta y_1}\big((1-p)\mathrm{e}^{\eta y_2}\big)\big((1-p)\mathrm{e}^{\eta y_2}+p\big)}{((1-p)\mathrm{e}^{\eta y_1}+p)^2((1-p)\mathrm{e}^{\eta y_2}+p)^2(f_y(y_2,p))^2} \\ &\quad - \frac{-\mathrm{e}^{\eta y_2}\big((1-p)\mathrm{e}^{\eta y_1}\big)\big((1-p)\mathrm{e}^{\eta y_1}+p\big)}{((1-p)\mathrm{e}^{\eta y_1}+p)^2((1-p)\mathrm{e}^{\eta y_2}+p)^2(f_y(y_2,p))^2} \\ &= \frac{\mathrm{e}^{\eta(y_1+y_2)}}{((1-p)\mathrm{e}^{\eta y_1}+p)^2((1-p)\mathrm{e}^{\eta y_2}+p)^2(f_y(y_2,p))^2} \\ &\quad \times \Big((1-p)\mathrm{e}^{\eta y_1}\big(1-p+p\mathrm{e}^{-\eta y_1}\big) - \big((1-p)\mathrm{e}^{\eta y_2}\big)\big(1-p+p\mathrm{e}^{-\eta y_2}\big)\Big). \end{aligned}$$

From this, we obtain $\frac{\partial}{\partial p}\frac{f_y(y_1,p)}{f_y(y_2,p)} \geq 0$ because

$$(1-p)\mathrm{e}^{\eta y_1}\big(1-p+p\mathrm{e}^{-\eta y_1}\big) - \big((1-p)\mathrm{e}^{\eta y_2}\big)\big(1-p+p\mathrm{e}^{-\eta y_2}\big) = (1-p)^2\big(\mathrm{e}^{\eta y_1} - \mathrm{e}^{\eta y_2}\big) \geq 0,$$

using $y_1 \geq y_2$. This concludes the proof of (24).

We now consider each of the two conditions (a) and (b) of the proposition.

Condition (a). We apply (24) choosing $p = p_i$, $y_1 = \sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell$, and $y_2 = \gamma_{n,i} s_i$. This implies

$$\frac{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)}{f_y(\gamma_{n,i} s_i, p_i)} \leq \frac{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_j\big)}{f_y(\gamma_{n,i} s_i, p_j)} \leq \frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y(\gamma_{n,j} s_j, p_j)},$$

where we use (23) and the convexity of $y \mapsto f(y, p_j)$ for the second inequality.

Condition (b). This time, we apply (24) choosing $p = p_j$, $y_1 = \sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell$, and $y_2 = \gamma_{n,j} s_j$. We obtain

$$\frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_j\big)}{f_y(\gamma_{n,j} s_j, p_j)} \geq \frac{f_y\big(\sum_{\ell:\gamma_{j,\ell}\geq 0} \gamma_{j,\ell} s_\ell, p_i\big)}{f_y(\gamma_{n,j} s_j, p_i)} \geq \frac{f_y\big(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\big)}{f_y(\gamma_{n,i} s_i, p_i)},$$

where we again use (23) and the convexity of $y \mapsto f(y, p_i)$ for the second inequality.

Under each of the two conditions (a) and (b), we obtain a contradiction to (22). Hence, $\Omega_i < \Omega_j$ cannot hold, which concludes the proof of (C2).

Claim 3. For two banks i and j, we have

$$\omega_i > \omega_j,\ p_j \leq p_i,\ s_j \leq s_i \Longrightarrow \Omega_i - \omega_i < \Omega_j - \omega_j.$$

Proof of Claim 3. We proceed similarly to the proof of (C2). We prove the claim by contradiction and assume that $\Omega_i - \omega_i \geq \Omega_j - \omega_j$. This implies $\Omega_i > \Omega_j$; hence, $\gamma_{i,n} \leq \gamma_{j,n}$

for all $\gamma_{i,n} \le 0$ by (C1) and $\gamma_{i,j} \le 0 \le \gamma_{j,i}$ by (C1a), and thus

$$\begin{aligned} f\left(\sum_{\ell:\gamma_{j,\ell}\ge 0} \gamma_{j,\ell} s_\ell, p_j\right) &= \Omega_j - \omega_j - \sum_{n:\gamma_{j,n}<0} f(\gamma_{j,n} s_n, p_n) \\ &< \Omega_i - \omega_i - \sum_{n:\gamma_{i,n}<0} f(\gamma_{i,n} s_n, p_n) \\ &= f\left(\sum_{\ell:\gamma_{i,\ell}\ge 0} \gamma_{i,\ell} s_\ell, p_i\right) \\ &\le f\left(\sum_{\ell:\gamma_{i,\ell}\ge 0} \gamma_{i,\ell} s_\ell, p_j\right) \end{aligned}$$

using $p_j \le p_i$ and (20), which yields $\sum_{\ell:\gamma_{j,\ell}\ge 0} \gamma_{j,\ell} s_\ell < \sum_{\ell:\gamma_{i,\ell}\ge 0} \gamma_{i,\ell} s_\ell$ because $y \mapsto f(y, p_j)$ is strictly increasing by Lemma IV.1. We conclude the proof in the same way as the proof of (C2) after (21), with i and j interchanged. □

Theorem A.4 (Theorem IV.6)**.** *Assume that the trade size limit is not binding and that there are at least two safe banks. Then*

1. *There exists the following relation between banks' creditworthiness, initial exposures and post-trade exposures:*

   (a) *All safe banks have the same post-trade exposure, say,* $\bar{\Omega}$*.*

   (b) *Risky banks with initial exposure above some level* α *also have the same post-trade exposure* $\bar{\Omega}$*. The level* α *is greater than* $\bar{\Omega}$ *and depends only on the distribution of initial exposures and sizes, but not on the banks' default probabilities.*

   (c) *Risky banks with initial exposure below* α *will have post-trade exposures strictly smaller than* $\bar{\Omega}$*.*

2. *Risky banks with initial exposure above α trade as follows:*

   (a) *They do not trade between each other.*

   (b) *They do not sell protection.*

   (c) *Their purchases depend only on their initial exposure, but not on their default probabilities.*

3. *Risky banks with initial exposure below α trade as follows:*

   (a) *Any two risky banks i and n with initial exposure below α do not trade between each other if their exposures $\tilde{\Omega}_i$ and $\tilde{\Omega}_n$ after trading with other banks but before trading between themselves satisfy $\tilde{\Omega}_i \leq \tilde{\Omega}_n$ and*

$$\frac{\Xi'(\tilde{\Omega}_n)}{\Xi'(\tilde{\Omega}_i)} \leq \frac{f_y\big(\sum_{\ell \neq n : \gamma_{i,\ell} \geq 0} \gamma_{i,\ell} s_\ell, p_i\big)}{1 - p_i}. \tag{25}$$

   (b) *They do not purchase protection from safe banks or risky banks with initial exposures above α.*

   (c) *If all banks have the same size, then they sell the same amount of protection to each safe bank and risky bank with initial exposure above α.*

Proof. To prove the first part, we define $\bar{k}_1$ by

$$\bar{k}_1 = \inf\big\{k > 0 : \Omega_i = \Omega_j \text{ for all } i, j \text{ with } p_i = p_j = 0\big\}. \tag{26}$$

We can prove that $0 < \bar{k}_1 < \infty$ and that the infimum in (26) is attained along the same lines as on page 2273 of Atkeson et al. (2015), restricting their arguments to the safe banks.

We choose $\bar{k}$ as the smallest number $k \geq \bar{k}_1$ such that

$$\Omega_i \leq \Omega_j \tag{27}$$

for all i, j with $p_i > 0$ and $p_j = 0$. We next show that such a finite $\bar{k}$ exists. If (27) holds for $k = \bar{k}_1$, we set $\bar{k} = \bar{k}_1$. Moreover, (27) always holds for k big enough. To see this, let i be such that $p_i > 0$ and, working towards a contradiction, assume that

$$\Omega_i > \Omega_j \tag{28}$$

for some j with $p_j = 0$. From (C1a) and (C1) in the proof of Proposition IV.5 with $p_j = 0$, it follows that $\gamma_{i,j} \leq 0$ and $\gamma_{i,n} \leq \gamma_{j,n}$ for all n; hence,

$$\begin{aligned}\Gamma^j_{y_i}(\gamma_j s) &= \Xi'(\Omega_j)\eta f_y\left(\sum_{n:\gamma_{j,n}\geq 0} \gamma_{j,n} s_n, p_j\right) = \Xi'(\Omega_j)\eta \\ &< \Xi'(\Omega_i)\eta = \Xi'(\Omega_i)\eta f_y(\gamma_{i,j} s_j, p_j) = \Gamma^i_{y_j}(\gamma_i s)\end{aligned}$$

using that $f_y(y, p_j) = 1$ because $p_j = 0$, Ξ is strictly increasing and strictly convex, and $\Omega_i > \Omega_j$. Then $\gamma_{i,j} = -k$ follows from $\Gamma^j_{y_i}(\gamma_j s) < \Gamma^i_{y_j}(\gamma_i s)$ by (4), and thus

$$\begin{aligned}\Omega_j &= \omega_j + f\left(\sum_{n:\gamma_{j,n}\geq 0} \gamma_{j,n} s_n, p_j\right) + \sum_{n:\gamma_{j,n}<0} f(\gamma_{j,n} s_n, p_n) \\ &\geq k s_i + \omega_j + f\left(\sum_{n:\gamma_{i,n}\geq 0} \gamma_{i,n} s_n, p_i\right) + \sum_{n:\gamma_{i,n}<0} f(\gamma_{i,n} s_n, p_n) \\ &= k s_i + \omega_j - \omega_i + \Omega_i.\end{aligned}$$

However, for $k \geq (\omega_i - \omega_j)/s_i$, this gives $\Omega_j \geq \Omega_i$ in contradiction to (28). Hence, we have that (27) holds for k big enough. By a compactness argument similar to page 2273 of Atkeson et al. (2015), we deduce that (27) holds for $k = \bar{k}$. By definition of $\bar{k}$, for $k < \bar{k}$, there exist i and j with $p_j = 0$ such that $\Omega_i > \Omega_j$.

We now consider $k \geq \bar{k}$ and

$$\beta(p,s) = \max_{i:p_i=p,s_i=s} \Omega_i, \quad \bar{i}(p,s) = \begin{cases} \arg\max\limits_{i:p_i=p,s_i=s} \Omega_i & \text{if } \beta(p,s) = \Omega_j \text{ for } j \text{ with } p_j = 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

$$\overline{\delta}(p,s) = \min_{i \in \bar{i}(p,s)} \omega_i, \quad \underline{\delta}(p,s) = \max_{\{i:p_i=p,s_i=s\} \setminus \bar{i}(p,s)} \omega_i$$

for $p \in \{p_1, \ldots, p_M\}$ and $s \in \{s_1, \ldots, s_M\}$ where the minimum (and maximum) over an empty set equals $+\infty$ and $-\infty$ by the usual convention. Several p_j and s_j for different j can take the same values, and thus $\bar{i}(p,s)$ can be a set with several entries because the maximum does not need to be attained at a unique i. We can choose a function $\bar{\alpha} : (0,1] \times [0,1] \to [0,\infty)$ for all s such that $\underline{\delta}(p,s) < \bar{\alpha}(p,s) \leq \overline{\delta}(p,s)$ for all $p \in \{p_1, \ldots, p_M\}$ and $s \in \{s_1, \ldots, s_M\}$. Note that $\bar{\alpha}(p,s)$ may depend here on both arguments p and s, but in the next paragraph, we will show that $\bar{\alpha}$ can be chosen independently of p. From $\bar{\alpha}(p,s) \leq \overline{\delta}(p,s)$, it follows that $A(\bar{\alpha})$ defined by

$$A(\bar{\alpha}) = \{i : \omega_i \geq \bar{\alpha}(p_i, s_i) \text{ or } p_i = 0\}$$

contains all indices i with $\Omega_i = \Omega_j$ for j with $p_j = 0$. To show that $A(\bar{\alpha})$ contains only such indices i, assume that there exists $i \in A(\bar{\alpha})$ with $\Omega_i < \Omega_j$ for j with $p_j = 0$. This implies

$$\omega_i \geq \bar{\alpha}(p_i, s_i) > \underline{\delta}(p_i, s_i);$$

hence, $\omega_i > \omega_\ell$ for all ω_ℓ with $\Omega_\ell < \Omega_j$, which contradicts $\Omega_i < \Omega_j$. Therefore, all banks $i \in A(\bar{\alpha})$ have the same post-trade exposure Ω_i while banks $i \notin A(\bar{\alpha})$ have a strictly smaller post-trade exposure. Thus, we can set $\bar{\Omega} = \Omega_i$ for some $i \in A(\bar{\alpha})$.

We next show that $\bar{\alpha}$ can be chosen independently of p, consider $k \geq \bar{k}$ and i with $p_i > 0$ and $\Omega_i = \Omega_j$ for j with $p_j = 0$. Because $k \geq \bar{k}$, it follows from (27) that $\Omega_i \geq \Omega_\ell$ for all ℓ. In the case $\Omega_i > \Omega_\ell$, we obtain $\gamma_{i,\ell} \leq 0$ by (C1a). In the case $\Omega_i = \Omega_\ell$, we argue similarly to the proof of (C1a) to show $\gamma_{i,\ell} \leq 0$. Indeed, to derive a contradiction, we assume that $\gamma_{i,\ell} > 0$ and $\Omega_i = \Omega_\ell$, which implies

$$\begin{aligned}
\Gamma^i_{y_\ell}(\gamma_i s) &= \Xi'(\Omega_i)\eta f_y\left(\sum_{n:\gamma_{i,n}\geq 0} \gamma_{i,n} s_n, p_i\right) \\
&= \Xi'(\Omega_\ell)\eta f_y\left(\sum_{n:\gamma_{i,n}\geq 0} \gamma_{i,n} s_n, p_i\right) \\
&> \Xi'(\Omega_\ell)\eta f_y(\gamma_{\ell,i} s_i, p_i) \\
&= \Gamma^\ell_{y_i}(\gamma_\ell s)
\end{aligned}$$

by strict convexity of $f(., p_i)$ from Lemma IV.1, using that $p_i > 0$. However, this implies $\gamma_{i,\ell} = -k$ by (4) in contradiction to the assumption $\gamma_{i,\ell} > 0$. Hence, we have $\gamma_{i,\ell} \leq 0$, and the trading choices of bank i do not depend on p_i. Using Lemma IV.1, we then deduce that, for all ℓ, $\Gamma^\ell(\gamma_\ell s)$ does not depend on p_i if $\gamma_{i,\ell} \leq 0$, and thus the objective function

$\sum_{\ell=1}^{M} s_\ell \Gamma^\ell(\gamma_\ell s)$ in (15) does not depend on p_i in the optimum. Therefore, $\bar{\alpha}$ can be chosen independently of p. From $\gamma_{i,\ell} \leq 0$ for all ℓ, we also deduce that $\bar{\alpha} \geq \bar{\Omega}$. This concludes the proof of the first part of the theorem.

To prove part 2a of the theorem, we consider two risky banks i and n with $\omega_i \geq \alpha$ and $\omega_n \geq \alpha$. From part 1b of the theorem, we know that the banks' post-trade exposures are $\Omega_i = \Omega_n = \bar{\Omega}$. Working towards a contradiction, we assume that $\gamma_{i,n} > 0$ so that bank i sells protection to bank n. We then have $\Gamma^i_{y_n}(\gamma_i) = \Gamma^n_{y_i}(\gamma_n)$ by (4), which implies

$$\Xi'(\Omega_i)\eta f_y\left(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\right) = \Xi'(\Omega_n)\eta f_y(\gamma_{n,i} s_n, p_i).$$

This equality cannot hold because $\Xi'(\Omega_i) = \Xi'(\Omega_n) = \Xi'(\bar{\Omega})$ and

$$f_y\left(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell} s_\ell, p_i\right) > f_y(\gamma_{n,i} s_n, p_j).$$

Therefore, we deduce that $\gamma_{i,n} > 0$ leads to a contradiction and so does $\gamma_{i,n} < 0$ by symmetry. Hence, we must have $\gamma_{i,n} = 0$, proving part 2a. Moreover, any bank i with initial exposure $\omega_i \geq \alpha$ will have a post-trade exposure $\bar{\Omega}$, which is strictly greater than the post-trade exposure of any risky bank n that has initial exposure $\omega_n < \alpha$. Therefore, bank i does not sell to bank n by (C1a). Similarly to part 2a, we can show that bank i does not sell to any safe bank, either, establishing part 2b. Consequently, the post-trade exposure of banks with initial exposure greater than or equal to α does not depend on the banks' own default probabilities, and neither does their purchased quantities. This shows part 2c.

For part 3, we consider two banks i and n that satisfy (25) and have exposures

$\tilde{\Omega}_i \leq \tilde{\Omega}_n$ after trading with other banks, but before trading between themselves. We deduce $\gamma_{i,n} \geq 0$ from (C1a). Working towards a contradiction, we assume that $\gamma_{i,n} > 0$. We then have $\Gamma^i_{y_n}(\gamma_i) = \Gamma^n_{y_i}(\gamma_n)$ by (4), which implies

$$\Xi'(\Omega_i)\eta f_y\Bigg(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell}s_\ell, p_i\Bigg) = \Xi'(\Omega_n)\eta f_y(\gamma_{n,i}s_n, p_i)$$

so that

$$\Xi'(\Omega_i)\eta f_y\Bigg(\sum_{\ell\neq n:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell}s_\ell, p_i\Bigg) < \Xi'(\Omega_n)\eta f_y(0, p_i) = \Xi'(\Omega_n)\eta(1-p_i),$$

which is a contradiction to

$$\frac{\Xi'(\Omega_n)}{\Xi'(\Omega_i)} \leq \frac{f_y\big(\sum_{\ell\neq n:\gamma_{i,\ell}\geq 0}\gamma_{i,\ell}s_\ell, p_i\big)}{1-p_i}.$$

which is implied by (25), $\gamma_{i,n} > 0$ and the convexity of Ξ by Lemma IV.1. This shows that $\gamma_{i,n} = 0$ so that banks i and n do not trade with each other, proving part 3a. Moreover, any bank i with initial exposure $\omega_i < \alpha$ will have a post-trade exposure smaller than $\bar{\Omega}$. Hence, its post-trade exposure is strictly smaller than that of any bank n that has initial exposure $\omega_n \geq \alpha$. Therefore, bank i does not buy from bank n by (C1a), proving part 3a. Finally, we note that safe banks and risky banks with initial exposures above α have all the same post-trade exposure $\bar{\Omega}$. Therefore, when traders of these banks meet traders of a risky bank i with initial exposure $\omega_i < \alpha$, they all have the same incentive compatibility condition, namely, either they do not trade or

$$\Xi'(\Omega_i)\eta f_y\Bigg(\sum_{\ell:\gamma_{i,\ell}\geq 0} \gamma_{i,\ell}s_\ell, p_i\Bigg) = \Xi'(\bar{\Omega})\eta f_y(\gamma_{n,i}s_n, p_j),$$

where n refers to any of the safe banks or risky banks with initial exposures above α. If all sizes s_n are equal, each bank n satisfies the same condition, thus each of them buys the same amount from bank i, as stated in part 3c. □

Proof of Corollary IV.8. Part 2b of Theorem A.4 implies that risky banks with initial exposures above α only purchase protection, hence they are not intermediaries.

Next we consider a bank n with initial exposure below α and that satisfies $\frac{\Xi'(\tilde{\Omega}_n)}{\Xi'(\tilde{\Omega}_i)} \leq \frac{f_y(\Gamma_i, p_i)}{1-p_i}$ for any risky bank i with $\tilde{\Omega}_i < \tilde{\Omega}_n$, where $\Gamma_i = \sum_{\ell \neq n: s_i \gamma_{i,\ell} \geq 0} \gamma_{i,\ell}$ is the sum of contracts sold by bank i, and $\tilde{\Omega}_i$ and $\tilde{\Omega}_n$ are the exposures of banks i and n, respectively, after trading with other banks but before trading between themselves. From part 3a of Theorem A.4, we obtain $\gamma_{i,n} = 0$ so that bank i and n do not trade between themselves. Moreover, part 3b of Theorem A.4 implies that bank n does not purchase protection from safe banks or risky banks with initial exposures above α. Therefore, bank n only sells protection, hence it is not an intermediary. □

A.D Results of Section V and their Proofs

Lemma A.5 (Lemma V.1)**.** *For given $s_1, \ldots, s_M$, the value of $x_i(p_1, \ldots, p_M)$ is uniquely determined.*

Proof. For general s_i, (9) becomes

$$x_i(p_1, \ldots, p_M) = \omega_i + \sum_{n \neq i} \gamma_{i,n} s_n R_{i,n} - \Gamma^i(\gamma_i s).$$

Using the definition (5) of $R_{i,n}$ and (4), we can write

$$
\begin{aligned}
x_i(p_1,\dots,p_M) &= \omega_i - \Gamma^i(\gamma_i s) + \sum_{n:\gamma_{i,n}>0} \gamma_{i,n} s_n \big(\nu \Gamma^n_{y_i}(\gamma_n s) + (1-\nu)\Gamma^i_{y_n}(\gamma_i s)\big) \\
&\quad + \sum_{n:\gamma_{i,n}<0} \gamma_{i,n} s_n \big(\nu \Gamma^i_{y_n}(\gamma_i s) + (1-\nu)\Gamma^n_{y_i}(\gamma_n s)\big) \\
&= \omega_i - \Gamma^i(\gamma_i s) + \nu \sum_{n:\gamma_{i,n}>0} \gamma_{i,n} s_n \big(\Gamma^n_{y_i}(\gamma_n s) - \Gamma^i_{y_n}(\gamma_i s)\big) \\
&\quad + (1-\nu) \sum_{n:\gamma_{i,n}<0} \gamma_{i,n} s_n \big(\Gamma^n_{y_i}(\gamma_n s) - \Gamma^i_{y_n}(\gamma_i s)\big) + \sum_{n\neq i} \gamma_{i,n} s_n \Gamma^i_{y_n}(\gamma_i s) \\
&= \omega_i - \Gamma^i(\gamma_i s) + \nu k \sum_{n:\gamma_{i,n}>0} s_n \big(\Gamma^n_{y_i}(\gamma_n s) - \Gamma^i_{y_n}(\gamma_i s)\big) \\
&\quad - (1-\nu) k \sum_{n:\gamma_{i,n}<0} s_n \big(\Gamma^n_{y_i}(\gamma_n s) - \Gamma^i_{y_n}(\gamma_i s)\big) + \sum_{\substack{p_n>0,\,\gamma_{i,n}<0,\text{ or} \\ p_i>0,\,\gamma_{i,n}>0}} \gamma_{i,n} s_n \Gamma^i_{y_n}(\gamma_i s) \\
&\quad + \overline{\Gamma}^i(\gamma_i s) \sum_{\substack{p_n=0,\,\gamma_{i,n}<0,\text{ or} \\ p_i=0,\,\gamma_{i,n}>0}} \gamma_{i,n} s_n,
\end{aligned}
$$

where

$$
\overline{\Gamma}^i(y) := \Gamma^i_{y_n}(y) = \frac{q e^{\eta\omega_i + \eta f(\sum_{n:y_n\geq 0} y_n, p_i) + \eta \sum_{n:y_n<0} f(y_n,p_n)}}{1 - q + q e^{\eta\omega_i + \eta f(\sum_{n:y_n\geq 0} y_n, p_i) + \eta \sum_{n:y_n<0} f(y_n,p_n)}}
$$

does not depend on the specific n for all n with $p_n = 0$ and $\gamma_{i,n} < 0$, or $p_i = 0$ and $\gamma_{i,n} > 0$. This means that $\Gamma^i_{y_n}$ is the same for all banks n that are (I) default-free protection sellers to i, or (II) protection buyers from i, and i is default-free. All these pairwise transactions do not bear any counterparty risk. Uniqueness of $x_i(p_1,\dots,p_M)$ now follows from Theorem A.2. □

Proof of Proposition V.3. We first note that the mapping $p_i \mapsto x_i(p_1,\dots,p_M)$ is continuous. This follows from the Envelope theorem using that Γ^i and its partial derivatives are differentiable. For $p_{-i} = (p_j)_{j \neq i}$, we define set-valued functions

$$r_i(p_{-i}) = \arg\max_{p_i \in [0, \bar{p}_i]} \big(x_i(p_1, \ldots, p_M) - C_i(p_i) \big), \quad r(p) = \big(r_1(p_{-1}), \ldots, r_M(p_{-M}) \big)$$

so that r is a mapping from $[0, \bar{p}_1] \times \cdots \times [0, \bar{p}_m]$ onto its power set. It has the following properties:

- $[0, \bar{p}_1] \times \cdots \times [0, \bar{p}_m]$ is compact, convex, and nonempty.
- For each p, $r(p)$ is nonempty because a continuous function over a compact set has always a maximizer.
- $r(p)$ is convex by assumption.
- It follows from Berge's maximum theorem that $r(p)$ has a closed graph.

Thanks to these properties, Kakutani's fixed point theorem implies that there exists a fixed point of the mapping r, which means that there exists an equilibrium. □

Proof of Proposition V.4. Because the function

$$\sum_{i=1}^{M} s_i \Gamma^i(\gamma_i s, p) + \sum_{i=1}^{M} s_i C_i(p_i)$$

is continuous over the compact set $[0, \bar{p}_1] \times \cdots \times [0, \bar{p}_M]$, it has a maximum, which shows the statement of the proposition, using that the social planner's optimization problem over $(\gamma_{i,n})_{i,n=1,\ldots,M}$ conditional on the choice of the default probabilities has a solution by Theorems IV.3 and IV.4. □

Theorem A.6 (Theorem V.5)**.** *The externality imposed by bank i on the system equals*

$$MSV_i(p) - MPV_i(p) = \frac{\partial S_i}{\partial p_i}(p) + k(1-\nu)\frac{\partial T_i}{\partial p_i}(p), \tag{29}$$

where

(30)

$$S_i(p) := \sum_{n\neq i} s_n\left(\gamma_{i,n}\Gamma^n_{y_i}(\gamma_n s, p) + \frac{1}{s_i}\Gamma^n(\gamma_n s, p)\right), \quad T_i(p) := \sum_{n\neq i} s_n\big(\Gamma^i_{y_n}(\gamma_i s, p) - \Gamma^n_{y_i}(\gamma_n s, p)\big).$$

For small enough p_i and large enough q, we have $\frac{\partial S_i}{\partial p_i}(p) < 0$ and $\frac{\partial T_i}{\partial p_i}(p) \geq 0$ with strict inequality when the trade size limit is binding for at least one bilateral trading relationship.

Proof. 1. part: proof of (29)*.*

For arbitrary bank sizes, marginal private and social values for bank i are given by

$$\begin{aligned} MSV_i(p) &= \sum_{n=1}^{M} s_n \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) + s_i C_i'(p_i), \\ MPV_i(p) &= s_i \frac{\partial \Gamma^i}{\partial p_i}(\gamma_i s, p) - s_i \sum_{n\neq i} \gamma_{i,n} s_n \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) + s_i C_i'(p_i) \end{aligned}$$

so that its difference is

$$MSV_i(p) - MPV_i(p) = \sum_{n\neq i} s_n \left(s_i \gamma_{i,n} \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p)\right).$$

If $\gamma_{i,n} \leq 0$, then we obtain from (5) that

$$R_{i,n}(\gamma_i s, \gamma_n s, p) = \nu \Gamma^i_{y_n}(\gamma_i s, p) + (1-\nu)\Gamma^n_{y_i}(\gamma_n s, p).$$

We then have that $\frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) = 0$ and $\frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) = 0$ because $\Gamma^n(\gamma_n s, p)$, $\Gamma^i_{y_n}(\gamma_i s, p)$, and $R_{i,n}(\gamma_i s, \gamma_n s, p)$ do not depend on p_i for $\gamma_{i,n} \leq 0$; if traders of bank i are buying CDSs from bank n, the default probability of bank i does not affect the terms of trade between traders of banks i and n. For $\gamma_{i,n} > 0$, we find

$$R_{i,n}(\gamma_i s, \gamma_n s, p) = \nu \Gamma^n_{y_i}(\gamma_n s, p) + (1-\nu)\Gamma^i_{y_n}(\gamma_i s, p)$$

by (4) and (5) so that

$$\begin{aligned}
MSV_i(p) - MPV_i(p) &= \sum_{n:\gamma_{i,n}>0} s_n \left(s_i \gamma_{i,n} \frac{\partial R_{i,n}}{\partial p_i}(\gamma_i s, \gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) \right) \\
&= \sum_{n:\gamma_{i,n}>0} s_n \left(s_i \gamma_{i,n} \frac{\partial \Gamma^n_{y_i}}{\partial p_i}(\gamma_n s, p) + \frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) \right) \\
&\quad + \sum_{n:\gamma_{i,n}>0} s_n s_i \gamma_{i,n}(1-\nu) \left(\frac{\partial \Gamma^i_{y_n}}{\partial p_i}(\gamma_i s, p) - \frac{\partial \Gamma^n_{y_i}}{\partial p_i}(\gamma_n s, p) \right) \\
&= \frac{\partial}{\partial p_i} \sum_{n \neq i} s_n \big(s_i \gamma_{i,n} \Gamma^n_{y_i}(\gamma_n s, p) + \Gamma^n(\gamma_n s, p) \big) \\
&\quad + \frac{\partial}{\partial p_i} \sum_{n \neq i} s_n s_i k (1-\nu) \big(\Gamma^i_{y_n}(\gamma_i s, p) - \Gamma^n_{y_i}(\gamma_n s, p) \big),
\end{aligned}$$

using for the last equality that $\frac{\partial \Gamma^n}{\partial p_i}(\gamma_n s, p) = 0$, $\frac{\partial \Gamma^n_{y_i}}{\partial p_i}(\gamma_n s, p) = 0$ and $\frac{\partial \Gamma^i_{y_n}}{\partial p_i}(\gamma_i s, p) = 0$ for $\gamma_{i,n} \leq 0$ and $\Gamma^i_{y_n}(\gamma_i s, p) = \Gamma^n_{y_i}(\gamma_n s, p)$ for $\gamma_{i,n} \in (-k, k)$. Combining this with (30), we conclude the proof of (29).

2. part: $\frac{\partial S_i}{\partial p_i}(p) < 0$ for small enough p_i and large enough q.

We recall from Lemma IV.1 that

$$\Gamma^n(y,p) = \frac{1}{\eta}\log\Big(1-q+q\mathrm{e}^{\eta\omega_n+\eta f(\sum_{\ell:y_\ell\geq 0} y_\ell,p_n)+\eta\sum_{\ell:y_\ell<0} f(y_\ell,p_\ell)}\Big)$$
$$= \frac{1}{\eta}\log\Big(1-q+a(1-p_i)\mathrm{e}^{\eta y_i}+ap_i\mathrm{e}^{\eta r y_i}\Big)$$

for $y_i<0$, where we use the abbreviation $a = q\mathrm{e}^{\eta\omega_n+\eta f(\sum_{\ell:y_\ell\geq 0} y_\ell,p_n)+\eta\sum_{\ell\neq i:y_\ell<0} f(y_\ell,p_\ell)}$ in this proof. We compute

$$\Gamma^n_{p_i}(y,p) = \frac{1}{\eta}\frac{-a\mathrm{e}^{\eta y_i}+a}{1-q+a(1-p_i)\mathrm{e}^{\eta y_i}+ap_i} = \frac{1}{\eta}\frac{1-\mathrm{e}^{\eta y_i}}{b+(1-p_i)\mathrm{e}^{\eta y_i}+p_i},$$

using the abbreviation $b=(1-q)/a$. Next, we find

$$\Gamma^n_{y_i,p_i}(y,p) = \frac{(b+(1-p_i)\mathrm{e}^{\eta y_i}+p_i)(-\mathrm{e}^{\eta y_i})-(1-\mathrm{e}^{\eta y_i})(1-p_i)\mathrm{e}^{\eta y_i}}{(b+(1-p_i)\mathrm{e}^{\eta y_i}+p_i)^2}$$
$$= \frac{b(-\mathrm{e}^{\eta y_i})-p_i\mathrm{e}^{\eta y_i}-(1-p_i)\mathrm{e}^{2\eta y_i}-(1-p_i)\mathrm{e}^{\eta y_i}+(1-p_i)\mathrm{e}^{2\eta y_i}}{(b+(1-p_i)\mathrm{e}^{\eta y_i}+p_i)^2}$$
$$= \frac{b(-\mathrm{e}^{\eta y_i})-\mathrm{e}^{\eta y_i}}{(b+(1-p_i)\mathrm{e}^{\eta y_i}+p_i)^2}. \tag{31}$$

For $p_i=0$, $q=1$ and $\gamma_{i,n}>0$, we obtain

$$\frac{\partial}{\partial p_i}\big(s_i\gamma_{i,n}\Gamma^n_{y_i}(\gamma_n s,p)+\Gamma^n(\gamma_n s,p)\big)\Big|_{p_i=0,q=1} = \big(s_i\gamma_{i,n}\Gamma^n_{y_i,p_i}(\gamma_n s,p)+\Gamma^n_{p_i}(\gamma_n s,p)\big)\big|_{p_i=0,q=1}$$
$$= \left(s_i\gamma_{i,n}\frac{b(-\mathrm{e}^{-\eta s_i\gamma_{i,n}})-\mathrm{e}^{-\eta s_i\gamma_{i,n}}}{(b+(1-p_i)\mathrm{e}^{-\eta s_i\gamma_{i,n}}+p_i)^2}+\frac{1}{\eta}\frac{1-\mathrm{e}^{-\eta s_i\gamma_{i,n}}}{b+(1-p_i)\mathrm{e}^{-\eta s_i\gamma_{i,n}}+p_i}\right)\Bigg|_{p_i=0,q=1}$$
$$= \frac{-s_i\gamma_{i,n}}{\mathrm{e}^{-\eta s_i\gamma_{i,n}}}+\frac{1-\mathrm{e}^{-\eta s_i\gamma_{i,n}}}{\eta\mathrm{e}^{-\eta s_i\gamma_{i,n}}} < \frac{-s_i\gamma_{i,n}}{\mathrm{e}^{-\eta s_i\gamma_{i,n}}}+\frac{\eta s_i\gamma_{i,n}}{\eta\mathrm{e}^{-\eta s_i\gamma_{i,n}}} = 0.$$

Using that $\gamma_{i,n}\Gamma^n_{y_i,p_i}(\gamma_n s,p)=0$ and $\Gamma^n_{p_i}(\gamma_n s,p)=0$, we deduce $\frac{\partial S_i}{\partial p_i}(p)\big|_{p_i=0,q=1}<0$ by the definition (30) of $S_i(p)$, which implies $\frac{\partial S_i}{\partial p_i}(p) < 0$ for small enough p_i and large enough q by continuity.

3. part: $\frac{\partial T_i}{\partial p_i}(p) \geq 0$ for small enough p_i and large enough q with strict inequality when the trade size limit is binding for at least one bilateral trading relationship.

We compare $\Gamma^n_{y_i,p_i}(\gamma_n, p)$ and $\Gamma^i_{y_n,p_i}(\gamma_i, p)$. We first note that $\Gamma^n_{y_i,p_i}(\gamma_n, p) = 0$ and $\Gamma^i_{y_n,p_i}(\gamma_i, p) = 0$ for $\gamma_{n,i} = -\gamma_{i,n} \geq 0$. For $p_i = 0$ and $q = 1$, we obtain from (31) that

$$\Gamma^n_{y_i,p_i}(y, p)\big|_{p_i=0,q=1} = \left.\frac{-b\mathrm{e}^{\eta y_i} - \mathrm{e}^{\eta y_i}}{(b + (1-p_i)\mathrm{e}^{\eta y_i} + p_i)^2}\right|_{p_i=0,q=1} = -\mathrm{e}^{-\eta y_i}$$

for $y_i < 0$. A calculation similar to (31) gives

$$\Gamma^i_{y_n,p_i}(y, p)\big|_{p_i=0,q=1} = \left.\frac{-\tilde{b}\mathrm{e}^{\eta\sum_{\ell:y_\ell\geq 0} y_\ell} - \mathrm{e}^{\eta\sum_{\ell:y_\ell\geq 0} y_\ell}}{(\tilde{b} + (1-p_i)\mathrm{e}^{\eta\sum_{\ell:y_\ell\geq 0} y_\ell} + p_i)^2}\right|_{p_i=0,q=1} = -\mathrm{e}^{-\eta\sum_{\ell:y_\ell\geq 0} y_\ell}$$

for $y_n > 0$, where $\tilde{b} = (1-q)/\big(q\mathrm{e}^{\eta\omega_i + \eta\sum_{\ell:y_\ell<0} f(y_\ell,p_\ell)}\big)$. Therefore, for $\gamma_{n,i} = -\gamma_{i,n} < 0$, we obtain

$$\Gamma^n_{y_i,p_i}(\gamma_n s, p)\big|_{p_i=0,q=1} < \Gamma^i_{y_n,p_i}(\gamma_i s, p)\big|_{p_i=0,q=1}$$

and thus $\frac{\partial T_i}{\partial p_i}(p) > 0$ in this case when k is binding for some bilateral trades. □

Other than these four restrictions, we do not make any further adjustments. In particular, our data set also includes settlement locations outside of the United States, which allows for a more complete coverage of CDS trades and, importantly, guarantees symmetry in the inclusion of CDS trades (the transactions of both buyers and sellers are accounted for). The resulting set consists of CDS data for 81 banks.

Initial exposure. For each of these 81 banks, we compute its initial exposure by using 2011 data from the Federal Financial Institutions Examination Council (FFIEC) form 031 ("call report"), as in Begenau, Piazzesi, and Schneider (2015). We compute the initial exposure of each bank as the discounted valuation of its securities and loan portfolio, including CDSs traded with nonbanks as explained above. For large banks that book their assets mainly in holding companies, we use securities and loan portfolios at the bank holding company level. We group the securities and loans into three categories and use a specific discount factor for each group: less than 1 year (using the 6-month U.S. Treasury rate to discount), 1 to five years (using the 2-year U.S. Treasury rate to discount), and more than five years (using the 7-year U.S. Treasury rate to discount). Given the low interest rate environment in 2011, the precise choice of the discounting date and rate does not have a significant effect on our results. For foreign banks that do not report to the FFIEC, we analyze individual annual reports from 2011 to find the maturity profile of their securities and loans. Most of these annual reports are dated Dec. 31, 2011, making them consistent with the domestic bank data. Some of them were released in March, June, or October of 2011, in line with the respective country's regulatory guidelines.

Default probabilities. The banks' default probabilities are calculated using CDS spread data from IHS Markit Ltd. (2018) via Wharton Research Data Services (WRDS). Because the default probabilities that are relevant for the analysis are those around the time of the transaction, we fix Jan. 3, 2011 as the proxy date for CDS transactions and use the spread on this date to infer the default probability. We use the average 5-year spread for Senior Unsecured Debt (Corporate/Financial) and Foreign Currency Sovereign Debt (Government) (SNRFOR). We compute the default probabilities from the CDS spreads applying standard techniques (credit triangle relation). For 19 among the 81 banks, CDS spread data were not available. For each of these banks, we instead use Moody's credit rating as of Jan. 2011 for its Senior Unsecured Debt, and relate the ratings to default probabilities by using corporate default rates over the 1982–2010 period from Moody's.

Intermediation volume. For each bank i, we compute the intermediation volume as $I_i = \min\{G_i^+, G_i^-\}$, where $G_i^+ = \sum_{n \neq i} \max\{\gamma_{i,n}, 0\}$ and $G_i^- = \sum_{n \neq i} \max\{-\gamma_{i,n}, 0\}$, following the definition in Section IV.C.