Network Centrality and Managerial Market Timing Ability

Online Appendix

Theodoros Evgeniou, Joel Peress, Theo Vermaelen, and Ling Yue *

Proofs of the Model

1 Proof of Lemma: Outsider's information choice and firm centrality

The model solved by backward induction starting from period 1, then proceeding to period 0.

1.1 The Period-1 Stock Price

In period 1, the stock price is pinned down in equilibrium by the representative outsider's expectation of the firm's cashflow, since he is risk neutral and unconstrained: $P = E(F|\mathcal{F}_O)$, where $\mathcal{F}_O = \{s_{nj}, n = 1, \ldots, N\}$ denotes the outsider's information set.

We conjecture that the outsider chooses precisions that are identical across shocks, i.e.,

*Evgeniou, theodoros.evgeniou@insead.edu, INSEAD; Peress, joel.peress@insead.edu, INSEAD; Vermaelen, theo.vermaelen@insead.edu, INSEAD; and Yue, ling.yue@insead.edu, National University of Singapore. We would like to thank Judson Caskey (the referee) for very helpful and insightful comments.

$\tau^{\varepsilon}_{nO} = \tau^{\varepsilon}_{O}$ for $n = 1, \ldots, N$. We shall confirm this conjecture later. From Bayes law:

$$E(F|\mathcal{F}_O) = \sum_{n=1}^{N} \frac{\tau^{\varepsilon}_{O}}{\tau^f + \tau^{\varepsilon}_{O}} s_{nO}, \tag{A.1}$$

and $\tau_O^{-1} \equiv Var(F|\mathcal{F}_O) = Var(\varphi + \sum_{n=1}^{N} f_n|\mathcal{F}_O) = Var(\varphi) + \sum_{n=1}^{N} Var(f_n|\mathcal{F}_O) = \tau^{\varphi^{-1}} + N(\tau^f + \tau^{\varepsilon}_{O})^{-1}$.

1.2 The Period-0 Learning Problem of Outsiders

We turn to the determination of the outsider's signal precisions. We assume that he chooses signal precisions that maximize his posterior precision, τ_O, about the firm's cash flow, subject to his capacity constraint, either equation (6), (7), or (8) in the paper, taking other outsiders' behaviour as given. We prove below that this intuitive objective indeed maximises the outsider's expected profit (see Section .1.3).

Since $\tau_O \equiv 1/Var(F|\mathcal{F}_O) = [\tau^{\varphi^{-1}} + \sum_{n=1}^{N}(\tau^f + \tau^{\varepsilon}_{nO})^{-1}]^{-1}$, maximizing the posterior precision is equivalent to minimizing the variance of the sum of the link-related cashflows, $\sum_{n=1}^{N} Var(f_n|\mathcal{F}_O) = \sum_{n=1}^{N}(\tau^f + \tau^{\varepsilon}_{nO})^{-1}$.

We solve this optimization problem for each of the three learning technologies we consider.

1.2.1 Variance capacity constraint

Under the variance capacity constraint, the outsider's optimization problem is

$$max_{\{\tau^{\varepsilon}_{nO}\}_{n=1}^{N}} - \sum_{n=1}^{N} Var(f_n|\mathcal{F}_O)$$

subject to: $\sum_{n=1}^{N} Var(f_n|\mathcal{F}_O) \geq N/\tau^f - k$ and $\tau^{\varepsilon}_{nO} \geq 0$ $for\ n = 1, \ldots, N.$

The constraint binds at the optimum, leading to a posterior variance and precision equal to $\sum_{n=1}^{N} Var(f_n|\mathcal{F}_O) = Max(0, N/\tau^f - k)$ and $1/Var(F|\mathcal{F}_O) = (\tau^{\varphi^{-1}} + N/\tau^f - k)^{-1}$, respectively. Individual variances are not determined, only their sum is; focusing on a symmetric equilibrium (i.e., identical τ^{ε}_{nO} across links), we obtain the following optimal precisions:

(A.2)

- $N \leq k\tau^f$, $\quad \tau^{\varepsilon}_{nO} = +\infty$ for $n = 1, \ldots, N$, and $\tau_O = \tau^{\varphi}$
- $N > k\tau^f$, $\quad \tau^{\varepsilon}_{nO} = \tau^f(\frac{N}{k\tau^f} - 1)^{-1}$ for $n = 1, \ldots, N$, and $\tau_O^{-1} = \tau^{\varphi^{-1}} + \frac{N}{\tau^f} - k$.

With this learning technology, it is possible for the outsider to know the $f_n's(n = 1, \ldots, N)$ without error, provided his capacity is large enough relative to the number of links ($N \leq k\tau^f$).

1.2.2 Linear precision constraint

Under the linear precision constraint, the outsider's optimization problem is

$$max_{\{\tau^{\varepsilon}_{nO}\}_{n=1}^{N}} - \sum_{n=1}^{N} (\tau^f + \tau^{\varepsilon}_{nO})^{-1}$$

subject to: $\quad \sum_{n=1}^{N} \tau^{\varepsilon}_{nO} \leq k'$ $\quad$ and $\quad \tau^{\varepsilon}_{nO} \geq 0 \quad for\, n = 1, \ldots, N.$

Maximizing the Lagrangian leads to the following first-order conditions

$$(\tau^f + \tau^{\varepsilon}_{nO})^{-2} = \nu' \qquad for \quad n = 1, \ldots, N,$$

where ν' is the Lagrange multiplier on the capacity constraint. This system of equations implies that the $\tau^{\varepsilon}_{nO}{'s}$ are equated across links n:

(A.3) $$\tau^{\varepsilon}_{nO} = \frac{k'}{N} \qquad for \quad n = 1, \ldots, N.$$

It follows that $\tau_O^{-1} = \tau^{\varphi^{-1}} + N(\tau^f + k'/N)^{-1}$.

1.2.3 Entropy constraint

Because all random variables (shocks and signal errors) are i.i.d., the prior and posterior variance-covariance matrices are diagonal. The determinant of these matrices is simply the product of their diagonal elements: $|\sum| = \prod_{n=1}^{N} \tau^{f^{-1}} = \tau^{f^{-N}}$ and $|\widehat{\sum}| = \prod_{n=1}^{N}(\tau^f + \tau^{\varepsilon}_{nO})^{-1}$.

The outsider's optimization problem is

$$max_{\{\tau^{\varepsilon}_{nO}\}_{n=1}^{N}} - \sum_{n=1}^{N}(\tau^f + \tau^{\varepsilon}_{nO})^{-1}$$

subject to: $\prod_{n=1}^{N}(\tau^f + \tau^{\varepsilon}_{nO}) \leq k''\tau^{f^N}$ and $\tau^{\varepsilon}_{nO} \geq 0$ $for\, n = 1, \ldots, N.$

Maximizing the Lagrangian leads to the following first-order conditions

$$(\tau^f + \tau^{\varepsilon}_{nO})^{-2} = \nu'' \prod_{m=1, m\neq n}^{N} (\tau^f + \tau^{\varepsilon}_{mO}) \qquad for \quad n = 1, \ldots, N,$$

where ν'' is the Lagrange multiplier on the capacity constraint. The first-order conditions imply

$$(\tau^f + \tau^{\varepsilon}_{nO})^{-1} = \nu'' \prod_{m=1}^{N} (\tau^f + \tau^{\varepsilon}_{mO})^{-1} = \nu'' k'' \tau^{f^N} \qquad for \quad n = 1, \ldots, N,$$

where the second equality results from the capacity constraint being binding. This system of equations implies that the τ^{ε}_{nO}'s are equated across links n:

(A.4) $$\tau^{\varepsilon}_{nO} = \tau^f(k''^{1/N} - 1) \quad for \quad n = 1, \ldots, N.$$

It follows that $\tau_O^{-1} = \tau^{\varphi^{-1}} + N(\tau^f k''^{1/N})^{-1}$.

Despite their differences, all three specifications imply that i) outsiders' precision is (weakly) decreasing in the number of links N (i.e., their information about each single link is less precise when there are more links to investigate), and ii) this precision is increasing in the learning capacity (k, k', or k'').

1.3 Proof that the outsider's optimal decision is to maximize $\tau_O \equiv 1/Var(F|\mathcal{F}_O)$*, the precision of his information about the firm's total cashflow*

When solving the outsider's learning problem, we postulated that he chooses signals such that the precision of his information about the firm's total cashflow, $\tau_O \equiv 1/Var(F|\mathcal{F}_O)$, is maximised. Here, we demonstrate that this intuitive rule is indeed optimal.

Under risk neutrality and in the absence of any restriction on trading (e.g., on borrowing or short-selling), the outsider's learning strategy is undetermined. Indeed, consider an outsider, labelled $O*$, who takes as given the information choice of the representative outsider. His expected profit in period 1 is infinite since he will buy (respectively, sell) an infinite number of shares if his expectation of the firm's cashflow F is greater (respectively, smaller) than that of the representative outsider. [A1] It follows that his profit expected in period 0 is also infinite, regardless of his precision choices, which therefore are indeterminate.

To break this indeterminacy, we solve the learning problem faced by a risk averse outsider and then drive his risk aversion to zero. We will establish that a risk averse outsider finds it optimal to maximise τ_O regardless of his degree of risk aversion. It follows that a risk averse outsider whose risk aversion is infinitesimally small—in other words, a risk neutral outsider—also finds this rule optimal.

We assume that the outsider's utility exhibits constant absolute risk aversion (CARA), where the coefficient of risk aversion is denoted γ. Risk neutrality corresponds to $\gamma = 0$. We normalize the outsider's initial wealth to 0, without loss of generality. Hence his terminal wealth is equal to the profit earned from portfolio investments, $\pi_O = X_O(F - P)$, where X_O denotes his stockholding. Thus, his objective is to maximize his expectation of $U \equiv -e^{-\gamma X_O(F-P)}$.

We proceed by backward induction as before, solving first for the equilibrium price in period 1 given arbitrary precisions, and then progressing to period 0 to determine optimal precisions.

[A1] Of course, in equilibrium, the expectations of $O*$ and of the representative outsider are identical, so that the price equals the expectation of the representative outsider.

1.3.1 The Period-1 Portfolio Problem of the Outsider

We solve for the outsider's optimal portfolios decision, taking his information choices as given. At this point, we conjecture that he chooses precisions that are identical across shocks, i.e., $\tau^{\varepsilon}_{nO} = \tau^{\varepsilon}_{O}$ for $n = 1, \ldots, N$. We shall confirm this conjecture later.

The outsider optimal portfolio is given by $X_O = \frac{\tau_O[E(F|\mathcal{F}_O)-P]}{\gamma}$, where τ_O and $E(F|\mathcal{F}_O)$ are given in equation (A.1). Aggregating asset demands across investors, neglecting the insider's demand who is assumed infinitesimal, and imposing market clearing, leads to the following equilibrium price P:

$$P = E(F|\mathcal{F}_O) - \frac{\gamma \bar{X}}{\tau_O} \tag{A.5}$$

Note that by setting γ to zero in the portfolio holding and price equations above, one reverts to the economy with risk neutral outsiders.

To solve for the outsider's information choice in period 0, we consider an outsider, labelled $O*$, who takes as given the information choice of the representative outsider. (At this stage outsider $O*$ differs from the representative outsider, but in equilibrium, they will be identical). It will prove useful to define the outsider $O*$'s Sharpe Ratio:

$$SR_O* \equiv [E(F|\mathcal{F}_{O*}) - P]\sqrt{\tau_{O*}} = \gamma X_{O*}/\sqrt{\tau_{O*}}, \tag{A.6}$$

Substituting in this formula the expression for the price yields:

$$SR_O* \equiv [E(F|\mathcal{F}_{O*}) - E(F|\mathcal{F}_O) + \frac{\gamma \bar{X}}{\tau_O}]\sqrt{\tau_{O*}} \tag{A.7}$$

$$= [\sum_{n=1}^{N} \frac{\tau^{\varepsilon}_{O*}}{\tau^f + \tau^{\varepsilon}_{O*}} s_{nO*} - \sum_{n=1}^{N} \frac{\tau^{\varepsilon}_{O}}{\tau^f + \tau^{\varepsilon}_{O}} s_{nO} + \frac{\gamma \bar{X}}{\tau_O}]\sqrt{\tau_{O*}}$$

$$= [\sum_{n=1}^{N} (\frac{\tau^{\varepsilon}_{O*}}{\tau^f + \tau^{\varepsilon}_{O*}} - \frac{\tau^{\varepsilon}_{O}}{\tau^f + \tau^{\varepsilon}_{O}}) f_n + \sum_{n=1}^{N} \frac{\tau^{\varepsilon}_{O*}}{\tau^f + \tau^{\varepsilon}_{O*}} \varepsilon_{nO*} - \sum_{n=1}^{N} \frac{\tau^{\varepsilon}_{O}}{\tau^f + \tau^{\varepsilon}_{O}} \varepsilon_{nO} + \frac{\gamma \bar{X}}{\tau_O}]\sqrt{\tau_{O*}}$$

Trading profits (and terminal wealth) equal $\pi_{O*} = X_{O*}(F - P)$. The mean and variance of trading profits, as of period 1, are given by the following expressions, after substituting out X_{O*}:

Substituting in this formula the expression for the price yields:

$$E(\pi_{O*}|\mathcal{F}_{O*}) = [E(F|\mathcal{F}_{O*}) - P]X_{O*} = \frac{SR^2_{O*}}{\gamma}; \tag{A.8}$$

and

$$Var(\pi_{O*}|\mathcal{F}_{O*}) = \frac{X^2_{O*}}{\tau_{O*}} = \frac{SR^2_{O*}}{\gamma^2}. \tag{A.9}$$

Because π_{O*} is normally distributed conditional on period-1 information, the outsider's expected utility equals:

$$E(U|\mathcal{F}_{O*}) = E(-e^{-\gamma W}|\mathcal{F}_{O*}) = E(-e^{-\gamma\pi_{O*}}|\mathcal{F}_{O*}) \tag{A.10}$$

$$= -e^{-\gamma E(\pi_{O*}|\mathcal{F}_{O*}) + \gamma^2 Var(\pi_{O*}|\mathcal{F}_{O*})/2} = -e^{-\frac{SR_{O*}{}^2}{2}}.$$

1.3.2 The Period-0 Learning Problem of the Outsider

In period 0, outsider $O*$ has expected utility:

$$E[E(U|\mathcal{F}_{O*})] = -E(e^{-\frac{SR^2_{O*}}{2}}).$$

At that time, SR_{O*} is normally distributed so this expected utility is the mean of the exponential of a chi-square distributed random variable. Hence,

$$E[E(U|\mathcal{F}_{O*})] = -\frac{1}{\sqrt{Var(SR_{O*}) + 1}} e^{-\frac{1}{2}\frac{E(SR_{O*})^2}{Var(SR_{O*})+1}}$$

We compute next the mean and variance of SR_{O*} for an outsider with arbitrary signal precisions τ^{ε}_{nO*} for $n = 1, \ldots, N$ which might differ across shocks. Taking the expectation of equation (A.7) yields

$$E(SR_{O*}) = \sqrt{\tau_j}\frac{\gamma \bar{X}}{\tau_O} \tag{A.11}$$

since all random variables have mean zero. Likewise, taking the variance of equation (A.7) yields

$$Var(SR_{O*}) = \tau_{O*}[\sum_{n=1}^{N}(\frac{\tau^{\varepsilon}_{nO*}}{\tau^f + \tau^{\varepsilon}_{nO*}} - \frac{\tau^{\varepsilon}_{O}}{\tau^f + \tau^{\varepsilon}_{O}})^2 \frac{1}{\tau^f} + \sum_{n=1}^{N} \frac{{\tau^{\varepsilon}_{nO*}}^2}{(\tau^f + \tau^{\varepsilon}_{nO*})^2} \frac{1}{\tau^{\varepsilon}_{nO*}} + \frac{{\tau^{\varepsilon}_{O}}^2}{(\tau^f + \tau^{\varepsilon}_{O})^2} \frac{N}{\tau^{\varepsilon}_{O}}]$$

Expanding the square, rearranging and simplifying leads to

$$Var(SR_{O*}) = \frac{\tau_{O*}}{\tau_f}[\sum_{n=1}^{N} \frac{\tau^{\varepsilon}_{nO*}}{\tau^f + \tau^{\varepsilon}_{nO*}} + \frac{N\tau^{\varepsilon}_{O}}{\tau^f + \tau^{\varepsilon}_{O}} - 2\frac{\tau^{\varepsilon}_{O}}{\tau^f + \tau^{\varepsilon}_{O}} \sum_{n=1}^{N} \frac{\tau^{\varepsilon}_{nO*}}{\tau^f + \tau^{\varepsilon}_{nO*}}]$$

Rearranging further implies

$$Var(SR_{O*}) = \frac{1}{\tau^f + \tau^{\varepsilon}_{O}}[a\tau_{O*} - \tau^f + \tau^{\varepsilon}_{O}] \tag{A.12}$$

where $a \equiv N + \frac{\tau^f - \tau^{\varepsilon}_{O}}{\tau^{\varphi}}$.

Plugging in the expressions for $E(SR_{O*})$ and $Var(SR_{O*})$ yields the following expression for outsider $O*$'s expected utility:

$$E[E(U|\mathcal{F}_{O*})] = -\frac{\sqrt{\tau^f + \tau^{\varepsilon}_{O}}}{\sqrt{a\tau_{O*} + 2\tau^{\varepsilon}_{O}}} e^{-\frac{c}{2}\frac{\tau_{O*}}{a\tau_{O*} + 2\tau^{\varepsilon}_{O}}}$$

where $c \equiv (\gamma \bar{X}/\tau_O)^2(\tau^f + \tau^{\varepsilon}_{O}) \geq 0$.

Outsider $O*$ maximises this expression with respect to his signals precisions, τ^{ε}_{nO*} (for $n = 1, \ldots, N$), subject to his capacity constraint, either equation (6), (7), or (8) in the paper, taking other outsiders' behaviour, represented by τ^{ε}_{O}, as given.

$E[E(U|\mathcal{F}_{O*})]$ is increasing in the outsider's posterior precision τ_{O*}. To see why, first note that $E[E(U|\mathcal{F}_{O*})]$ increasing in τ_{O*} is equivalent to

$$f(\tau_{O*}) \equiv -2ln[E[E(U|\mathcal{F}_{O*})]] = ln(a\tau_{O*} + 2\tau_O^{\varepsilon}) + \frac{c\tau_{O*}}{a\tau_{O*} + 2\tau_O^{\varepsilon}} - ln(\tau^f + \tau_O^{\varepsilon})$$

increasing in τ_{O*}. Since $f'(\tau_{O*}) = \frac{a(\tau_{O*}+2\tau_O^{\varepsilon})+2c\tau_O^{\varepsilon}}{(a\tau_{O*}+2\tau_O^{\varepsilon})^2}$ where $c \geq 0$, f is increasing in τ_{O*} if $a > 0$. It is a priori unclear what the sign of a is. Let's suppose $a < 0$. In that case, the outsider's expected utility is decreasing in his posterior precision τ_{O*}, leading to an optimal signal precision, τ_{nO*}^{ε}, of zero across all shocks n. As a result, $\tau_O^{\varepsilon} = 0$ in equilibrium, which in turn leads to $a > 0$ and contradicts our premise. Thus, a must be positive in equilibrium, and the outsider's expected utility is increasing in his posterior precision τ_{O*}.

Hence, a risk averse outsider's optimization problem amounts to choosing signals precisions, τ_{nO}^{ε} (for $n = 1, \ldots, N$) that maximize his posterior precision τ_O subject to his capacity constraint. By continuity, this rule remains optimal for a risk neutral outsider—one with an infinitesimally small risk aversion.

2 Proof of Proposition: Insider's profit and firm centrality

We study the (period-0) expectation of the insider's profit, conditional on the insider buying shares. The insider's profit per share purchased equals $F - P$. She purchases shares if and only if she considers them underpriced, i.e. if $E(F|\mathcal{F}_I) - P > 0$. Hence, her expected profit, denoted π_I, is given by:

$$\pi_I = E[F - P|E(F|\mathcal{F}_I) - P > 0].$$

Substituting in the expressions for the cash flow and the equilibrium price yields:

$$F - P = (\varphi + \sum_{n=1}^{N} f_n) - E(F|\mathcal{F}_O) = (\varphi + \sum_{n=1}^{N} f_n) - (\sum_{n=1}^{N} \tau_O^{\varepsilon}(\tau^f + \tau_O^{\varepsilon})^{-1} s_{nO})$$

$$= \varphi + \sum_{n=1}^{N} \tau^f (\tau^f + \tau^\varepsilon_O)^{-1} f_n - \sum_{n=1}^{N} \tau^\varepsilon_O (\tau^f + \tau^\varepsilon_O)^{-1} \varepsilon_{nO},$$

and

$$E(F|\mathcal{F}_I) - P = E(F|\mathcal{F}_I) - E(F|\mathcal{F}_O)$$

$$= \varphi + \sum_{n=1}^{N} \frac{\tau^f(\delta - \tau^\varepsilon_O)}{(\tau^f + \delta)(\tau^f + \tau^\varepsilon_O)} f_n - \sum_{n=1}^{N} \frac{\tau^\varepsilon_O}{\tau^f + \tau^\varepsilon_O} \varepsilon_{nO} + \sum_{n=1}^{N} \frac{\delta}{\tau^f + \delta} \varepsilon_{nI},$$

where we used that $\tau^\varepsilon_{nI} = \delta$ for $n = 1, \ldots, N$, by assumption.

The expectation of a mean-zero random variable x conditioned on another mean-zero random variable y being positive is

$$E(x|y > 0) = q \frac{Cov(x, y)}{\sqrt{Var(y)}},$$

where $q \equiv \frac{\phi(O)}{1-\Phi(O)} \approx 0.8$, ϕ and Φ are the probability and cumulative density functions of the standard normal distribution. Applying this formula to the insider's expected profit yields:

$$\pi_I = q \frac{Cov(F - P, E(F|\mathcal{F}_I) - P)}{\sqrt{Var(E(F|\mathcal{F}_I) - P)}}$$

where

$$Cov(F - P, E(F|\mathcal{F}_I) - P) = \frac{1}{\tau^\varphi} + \frac{N\delta}{(\tau^f + \delta)(\tau^f + \tau^\varepsilon_O)}$$

and

$$Var(E(F|\mathcal{F}_I) - P) = \frac{1}{\tau^\varphi} + \frac{N\tau^f(\delta - \tau^\varepsilon_O)^2}{(\tau^f + \delta)^2(\tau^f + \tau^\varepsilon_O)^2} + \frac{N\tau^\varepsilon_O}{(\tau^f + \tau^\varepsilon_O)^2} + \frac{N\delta}{(\tau^f + \delta)^2}$$

$$= Cov(F - P, E(F|\mathcal{F}_I) - P) + \frac{N\tau^\varepsilon_O}{(\tau^f + \delta)(\tau^f + \tau^\varepsilon_O)}.$$

The effect of the number of links N on π_I can be decomposed into two parts, as shown in the following equation:

$$\frac{d\pi_I}{dN} = \frac{\partial \pi_I}{\partial N} + \frac{\partial \pi_I}{\partial \tau^\varepsilon_O} \frac{d\tau^\varepsilon_O}{dN}.$$

The first term,

$$\frac{\partial \pi_I}{\partial N} = \frac{q}{Var(E(F|\mathcal{F}_I) - P)^{3/2}} \frac{(\delta - \tau_O^\varepsilon)(\tau^f + \delta)/\tau^\varphi + N\delta}{(\tau^f + \delta)^2(\tau^f + \tau_O^\varepsilon)},$$

represents the direct effect of N on π_I, i.e., keeping τ_O^ε, the precision of outsiders' signals, constant. Its sign depends on τ_O^ε.

- If $\tau_O^\varepsilon > \delta(1 + N\tau^\varphi/(\tau^f + \delta))$, then $\frac{\partial \pi_I}{\partial N} < 0$. Intuitively, increasing the number of link-related shocks increases uncertainty for the insider more than that for outsiders, since the latter knows each shock much better than the former does. Therefore, the insider's information advantage relative to the outsider, and hence her expected profit, decrease.
- If instead $\tau_O^\varepsilon < \delta(1 + N\tau^\varphi/(\tau^f + \delta))$, then $\frac{\partial \pi_I}{\partial N} > 0$. The intuition is now reversed: as the number of links grows, the insider's information advantage relative to outsiders, and hence her expected profit, increase.

The second term, $\frac{\partial \pi_I}{\partial \tau_O^\varepsilon}\frac{d\tau_O^\varepsilon}{dN}$, represents the indirect effect of the number of links N on π_I, through outsiders' information choice. Since

$$\frac{\partial \pi_I}{\partial \tau_O^\varepsilon} = -\frac{q}{Var(E(F|\mathcal{F}_I) - P)^{3/2}} \frac{N}{(\tau^f + \delta)(\tau^f + \tau_O^\varepsilon)^2}\left[\frac{\tau^f + \delta}{\tau^\varphi} + \frac{N\delta}{\tau^f + \tau_O^\varepsilon} + \frac{2N\delta\tau_O^\varepsilon}{(\tau^f + \delta)(\tau^f + \tau_O^\varepsilon)}\right]$$

is negative and $\frac{d\tau_O^\varepsilon}{dN} < 0$, this term is positive. Through this channel, the insider performs better as N increases because the outsider needs to reduce his signal precision as he spreads his scarce learning capacity more thinly across the N shocks (τ_O^ε lower); this improve the insider's information advantage relative to the outsider.

The net effect of N on the insider's profit depends on the sign and magnitude of these two channels. These, in turn, depend on the number of links and the outsider's learning capacity.

- If there are few links (e.g., $N = 1$) and a large capacity, then τ_O^ε is large, so $\frac{\partial \pi_I}{\partial N}$ is negative and large in absolute value, and hence $\frac{d\pi_I}{dN} < 0$. In words, adding links leads to a reduction in the insider's profit, because the outsider is better informed than the insider about link-related shocks.

- If instead N is large, then τ_O^ε is small, so $\frac{\partial \pi_I}{\partial N} > 0$ and hence $\frac{d\pi_I}{dN} > 0$. In that case, adding links increases the insider's expected profit, because her informational advantage grows.

Thus, the insider's profit is U-shaped as a function of the number of links N, provided the outsider's learning capacity is large enough. If instead this capacity is low, then the profit increases with N for all N.

We establish next this result formally for each of the three learning technologies. We start with the case of the variance capacity constraint:

- When $N \leq k\tau_f$, $\tau_O^\varepsilon = +\infty$; so taking limits in the above expressions yields

$$Var(E(F|\mathcal{F}_I) - P) \approx \frac{1}{\tau^\varphi} + \frac{N}{\tau^f + \delta}, \frac{\partial \pi_I}{\partial N} \approx -\frac{q}{Var(E(F|\mathcal{F}_I) - P)^{3/2}} \frac{1/\tau^\varphi}{\tau^f + \delta} < 0$$

  and $\frac{\partial \pi_I}{\partial \tau_O^\varepsilon} \approx 0$, leading to $\frac{d\pi_I}{dN} < 0$.
- When N is large, $\tau_O^\varepsilon = (N/k\tau_f - 1)^{-1}$ which converges towards zero as N grows to infinity. It follows that

$$Var(E(F|\mathcal{F}_I) - P) \approx \frac{N\delta}{\tau^f(\tau^f + \delta)}$$

  grows to infinity, and that

$$\frac{\partial \pi_I}{\partial N} \approx \frac{q}{Var(E(F|\mathcal{F}_I) - P)^{3/2}} \frac{N\delta}{(\tau^f + \delta)\tau_f}$$

  converges towards zero, leading to $\frac{d\pi_I}{dN} \geq 0$.

The resulting pattern is a U shape, provided that $k > 1 / \tau^f$. Intuitively, adding links favours the outsider as long as $N \leq k\tau^f$ because the outsider is able to learn each link-related shock perfectly; but beyond a number of links, he can no longer keep pace and his information per link deteriorates, giving a greater advantage to the insider.

The other two constraints also lead to similar U–shaped patterns for π_I. The only difference relative to the case of the variance capacity constraint is that the downward-sloping branch is now less pronounced because the outsider's information about link-related shocks is imperfect even at low levels of N: the outsider's precision about these shocks starts to deteriorate as links are added starting from the very first link (whereas in the variance capacity constraint case, it remains infinite up to τ^f links). Again, the U shape obtains only if the capacity is large enough, because it ensures that the outsider is well informed about link-related shocks when there are few links, and thus that adding links reduces his disadvantage relative to the insider (who knows perfectly the shock φ but not the link-related shocks f_n).

- Formally, substitute $N = 1$ into the expressions for $\frac{\partial \pi_I}{\partial N}$ and $\frac{\partial \pi_I}{\partial \tau_O^\varepsilon}$; the condition $\frac{d\pi_I}{dN} < 0$ is then equivalent to

$$\tau^\varphi \delta - (\tau^f + \delta + \frac{\tau^\varphi \delta}{\tau^f + \tau_O^\varepsilon} + \frac{2\tau^\varphi \delta \tau_O^\varepsilon}{(\tau^f + \delta)(\tau^f + \tau_O^\varepsilon)})\frac{d\tau_O^\varepsilon}{dN} < (\tau_O^\varepsilon - \delta)(\tau^f + \tau_O^\varepsilon), \tag{A.13}$$

where τ_O^ε and $\frac{d\tau_O^\varepsilon}{dN}$ are evaluated at $N = 1$. Under the linear precision constraint, $\tau_O^\varepsilon = \frac{k'}{N} = k'$ and $\frac{d\tau_O^\varepsilon}{dN} = -\frac{k'}{N^2} = -k'$ for $N = 1$ so condition (A.13) can be stated as:

$$\frac{\tau^\varphi \delta(\tau^f + k') + (\tau^f + \delta)(\tau^f + k')k' + \tau^\varphi \delta + \frac{2\tau^\varphi \delta}{\tau^f + \delta}k'^2}{(k' - \delta)(\tau^f + k')^2} < 1.$$

The numerator of the ratio on left-hand side of this inequality grows with k' to infinity at a rate k'^2 while its denominator grows to infinity at a rate k'^3. Hence, there exist a threshold, $\bar{k}'$, such that, for any $k' > \bar{k}'$, this ratio is smaller than 1. Likewise, under the entropy constraint, $\tau_O^\varepsilon = \tau^f(k''^{1/N} - 1) = \tau^f(k'' - 1)$ and

$$\frac{\tau_O^\varepsilon}{dN} = \frac{-\tau^f(ln(k''))k''^{1/N}}{N^2} = -\tau^f k'' ln(k'')$$

for $N = 1$ so condition (A.13) is equivalent to:

$$\frac{\tau^{\varphi}\delta + ((\tau^f + \delta)k'' + \frac{\tau^{\varphi}\delta}{\tau^f} + \frac{2\tau^{\varphi}\delta\tau^f(k''-1)}{(\tau^f+\delta)\tau^f})\tau^f ln(k'')}{\tau^f k''(\tau^f(k''-1)-\delta)} < 1.$$

The numerator of the ratio on left-hand side of this inequality grows with k'' to infinity at a rate $k''ln(k'')$ while its denominator grows to infinity at a rate k'^2. Hence, there exist a threshold, $\bar{k}''$, such that, for any $k'' > \bar{k}''$, this ratio is smaller than 1. Thus, under both the linear precision and entropy constraints, there exist a threshold such that condition (A.13) is satisfied for any capacity larger than this threshold. In words, if the outsider's learning capacity is large enough, the downward-sloping branch of the U shape obtains.

- The upward-sloping branch of the U shape obtains, as in the case of the variance capacity constraint, because τ_O^{ε} converges to 0, regardless of the learning technology employed. That is, eventually (i.e., for high enough number of links N), the outsider lacks the resources to investigate all links.