Internet Appendix "Pricing Liquidity Risk with Heterogeneous Investment Horizons"

Alessandro Beber, Joost Driessen, Anthony Neuberger, and Patrick Tuijp

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In this document we provide supplementary material to the paper. We start with proofs and derivations of the results in the paper. Next, we provide details on the estimation of the model. We conclude with a number of additional empirical results.

I. Proofs and Derivations

A. Main result

In this section we start with the proof that a unique stationary equilibrium exists for the setting that we consider. We consider N classes of investors as this shows the generality of the result.

Proof of Proposition 1: The solution to the model in Section III of the paper has risk averse, price-taking investors of cohort j choosing their demand vectors $y_{j,t}$ to maximize their expected utility at time t in the face of a price vector P_t (equation (1) in the paper) and prices being set to bring demand and supply into equilibrium (equation (2) in the paper). The search for the solution can be formalized as

Problem 1. To find $\left\{y_{j,t}, P_t \mid j = 1,\ldots,N;\ t \in \mathbb{Z}\right\}$ such that it solves

$$\text{(IA.1)} \qquad \max_{y_{j,t}} \mathbb{E}_t\left[W_{j,t+h_j}\right] - \frac{1}{2}A_j \mathrm{Var_t}\left(W_{j,t+h_j}\right), \text{ s.t. } y_{j,t} \geq 0$$

$$W_{j,t+h_j} = \left(P_{t+h_j} + \sum_{k=1}^{h_j} R_f^{h_j-k} D_{t+k} - C_{t+h_j}\right)' y_{j,t} + R_f^{h_j}\left(e_j - P_t' y_{j,t}\right)$$

and

$$\text{(IA.2)} \qquad \sum_{j=1}^{N} Q_j y_{j,t} = S - \sum_{j=1}^{N} \sum_{k=1}^{h_j-1} Q_j y_{j,t-k}.$$

As explained in the paper, we restrict ourselves to stationary equilibria. Our problem therefore becomes

Problem 2. To find $\left\{y_j, P \mid j = 1,\ldots,N\right\}$ such that it solves

$$\text{(IA.3)} \qquad \max_{y_j} \mathbb{E}\left[W_{j,t+h_j}\right] - \frac{1}{2}A_j \mathrm{Var}\left(W_{j,t+h_j}\right), \text{ s.t. } y_j \geq 0$$

$$W_{j,t+h_j} = y_j'\left(Z_{j,t+h_j} - \left(R_f^{h_j} - 1\right)P\right) + R_f^{h_j} e_j$$

$$\text{(IA.4)} \qquad Z_{j,t+h_j} = \sum_{k=1}^{h_j} R_f^{h_j-k} D_{t+k} - C_{t+h_j}$$

and

$$\text{(IA.5)} \qquad \sum_{j=1}^{N} h_j Q_j y_j = S.$$

We demonstrate the existence and uniqueness of the solution to Problem 2 indirectly, by first considering a related problem. Take the perspective of the social planner who seeks to allocate stocks to individuals to maximize aggregate utility. The social planner is not concerned about endowments and prices. Her problem is to maximize the weighted average of individual utilities (with some strictly positive weights θ) over dividends net of transaction costs. Her problem can be written as

Problem 3. To find $\{y_j \mid j = 1, \ldots, N\}$ such that it solves

(IA.6)
$$\max_{y_j} \sum_{j=1}^{N} \theta_j \left(\mathbb{E}\left[y_j' Z_{j,t+h_j}\right] - \frac{1}{2} A_j y_j' \mathrm{Var}\left(Z_{j,t+h_j}\right) y_j \right)$$

subject to

(IA.7)
$$\sum_{j=1}^{N} h_j Q_j y_j = S, \quad y_j \geq 0.$$

For any θ, Problem 3 has a unique solution. To see this, note that

1. The feasible set is non-empty. For example take $y_1 = S/h_1 Q_1$ with $y_j = 0$ for $j > 1$).
2. The objective function is bounded above, since it is quadratic in the y_j with the quadratic terms being positive definite matrices, with a negative sign.
3. It therefore has a solution.
4. The solution must be unique. Suppose there are two solutions y and y^*; then any convex linear combination $\alpha y + (1-\alpha) y^*$ with $\alpha \in (0,1)$ is also feasible, and will dominate y and y^* because the objective function is quadratic in α. So the two solutions must be identical.

Consider the case where the social planner sets weights according to

Problem 4. To find $\{y_j \mid j = 1, \ldots, N\}$ such that it solves

(IA.8)
$$\max_{y_j} \sum_{j=1}^{N} \theta_j \left(\mathbb{E}\left[y_j' Z_{j,t+h_j}\right] - \frac{1}{2} A_j y_j' \mathrm{Var}\left(Z_{j,t+h_j}\right) y_j \right)$$

subject to

(IA.9)
$$\sum_{j=1}^{N} h_j Q_j y_j = S, \quad y_j \geq 0,$$

with

$$\theta_j = \frac{h_j Q_j}{R_f^{h_j} - 1}. \tag{IA.10}$$

In this case, the first order condition on y_j is

$$\mathbb{E}\left[y_j' Z_{j,t+h_j}\right] - A_j \mathrm{Var}\left(Z_{j,t+h_j}\right) y_j - \left(R_f^{h_j} - 1\right)\lambda - \mu_j = 0, \tag{IA.11}$$

where λ is the Lagrange multiplier on the aggregate supply demand constraint, and μ_j is the Lagrange multiplier on the short selling constraint.

This is very similar to the first order condition for the investor's problem in Problem 2. Specifically, if the social planner decides not to allocate stocks, but instead sets prices according to the formula $P = \lambda$, investor j would decide of its own accord to buy y_j and demand and supply would balance. So the unique solution to Problem 4, which can be found using standard quadratic optimization software, generates the solution to Problem 2. It gives the unique stationary equilibrium solution to our model. Finally, we can multiply the entire objective function by $R_f - 1$ to ensure that the limit is finite as the net risk-free rate, $R_f - 1$, goes to zero. When we take the special case of $N = 2$ and $R_f = 1$, this gives us Proposition 1.

Finally, we can rewrite the quadratic problem into asset returns and percentage costs, which makes it suitable for empirical implementation. With equilibrium prices constant, the dividend yield and the net return on stocks are the same. So we can write the problem in terms of returns R and percentage transaction costs c. Define dollar supply as $\widetilde{S} = \mathrm{diag}(P) S$, relative dollar supply as $s = \widetilde{S}/\widetilde{S}'\iota$, and define $\tilde{y}_j = h_j Q_j \mathrm{diag}(P) y_j / (\widetilde{S}'\iota)$. Dividing the entire goal function by $\widetilde{S}'\iota$, we can rewrite Problem 4 as follows

Problem 5. To find $\{\tilde{y}_j \mid j = 1,\ldots,N\}$ such that it solves

$$\text{(IA.12)} \qquad \max_{\tilde{y}_j} \sum_{j=1}^{N} \frac{R_f - 1}{R_f^{h_j} - 1}\left(\mathbb{E}\left[\tilde{y}_j' \tilde{Z}_{j,t+h_j}\right] - \frac{1}{2}\frac{1}{h_j \gamma_j}\tilde{y}_j' \mathrm{Var}\left(\tilde{Z}_{j,t+h_j}\right)\tilde{y}_j\right)$$

$$\text{subject to } \sum_{j=1}^{N} \tilde{y}_j = s, \text{ and } \tilde{y}_j \geq 0,$$

where

$$\text{(IA.13)} \qquad \tilde{Z}_{j,t+h} = \left(\sum_{k=1}^{h_j} R_f^{h_j - k}\left(R_{t+k} - 1\right) - c_{t+h_j}\right).$$

Given the parameters $\gamma_j = Q_j/(A_j \widetilde{S}'\iota)$, this problem can be implemented empirically given estimates for the expectations and covariances using data on returns and percentage costs. As discussed in Section IV of the paper, for our empirical analysis we use 25 portfolios and an equal-weighted market portfolio, which corresponds to a relative supply s that is the same across portfolios and equal to $1/25$ for each portfolio.

Q.E.D.

We now proceed with the derivation of the equilibrium expected returns. We start by introducing sets B_j ($j = 1,\ldots,N$) that represent the assets that investor j optimally holds in his or her portfolio, as determined in Proposition 1.

Proof of Proposition 2: To derive the equilibrium, we first consider each investor's optimization problem. For the investors with horizon h_j it is given by

$$\text{(IA.14)} \qquad \max_{y_{j,t}} \mathbb{E}_t\left[W_{j,t+h_j}\right] - \frac{1}{2}A_j \mathrm{Var_t}\left(W_{j,t+h_j}\right), \text{ s.t. } y_{j,t} \geq 0$$

$$W_{j,t+h_j} = \left(P_{t+h_j} + \sum_{k=1}^{h_j} R_f^{h_j-k} D_{t+k} - C_{t+h_j}\right)' y_{j,t} + R_f^{h_j}\left(e_j - P_t' y_{j,t}\right).$$

We first repeat the notation used in the main text that will allow us to derive the equilibrium in the case where investor j holds only assets that are in B_j. For a $K \times K$ matrix M, we denote by $M_{y_j>0}$ the $|B_j| \times |B_j|$ matrix (with $|\cdot|$ the cardinality of a set) with the rows and columns that are not elements of B_j removed. We also introduce the notation $M^{-1}_{y_j>0,p}$ for the inverse of $M_{y_j>0}$ with zeros inserted at the locations where rows and columns of M were removed, so that $M^{-1}_{y_j>0,p}$ is a $K \times K$ matrix.

For example, let

$$M = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

and let $B_j = \{1,3\}$. Then

$$M_{y_j>0} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix},$$

so that

$$M^{-1}_{y_j>0} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}.$$

We then have

$$M^{-1}_{y_j>0,p} = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 0 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

If we apply this operation to the covariance matrix in the optimization problem of investor j, it yields a $K \times 1$ vector where elements corresponding to assets not in B_j are set to zero. The benefit is that it makes the solution vectors $y_{j,t}$ ($j = 1, \ldots, N$) conformable to addition, which allows us to derive the equilibrium.

We then use a result of De Roon, Nijman, and Werker (2001), who show that the solution to a utility maximization problem with short-sales constraints can be rewritten as the usual mean-variance solution for the subset of assets for which the short-sales constraint turns out not to be binding. Thus, given that the optimal portfolio of the investor consists only of assets that are elements of B_j, the solution is

$$
\text{(IA.15)} \qquad y_{j,t} = \frac{1}{A_j}\mathrm{Var_t}\left(P_{t+h_j} + \sum_{k=1}^{h_j} R_f^{h_j-k} D_{t+k} - C_{t+h_j} - R_f^{h_j} P_t\right)^{-1}_{y_j>0,p} \times \mathbb{E}_t\left[P_{t+h_j} + \sum_{k=1}^{h_j} R_f^{h_j-k} D_{t+k} - C_{t+h_j} - R_f^{h_j} P_t\right].
$$

Using the i.i.d. assumption for dividends and costs, we obtain a stationary equilibrium with constant prices and i.i.d. returns. With constant prices we get $R_{i,t+1} - 1 = D_{i.t+1}/P_i$, so that $y_{j,t}$ can be written as

$$
\text{(IA.16)} \qquad y_{j,t} = \frac{1}{A_j}\mathrm{diag}\left(P_t\right)^{-1}\mathrm{Var}\left(\sum_{k=1}^{h_j} R_f^{h_j-k} R_{t+k} - c_{t+h_j}\right)^{-1}_{y_j>0,p} \times \left(\mathbb{E}\left[\sum_{k=1}^{h_j} R_f^{h_j-k} R_{t+k} - c_{t+h_j}\right] - \sum_{k=0}^{h_j-1} R_f^{h_j-k}\right).
$$

Making further use of the i.i.d. assumption by which $\mathbb{E}\left[c_{t+h_j}\right] = \mathbb{E}\left[c_{t+k}\right]$ and $\mathbb{E}\left[R_{t+h_j}\right] = \mathbb{E}\left[R_{t+k}\right]$ for all j and k, and defining $\rho_j = \sum_{k=1}^{h_j} R_f^{h_j-k} = \frac{R_f^{h_j}-1}{R_f-1}$, the allocations can be written as

$$
\text{(IA.17)} \qquad y_{j,t} = \frac{1}{A_j}\mathrm{diag}\left(P_t\right)^{-1}\mathrm{Var}\left(\sum_{k=1}^{h_j} R_f^{h_j-k} R_{t+k} - c_{t+h_j}\right)^{-1}_{y_j>0,p} \times \left(\rho_j\left(\mathbb{E}\left[R_{t+1}\right] - R_f\right) - \mathbb{E}\left[c_{t+1}\right]\right).
$$

Each period a fixed quantity $Q_j > 0$ of type j investors enters the market. The equilibrium condition at time t is

$$
\text{(IA.18)} \qquad \sum_{j=1}^{N} Q_j y_{j,t} = S - \sum_{j=1}^{N}\sum_{k=1}^{h_j-1} Q_j y_{j,t-k}.
$$

Under the i.i.d. assumption we have $y_{j,t-k} = y_{j,t}$ for all k, so that

$$\sum_{j=1}^{N} h_j Q_j y_{j,t} = S. \tag{IA.19}$$

Scaling by price we obtain

$$\sum_{j=1}^{N} h_j Q_j \operatorname{diag}(P_t)\, y_{j,t} = \widetilde{S}_t, \tag{IA.20}$$

where $\widetilde{S}_t = \operatorname{diag}(P_t)S$. At this point it is useful to introduce the notation $R^m_{t+1} = \widetilde{S}'_t R_{t+1}/\widetilde{S}'_t \iota$, and $c^m_{t+1} = \widetilde{S}'_t c_{t+1}/\widetilde{S}'_t \iota$. We note that in the i.i.d. setting with constant prices, $\widetilde{S}_t$ is constant over time, hence we omit the time subscript and write $\widetilde{S}$ in what follows. This allows us to write

$$\operatorname{Var}(R_{t+1} - c_{t+1})\,\widetilde{S} = \widetilde{S}'\iota \operatorname{Cov}\left(R_{t+1} - c_{t+1}, R^m_{t+1} - c^m_{t+1}\right). \tag{IA.21}$$

Then, multiplying both sides of (IA.20) by $(1/\widetilde{S}'\iota)\operatorname{Var}(R_{t+1} - c_{t+1})$, and filling in the expression for the optimal allocations gives

$$\begin{aligned} \sum_{j=1}^{N} h_j \frac{Q_j}{A_j \widetilde{S}'\iota} \operatorname{Var}(R_{t+1} - c_{t+1}) \operatorname{Var}\left(\sum_{k=1}^{h_j} R_f^{h_j - k} R_{t+k} - c_{t+1}\right)^{-1}_{y_j > 0, p} \\ \times \left(\rho_j \left(\mathbb{E}[R_{t+1}] - R_f\right) - \mathbb{E}[c_{t+1}]\right) = \operatorname{Cov}\left(R_{t+1} - c_{t+1}, R^m_{t+1} - c^m_{t+1}\right). \end{aligned} \tag{IA.22}$$

We define $\gamma_j = Q_j/(A_j \widetilde{S}'\iota)$ and

$$V_j = h_j \operatorname{Var}(R_{t+1} - c_{t+1}) \operatorname{Var}\left(\sum_{k=1}^{h_j} R_f^{h_j - k} R_{t+k} - c_{t+h_j}\right)^{-1}_{y_j > 0, p}. \tag{IA.23}$$

This allows us to write

$$\sum_{j=1}^{N} \gamma_j V_j \left(\rho_j \left(\mathbb{E}\left[R_{t+1}\right] - R_f \right) - \mathbb{E}\left[c_{t+1}\right] \right) = \operatorname{Cov}\left(R_{t+1} - c_{t+1}, R_{t+1}^m - c_{t+1}^m \right). \tag{IA.24}$$

We can rewrite this equilibrium condition as

$$\begin{aligned} \mathbb{E}\left[R_{t+1}\right] - R_f = & \left(\sum_{j=1}^{N} \gamma_j \rho_j V_j \right)^{-1} \sum_{j=1}^{N} \gamma_j V_j \mathbb{E}\left[c_{t+1}\right] \\ & + \left(\sum_{j=1}^{N} \gamma_j \rho_j V_j \right)^{-1} \operatorname{Cov}\left(R_{t+1} - c_{t+1}, R_{t+1}^m - c_{t+1}^m \right). \end{aligned} \tag{IA.25}$$

Q.E.D.

B. Short-selling

In this Appendix we describe how one could incorporate short-selling in the model. As noted by Bongaerts, De Jong, and Driessen (2011), the key step is to realize that, for short positions, the net return equals $-(R+c)$. The mean-variance problem for the case without short-sale constraints can then be written as

$$\begin{aligned} & \max_{y_{j,t}} \; \mathbb{E}_t\left[W_{j,t+h_j}\right] - \frac{1}{2} A_j \operatorname{Var}_t\left(W_{j,t+h_j}\right), \\ & W_{j,t+h_j} = \left(P_{t+h_j} + \sum_{k=1}^{h_j} R_f^{h_j-k} D_{t+k} - \omega_j \cdot C_{t+h_j} \right)' y_{j,t} + R_f^{h_j}\left(e_j - P_t' y_{j,t} \right), \end{aligned} \tag{IA.26}$$

where ω_j is a K by 1 vector, with $\omega_j(i) = 1$ if $y_{j,t}(i) > 0$ and $\omega_j(i) = -1$ if $y_{j,t}(i) < 0$. Then, the optimal portfolio can consist of long positions, zero positions, and short positions. The vectors ω_j are determined endogenously as they depend on the optimal demand. One could then proceed in a similar way as in the benchmark model derivation to derive equilibrium expected returns and optimal demands.

C. Segmentation effects

Proof of Proposition 3: In this proposition, we have $N = 2$, $h_1 = 1$, $Var(c_t) = 0$, the h_1-investors optimally invest only in the most liquid assets, and the h_2-investors optimally invest in all assets. In this case we obtain $V_2 = I$. If we sort the assets by liquidity with the most liquid assets first, writing

$$\text{Var}\left(R_{t+1} - c_{t+1}\right) = \begin{bmatrix} V_{liq} & V_{liq,illiq} \\ V_{illiq,liq} & V_{illiq} \end{bmatrix}, \tag{IA.27}$$

we have

$$\begin{aligned} V_1 &= h_1 \text{Var}\left(R_{t+1}\right) \text{Var}\left(\sum_{k=1}^{h_1} R_f^{h_1-k} R_{t+k}\right)^{-1}_{y_1>0,p} \\ &= \begin{bmatrix} V_{liq} & 0 \\ 0 & V_{illiq} \end{bmatrix} \begin{bmatrix} V_{liq}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ V_{illiq,liq} V_{liq}^{-1} & 0 \end{bmatrix}. \end{aligned} \tag{IA.28}$$

Using $N = 2$ and $V_2 = I$ in (IA.25) leads to the equilibrium relation

$$\begin{aligned} \mathbb{E}\left[R_{t+1}\right] - R_f &= \left(\gamma_1 \rho_1 V_1 + \gamma_2 \rho_2 I\right)^{-1} \left(\gamma_1 V_1 + \gamma_2 I\right) \mathbb{E}\left[c_{t+1}\right] \\ &\quad + \left(\gamma_1 \rho_1 V_1 + \gamma_2 \rho_2 I\right)^{-1} \text{Cov}\left(R_{t+1}, R_{t+1}^m\right). \end{aligned} \tag{IA.29}$$

To find the liquidity risk effect, we focus on the factor

$$\text{(IA.30)} \qquad \left(\gamma_1\rho_1 V_1 + \gamma_2\rho_2 I\right)^{-1} = \begin{bmatrix} (\gamma_1\rho_1+\gamma_2\rho_2)I & 0 \\ \gamma_1\rho_1 V_{illiq,liq}V_{liq}^{-1} & \gamma_2\rho_2 I \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} (\gamma_1\rho_1+\gamma_2\rho_2)^{-1} I & 0 \\ -\gamma_1\rho_1(\gamma_2\rho_2)^{-1}(\gamma_1\rho_1+\gamma_2\rho_2)^{-1} V_{illiq,liq}V_{liq}^{-1} & (\gamma_2\rho_2)^{-1} I \end{bmatrix}.$$

In what follows, we will use the *liquidity spillover beta*, defined by

$$\text{(IA.31)} \qquad \beta = V_{illiq,liq}V_{liq}^{-1}$$
$$= \mathrm{Cov}\left(R_{t+1}^{illiq}, R_{t+1}^{liq}\right)\mathrm{Var}\left(R_{t+1}^{liq}\right)^{-1}.$$

For the impact of the level of liquidity we write

$$\text{(IA.32)} \qquad \left(\gamma_1\rho_1 V_1 + \gamma_2\rho_2 I\right)^{-1}\left(\gamma_1 V_1 + \gamma_2 I\right)$$
$$= \begin{bmatrix} (\gamma_1\rho_1+\gamma_2\rho_2)^{-1} I & 0 \\ -\gamma_1\rho_1(\gamma_2\rho_2)^{-1}(\gamma_1\rho_1+\gamma_2\rho_2)^{-1}\beta & (\gamma_2\rho_2)^{-1} I \end{bmatrix} \begin{bmatrix} (\gamma_1+\gamma_2)I & 0 \\ \gamma_1\beta & \gamma_2 I \end{bmatrix}$$
$$= \begin{bmatrix} (\gamma_1+\gamma_2)(\gamma_1\rho_1+\gamma_2\rho_2)^{-1} I & 0 \\ \left(\gamma_1(\gamma_2\rho_2)^{-1} - \gamma_1\rho_1(\gamma_2\rho_2)^{-1}(\gamma_1+\gamma_2)(\gamma_1\rho_1+\gamma_2\rho_2)^{-1}\right)\beta & \rho_2^{-1} I \end{bmatrix}.$$

We rewrite the scalar part of the spillover coefficient using the identity

$$\text{(IA.33)} \qquad \frac{\gamma_1}{\gamma_2\rho_2} - \frac{\gamma_1\rho_1(\gamma_1+\gamma_2)}{\gamma_2\rho_2(\gamma_1\rho_1+\gamma_2\rho_2)} = \frac{\rho_2-\rho_1}{\rho_2}\frac{\gamma_1}{\gamma_1\rho_1+\gamma_2\rho_2}.$$

Combining the results above, we can write the equilibrium relation for the liquid assets as

$$\text{(IA.34)} \qquad \mathbb{E}\left[R_{t+1}^{liq}\right] - R_f = \frac{\gamma_1+\gamma_2}{\gamma_1\rho_1+\gamma_2\rho_2}\mathbb{E}\left[c_{t+1}^{liq}\right] + \frac{1}{\gamma_1\rho_1+\gamma_2\rho_2}\mathrm{Cov}\left(R_{t+1}^{liq}, R_{t+1}^{m}\right).$$

and the equilibrium relation for the illiquid assets as

$$\text{(IA.35)} \qquad \mathbb{E}\left[R^{illiq}_{t+1}\right] - R_f = \frac{1}{\rho_2}\mathbb{E}\left[c^{illiq}_{t+1}\right] + \frac{\rho_2-\rho_1}{\rho_2}\frac{\gamma_1}{\gamma_1\rho_1+\gamma_2\rho_2}\beta\mathbb{E}\left[c^{liq}_{t+1}\right] + \frac{1}{\gamma_2\rho_2}\mathrm{Cov}\left(R^{illiq}_{t+1}, R^m_{t+1}\right) - \frac{\gamma_1\rho_1}{\gamma_2\rho_2\left(\gamma_1\rho_1+\gamma_2\rho_2\right)}\beta\mathrm{Cov}\left(R^{liq}_{t+1}, R^m_{t+1}\right).$$

The desired expressions now follow directly. Q.E.D.

D. Dynamic model

In this section, we derive the full dynamic extension in Section VI of the paper. As mentioned in the main text, in the liquid state, defined by $I_t = 0$, transaction costs are low with $C_{i,t} = C_i^0 + \eta^0_{i,t}$, where $\eta^0_{i,t}$ is a mean-zero, i.i.d. variable that captures variation in transaction costs within the regime. Similarly, in the illiquid state $(I_t = 1)$ we have $C_{i,t} = C_i^1 + \eta^1_{i,t}$, with $C_i^1 \geq C_i^0$. The liquidity state follows a Markov-switching process, with $\Pr(I_{t+1} = 0|I_t = 1) = \theta/2\pi$ and $\Pr(I_{t+1} = 1|I_t = 0) = \theta/2(1-\pi)$. By parameterizing the process in this way, the unconditional probability of being in the illiquid state is π and the unconditional probability of a change in liquidity state in any given period is θ. This modelling approach thus allows for persistence in transaction costs because the regimes are persistent, and also incorporates i.i.d. variation in transaction costs within each regime. We continue to assume that dividends are i.i.d. We again have short-horizon and long-horizon investors, where we normalize $h_1 = 1$ to save on notation.

To obtain tractable results in the presence of persistent transaction costs, we need to modify our assumptions concerning the behavior of long-term investors. First, we allow their horizon h_2 to tend to infinity. As shown below, this simplifies their demand function since the risk and return from investment is dominated by the dividend stream.

Second, and more significantly, we now allow long-term investors to rebalance at any time.

With time-varying liquidity, short-term investors change their asset holdings depending on the state. This will necessarily affect prices, so the investment opportunity set available to long-term investors varies over time. They therefore have a strong economic incentive to trade – to absorb assets which have suddenly become illiquid and which the short-term investors no longer wish to hold, or to supply assets which come into high demand when they become more liquid.

We therefore replace the market clearing condition in the benchmark model (equation (IA.18)) with

$$Q_1 y_{1,t} + Q_2 y_{2,t} = S. \tag{IA.36}$$

Now, the entire supply of assets is available for trading each period; in the benchmark model, it was only the supply made available by departing generations that was available for trading. This casts long-horizon agents in the role of value investors who buy or sell when the market price is low or high relative to the fundamental value of the asset, as measured by the present value of future dividends. Furthermore, given the infinite horizon of the long-term investors transaction costs do not affect their demand. Long horizon agents thus act as passive traders, and do not face transaction costs when they trade with the short-horizon investors.

The new market clearing assumption has a major technical advantage. With the supply of assets being constant, with demand dependent on beliefs about future dividends, prices and costs, and with the liquidity state following a Markov process, we can look for an equilibrium where prices depend on the liquidity state alone.

The final component of the time-varying liquidity model is to focus on the case where θ, the probability of a switch in the liquidity state per period, tends to zero. Note that the unconditional probability of being in the illiquid state remains fixed at π; the assumption is effectively saying that the average length of time the market is in any liquidity state is long compared with the horizon of the short-term investor.

Given these additional assumptions, we can then calculate demands, equate them to supply and calculate equilibrium expected returns. The demand of the short-term investors is given by

$$y_{1,t} = \frac{1}{A_1} \mathrm{Var}\left(P_{t+1} + D_{t+1} - C_{t+1} - R_f P_t \mid I_t\right)^{-1}_{y_{1,t}>0,p} \mathbb{E}\left[P_{t+1} + D_{t+1} - C_{t+1} - R_f P_t \mid I_t\right], \tag{IA.37}$$

while the long-term investors have demand

$$\begin{aligned} y_{2,t} = \frac{1}{A_2} \mathrm{Var}&\left(P_{t+h_2} + \sum_{k=1}^{h_2} R_f^{h_2-k} D_{t+k} - C_{t+h_2} - R_f^{h_2} P_t \mid I_t\right)^{-1}_{y_{2,t}>0,p} \\ &\times \mathbb{E}\left[P_{t+h_2} + \sum_{k=1}^{h_2} R_f^{h_2-k} D_{t+k} - C_{t+h_2} - R_f^{h_2} P_t \mid I_t\right]. \end{aligned} \tag{IA.38}$$

This demand function depends on the current liquidity state, as does the degree of segmentation that the investor chooses in equilibrium. Hence, short-term and long-term investors can endogenously choose to invest in a different set of assets in the liquid state compared to the illiquid state.

The net gains from the investment portfolio can conveniently be split into a net capital gain and a dividend stream

$$\begin{aligned} &P_{t+h_2} + \sum_{k=1}^{h_2} R_f^{h_2-k} D_{t+k} - C_{t+h_2} - R_f^{h_2} P_t \\ &\quad = \left(P_{t+h_2} - C_{t+h_2} - P_t\right) + \left(\sum_{k=1}^{h_2} R_f^{h_2-k} D_{t+k} - \left(R_f^{h_2} - 1\right) P_t\right). \end{aligned} \tag{IA.39}$$

With P_t being known at time t, and dividends being i.i.d. the variance and expectation of the dividend stream can be simplified as

$$\mathrm{Var}\left(\sum_{k=1}^{h_2} R_f^{h_2-k} D_{t+k} - \left(R_f^{h_2} - 1\right) P_t \mid I_t\right)_{y_{2,t}>0,p} = \sum_{k=1}^{h_2} R_f^{2h_2-2k} \mathrm{Var}\left(D_{t+1} \mid I_t\right)_{y_{2,t}>0,p}, \tag{IA.40}$$

and

$$\text{(IA.41)} \qquad \mathbb{E}\left[\sum_{k=1}^{h_2} R_f^{h_2-k} D_{t+k} - \left(R_f^{h_2}-1\right) P_t \mid I_t\right] = \sum_{k=1}^{h_2} R_f^{h_2-k}\mathbb{E}\left[D_{t+1} \mid I_t\right] - \left(R_f^{h_2}-1\right) P_t$$
$$= \frac{R_f^{h_2}-1}{R_f-1}\mathbb{E}\left[D_{t+1} - \left(R_f-1\right) P_t \mid I_t\right].$$

Equation (IA.38) can now be rewritten as

$$\text{(IA.42)} \qquad y_{2,t} = \frac{1}{A_2}\left(\mathrm{Var}\left(P_{t+h_2} - C_{t+h_2} - P_t \mid I_t\right) + \frac{R_f^{2h_2}-1}{R_f^2-1}\mathrm{Var}\left(D_{t+1} \mid I_t\right)\right)^{-1}_{y_{2,t}>0,p}$$
$$\times \left(\mathbb{E}\left[P_{t+h_2} - C_{t+h_2} - P_t \mid I_t\right] + \rho_2\mathbb{E}\left[D_{t+1} - \left(R_f-1\right) P_t \mid I_t\right]\right),$$

where $\rho_j = \sum_{k=1}^{h_j} R_f^{h_j-k} = (R_f^{h_j}-1)/(R_f-1)$. Note that both the variance and the expectation of the value of dividend grow at least as fast as the horizon, h_2. By contrast, both the variance and expectation of the net capital gain tend asymptotically to a finite limit. So for large h_2 we ignore the net capital gain and approximate the demand function of the long-term investors as

$$\text{(IA.43)} \qquad y_{2,t} \approx \frac{1}{A_2}\left(\frac{R_f^{2h_2}-1}{R_f^2-1}\mathrm{Var}\left(D_{t+1} \mid I_t\right)\right)^{-1}_{y_{2,t}>0,p} \rho_2\mathbb{E}\left[D_{t+1} - \left(R_f-1\right) P_t \mid I_t\right].$$

As mentioned above, we now allow the long-term investors to rebalance. The market clearing equation is given by

$$\text{(IA.44)} \qquad Q_1 y_{1,t} + Q_2 y_{2,t} = S.$$

Both demand and supply depend only on current and future dividends and transactions, and no longer depend on past transactions. Prices are thus a function of the regime. They jump when the regime switches, and while the regime remains unchanged, prices are unchanged. We may therefore write

$$P_t = P^{I_t}. \tag{IA.45}$$

We divide the numerator and the denominator of the demand equations by prices to write demand in terms of returns. For short-term investors

$$\begin{aligned} y_{1,t} &= \frac{1}{A_1} \operatorname{diag}\left(P^{I_t}\right)^{-1} \left(\operatorname{diag}\left(P^{I_t}\right)^{-1} \operatorname{Var}\left(P^{I_{t+1}} + D_{t+1} - C_{t+1} \mid I_t\right) \operatorname{diag}\left(P^{I_t}\right)^{-1}\right)^{-1}_{y_{1,t}>0,p} \\ &\qquad \times \operatorname{diag}\left(P^{I_t}\right)^{-1} \mathbb{E}\left[P^{I_{t+1}} + D_{t+1} - C_{t+1} - R_f P^{I_t} \mid I_t\right] \\ &= \frac{1}{A_1} \operatorname{diag}\left(P^{I_t}\right)^{-1} \operatorname{Var}\left(R_{t+1} - c_{t+1} \mid I_t\right)^{-1}_{y_{1,t}>0,p} \mathbb{E}\left[R_{t+1} - R_f - c_{t+1} \mid I_t\right], \end{aligned} \tag{IA.46}$$

We focus on the case where the unconditional probability of a change in liquidity state in any period is very small, that is the case where $\theta \to 0$. Then the probability of a price change is small, and we can approximate the net return on a stock by its dividend yield

$$\operatorname{diag}\left(P^{I_t}\right)^{-1} D_{t+1} \approx R_{t+1} - 1. \tag{IA.47}$$

This allows us to write the demand by the long-term investors as

$$\begin{aligned} y_{2,t} &= \frac{1}{A_2} \operatorname{diag}\left(P^{I_t}\right)^{-1} \left(\frac{R_f^{2h_2} - 1}{R_f^2 - 1} \operatorname{diag}\left(P^{I_t}\right)^{-1} \operatorname{Var}\left(D_{t+1} \mid I_t\right) \operatorname{diag}\left(P^{I_t}\right)^{-1}_{y_{2,t}>0,p}\right) \\ &\qquad \times \rho_2 \operatorname{diag}\left(P^{I_t}\right)^{-1} \mathbb{E}\left[D_{t+1} - \left(R_f - 1\right) P^{I_t} \mid I_t\right] \\ &\approx \frac{1}{A_2} \operatorname{diag}\left(P^{I_t}\right)^{-1} \left(\frac{R_f^{2h_2} - 1}{R_f^2 - 1} \operatorname{Var}\left(R_{t+1} \mid I_t\right)\right)^{-1} \rho_2 \mathbb{E}\left[R_{t+1} - R_f \mid I_t\right]. \end{aligned} \tag{IA.48}$$

Next, we let

$$S^{I_t} = \operatorname{diag}\left(P^{I_t}\right) S, \tag{IA.49}$$

and use (IA.46) and (IA.48) to rewrite the equilibrium condition (IA.44) similarly to the static case as

$$
\text{(IA.50)} \qquad \begin{aligned} S^{I_t} &= \operatorname{diag}\left(P^{I_t}\right)\left(Q_1 y_{1,t} + Q_2 y_{2,t}\right) \\ &= \frac{Q_1}{A_1} \operatorname{Var}\left(R_{t+1} - c_{t+1} \mid I_t\right)^{-1}_{y_{1,t}>0,p} \mathbb{E}\left[R_{t+1} - R_f - c_{t+1} \mid I_t\right] \\ &\quad + \frac{Q_2}{A_2} \left(\frac{R_f^{2h_2} - 1}{R_f^2 - 1} \operatorname{Var}\left(R_{t+1} \mid I_t\right)\right)^{-1}_{y_{2,t}>0,p} \rho_2 \mathbb{E}\left[R_{t+1} - R_f \mid I_t\right]. \end{aligned}
$$

Then we can make use of the fact that

$$
\text{(IA.51)} \qquad \begin{aligned} \operatorname{Var}\left(R_{t+1} - c_{t+1} \mid I_t\right) S^{I_t} &= \mathbb{E}\left[\left(R_{t+1} - c_{t+1}\right)\left(R_{t+1} - c_{t+1}\right)' \mid I_t\right] S^{I_t} \\ &= S^{I_t\prime}\iota \mathbb{E}\left[\left(R_{t+1} - c_{t+1}\right)\left(\frac{S^{I_t\prime} R_{t+1}}{S^{I_t\prime}\iota} - \frac{S^{I_t\prime} c_{t+1}}{S^{I_t\prime}\iota}\right)' \mid I_t\right] \\ &= S^{I_t\prime}\iota \operatorname{Cov}\left(R_{t+1} - c_{t+1}, R^m_{t+1} - c^m_{t+1} \mid I_t\right) \end{aligned}
$$

to obtain

(IA.52)

$$
\begin{aligned} &\operatorname{Cov}\left(R_{t+1} - c_{t+1}, R^m_{t+1} - c^m_{t+1} \mid I_t\right) \\ &\quad = \frac{Q_1}{A_1 S^{I_t\prime}\iota} \operatorname{Var}\left(R_{t+1} - c_{t+1} \mid I_t\right) \operatorname{Var}\left(R_{t+1} - c_{t+1} \mid I_t\right)^{-1}_{y_{1,t}>0,p} \mathbb{E}\left[R_{t+1} - R_f - c_{t+1} \mid I_t\right] \\ &\qquad + \frac{Q_2}{A_2 S^{I_t\prime}\iota} \operatorname{Var}\left(R_{t+1} - c_{t+1} \mid I_t\right) \left(\frac{R_f^{2h_2} - 1}{R_f^2 - 1} \operatorname{Var}\left(R_{t+1} \mid I_t\right)\right)^{-1}_{y_{2,t}>0,p} \rho_2 \mathbb{E}\left[R_{t+1} - R_f \mid I_t\right]. \end{aligned}
$$

If we now introduce

$$V_{1,t} = \mathrm{Var}\left(R_{t+1} - c_{t+1} \mid I_t\right) \mathrm{Var}\left(R_{t+1} - c_{t+1} \mid I_t\right)^{-1}_{y_{1,t}>0,p}, \tag{IA.53}$$

$$V_{2,t} = \mathrm{Var}\left(R_{t+1} - c_{t+1} \mid I_t\right) \left(\frac{R_f^{2h_2} - 1}{R_f^2 - 1} \mathrm{Var}\left(R_{t+1} \mid I_t\right)\right)^{-1}_{y_{2,t}>0,p}, \tag{IA.54}$$

and let $\gamma_j = Q_j/(A_j S^{I_t \prime} \iota)$, then we find that

$$\begin{aligned} \mathbb{E}\left[R_{t+1} - R_f \mid I_t\right] &= \left(\gamma_1 V_{1,t} + \gamma_2 \rho_2 V_{2,t}\right)^{-1} \gamma_1 V_{1,t} \mathbb{E}\left[c_{t+1} \mid I_t\right] \\ &\quad + \left(\gamma_1 V_{1,t} + \gamma_2 \rho_2 V_{2,t}\right)^{-1} \mathrm{Cov}\left(R_{t+1} - c_{t+1}, R^m_{t+1} - c^m_{t+1} \mid I_t\right), \end{aligned} \tag{IA.55}$$

which is the desired expression.

II. Estimation Methodology

A. Computing the long-term covariance matrix

We use the i.i.d. assumption to rewrite part of the moment conditions as follows

$$\begin{aligned} \mathrm{Var}\left(\sum_{k=1}^{h_j} R_f^{h_j-k} R_{t+k} - c_{t+h_j}\right)^{-1} &= \left(\left(\sum_{k=1}^{h_j-1} R_f^{2(h_j-k)}\right) \mathrm{Var}\left(R_{t+1}\right) + \mathrm{Var}\left(R_{t+1} - c_{t+1}\right)\right)^{-1} \\ &= \left(\frac{R_f^{2(h_j-1)} - 1}{R_f^2 - 1} \mathrm{Var}\left(R_{t+1}\right) + \mathrm{Var}\left(R_{t+1} - c_{t+1}\right)\right)^{-1}. \end{aligned} \tag{IA.56}$$

This allows us to compute the covariance terms using only one-period covariances.

B. Standard Errors and Model Testing

We denote the required moments that enter the asset pricing model by the vector ψ. This vector contains expected returns, expected costs, and all required covariances of returns and costs. It is straightforward to derive the asymptotic covariance matrix of the sample estimator of these moments (since covariances can be written as second moments plus products of first moments),

$$\sqrt{T}\,(\widehat{\psi}-\psi) \xrightarrow{d} \mathcal{N}\left(0, S_{\psi}\right). \tag{IA.57}$$

We can now use the delta method to find the standard errors for $\widehat{\gamma}$.

Consider the GMM minimization problem given by

$$\min_{\gamma}\; g(\widehat{\psi},\gamma)' g(\widehat{\psi},\gamma), \tag{IA.58}$$

for which the solution is implicitly given by

$$2G_{\gamma}(\widehat{\psi},\gamma)' g(\widehat{\psi},\gamma) = 0, \tag{IA.59}$$

where

$$G_{\gamma}(\psi,\gamma) = \frac{\partial g(\psi,\gamma)}{\partial \gamma}. \tag{IA.60}$$

Dividing both sides of (IA.59) by 2 and evaluating at $\widehat{\gamma}$, we may write

$$G_{\gamma}(\widehat{\psi},\widehat{\gamma})' g(\widehat{\psi},\gamma_0) + G_{\gamma}(\widehat{\psi},\widehat{\gamma})' \left(g(\widehat{\psi},\widehat{\gamma}) - g(\widehat{\psi},\gamma_0)\right) = 0. \tag{IA.61}$$

Next, we expand $g(\widehat{\psi}, \widehat{\gamma})$ around γ_0 to obtain

$$g(\widehat{\psi}, \widehat{\gamma}) - g(\widehat{\psi}, \gamma_0) \approx G_\gamma(\widehat{\psi}, \widehat{\gamma})\, (\widehat{\gamma} - \gamma_0)\,. \tag{IA.62}$$

It follows that

$$G_\gamma(\widehat{\psi}, \widehat{\gamma})' g(\widehat{\psi}, \gamma_0) + G_\gamma(\widehat{\psi}, \widehat{\gamma})' G_\gamma(\widehat{\psi}, \widehat{\gamma})\, (\widehat{\gamma} - \gamma_0) = 0. \tag{IA.63}$$

We now expand $g(\widehat{\psi}, \gamma_0)$ around ψ_0 and use the fact that $g(\psi_0, \gamma_0) = 0$ to find that

$$g(\widehat{\psi}, \gamma_0) \approx G_\psi(\widehat{\psi}, \widehat{\gamma})\, (\widehat{\psi} - \psi_0)\,, \tag{IA.64}$$

where

$$G_\psi(\psi, \gamma) = \frac{\partial g(\psi, \gamma)}{\partial \psi}. \tag{IA.65}$$

Hence

$$G_\gamma(\widehat{\psi}, \widehat{\gamma})' G_\gamma(\widehat{\psi}, \widehat{\gamma})\, (\widehat{\gamma} - \gamma_0) = -G_\gamma(\widehat{\psi}, \widehat{\gamma})' G_\psi(\widehat{\psi}, \widehat{\gamma})\, (\widehat{\psi} - \psi_0)\,. \tag{IA.66}$$

Using this result we obtain

$$\sqrt{T}\, (\widehat{\gamma} - \gamma_0) \approx -\left(G_\gamma(\widehat{\psi}, \widehat{\gamma})' G_\gamma(\widehat{\psi}, \widehat{\gamma})\right)^{-1} G_\gamma(\widehat{\psi}, \widehat{\gamma})' G_\psi(\widehat{\psi}, \widehat{\gamma}) \sqrt{T}\, (\widehat{\psi} - \psi_0)\,. \tag{IA.67}$$

It follows that

$$\sqrt{T}\, (\widehat{\gamma} - \gamma_0) \xrightarrow{d} \mathcal{N}\left(0, \left(G_\gamma' G_\gamma\right)^{-1} G_\gamma' G_\psi S_\psi G_\psi' G_\gamma \left(G_\gamma' G_\gamma\right)^{-1}\right). \tag{IA.68}$$

This result shows that the parameter estimates have an asymptotic normal distribution. We use a standard bootstrap method to calculate the asymptotic standard errors. Under standard regularity conditions, the bootstrap distribution converges to the asymptotic distribution (Shao and Tu, 1995). We prefer this approach over directly calculating G_γ and G_ψ because we have to solve the asset pricing model numerically and calculating the partial derivatives may render unstable results. The bootstrap method is also useful when testing models against each other, as discussed below.

In each bootstrap, we draw monthly observations for returns and transaction costs (for all portfolios) with replacement from the 1964-2009 sample period, to create a sample of the same size as the original data. We then re-estimate the model parameters given this bootstrap sample. We perform 5000 bootstrap simulations, and the standard error for a given parameter is then obtained by calculating the standard deviation of the parameter estimate across the 5000 simulations.

The bootstrap approach can also be used directly to implement the test statistic of Rivers and Vuong (2002) and Hall and Pelletier (2011). This test-statistic compares the GMM J values of two non-nested models, and tests whether one model has significantly lower pricing errors than an alternative model. The difference between the J values has an asymptotically normal distribution if both models are misspecified. We apply this test to our two-horizon model and the single-horizon AP model. In each bootstrap simulation, we estimate both models and calculate their J statistics. We then calculate the standard deviation of the difference between the two J values across 5000 simulations. The asymptotically normal test statistic equals the difference between the full-sample J values of the two models, divided by this standard deviation.

III. Additional Empirical Results

A. Atkins and Dyl (1997) Holding Period Estimates

In the absence of investor-level data, it is difficult to estimate the average holding period of investors in a single stock or portfolio of stocks. A proxy for the average holding period put forward by Atkins and Dyl (1997) is given by

$$\text{Holding Period}_{it} = \frac{\text{Shares Outstanding in Period } t}{\text{Trading Volume in Period } t}. \tag{IA.69}$$

This is a rough proxy, as not all investors hold the asset for the same period of time. As Næs and Ødegaard (2009) show, the holding period can be overstated if there are a few very long-term investors holding the stock. Therefore, we only use this proxy to obtain an impression of the average holding periods for the various portfolios in our sample. We compute the proxy for each stock in our sample, and then take the average across stocks to obtain a portfolio measure. Table IA.1 presents time-series averages of these estimates.

B. Results for Different Portfolios

We estimate our model for various different portfolio sorts. Table IA.2 shows the cross-sectional fit of our model for a sort on the variance of the stock-level liquidity: the R^2 equals 74.5% in the AP model versus 28.6% in the heterogenous horizon model for the case without a constant term. For the B/M-by-size portfolios the cross-sectional R^2 equals 28.4% in the AP model versus 46.3% in the heterogenous horizon model (without constant term). To investigate the importance of liquidity risk further, in addition to the analysis in Section V.C of the paper, we perform double sorts on liquidity level and liquidity risk. Table IA.2 has results for a 5x5 double sort, sorting on the daily covariance of the stock return and market-wide transaction costs (over the past 12 months), and on the average transaction costs. In addition, we present results for another 5x5 double sort, sorting on the standard deviation of daily transaction costs in the past year (liquidity risk) and on the average transaction costs over the past 12 months. The results show that the two-horizon model has a decent fit for both portfolio sorts, well above the fit of the AP model.

We also calculate the liquidity risk premium implied by the model and the parameter estimates in the same way as in Section V.C of the paper. Table IA.3 shows that, across all sorts, this liquidity risk premium is at most 2 basis points per year, similar to our benchmark estimates. Hence, the liquidity risk premium is small for all these sorts.

Finally, we assess how much segmentation there is when estimating the model on the basis of these different portfolio sorts. For each estimation, we calculate for how many of the 25 portfolios both the short-term and long-term investor have a nonzero investment. In the benchmark estimation, there are 3 of these integrated portfolios. Table IA.4 reports the number of integrated portfolios for the different portfolio sorts. For all sorts, there is a substantial degree of segmentation. The lowest level of segmentation is obtained for the book-to-market x size portfolios, where 9 out of the 25 portfolios are held by both investors.

C. Robustness to Sample Period

In addition to our full-sample results for the period 1964 until 2009, we also provide subsample results for the period 1964 until 1986 and for the period 1987 until 2009. We also provide the results for the Acharya and Pedersen (2005) sample, which runs from 1964 until 1999. We report the R^2 in Table IA.2, the liquidity risk premium in Table IA.3 and the level of segmentation in Table IA.4.

We find in all cases low levels for the liquidity risk premium and high levels of segmentation. In terms of fit, only for the second half of the sample the fit is much lower than in all other cases, both for the two-horizon model and the AP model. For these subsample the average returns across portfolios are less smooth and more noisy than for the full sample, leading to a lower R^2.

D. Robustness to Horizon

We then test the sensitivity of model performance to the choice of the investor horizons. We first fix the short-term investor horizon at 1 month and estimate the model for $h_2 = 36, 60, 240$ months. Similarly, we fix the long-term horizon at 120 months and vary the short-term horizon

($h_1 = 3, 6, 12$ months). As before, we report the R^2 in Table IA.2, the liquidity risk premium in Table IA.3 and the level of segmentation in Table IA.4.

We see that varying the long-term horizon has little influence on the results, in all dimensions. Varying the short-term horizon has more effects. For $h_1 = 3$ the results are quite similar to the benchmark case, but when we increase h_1 further, the fit decreases and the liquidity risk premium increases a bit, but remains small (at most 6 basis points per year). The parameter estimates (not reported) show that the role of the long-term investors becomes negligble when we increase the horizon of the short-term investors. Hence, we conclude that we need a sufficient difference in horizons of the two investor types to obtain a reasonable fit. Finally, we still find substantial segmentation when we vary the horizons, though a bit less than in the benchmark setting.

E. Determinants of the Regime Probabilities

In this subsection we study the time series of the regime probabilities implied by the regime-switching model in Table III of the paper. As described in Section VI of the paper, for each month in the 1964-2009 sample period we calculate the probability of being in the illiquid regime. In formulas, this probability is equal to $\mathbb{P}(I_t = 1 \mid c_t, c_{t-1}, \ldots)$, conditional upon all information at time t.

Figure IA.1 shows the time series of this illiquid regime probability, along with the NBER recession dummy. In 27% of the months the probability of an illiquid regime is above 50%. We see that the probability of the illiquid regime is high, for example, in two recessions in the 70s, around the 1987 crash, from the LTCM crisis in 1998 continuing into the recession in 2001, and October to December 2008. In addition, we regress the probability of being in the liquid regime at time t on a set of financial and economic variables that may capture distress and illiquidity.[1] Table IA.5

[1]The data on the default spread and the term spread, are from the Federal Reserve Bank of St. Louis, and the risk-free rate is from the website of Kenneth French. We define the default spread as the difference between the yield on Moody's Baa or better corporate bond yield index and the yield on a 10-year constant maturity Treasury bond. The

shows that the most important variables, in terms of statistical and economic significance, are the NBER dummy and stock market volatility (measured as the monthly standard deviation of daily market returns). Both these variables negatively affect the probability of a liquid regime, which is what one would intuitively expect.

F. Model with Regime Switches in Covariances

In Section VI of the paper we estimate a regime-switching model for transaction costs and use this to estimate dynamic version of our asset pricing model. In the benchmark analysis we focus on a model where only transaction cost levels change across regimes, and keep the covariances of costs and returns constant across regimes. In this subsection we present results when we estimate the dynamic asset pricing model when both transaction costs and covariances are allowed to change across regimes.

As described in Section VI of the paper, we use the regime probabilities to construct estimates of the conditional expectations and covariances of returns and transaction costs for all portfolios. Specifically, if $\mathbb{P}(I_t = 0 \mid c_t, c_{t-1}, \ldots) > 0.5$, the subsequent month $t+1$ is assigned to the set of "liquid months", and else it is assigned to the set of "illiquid months". We then calculate the means and covariances for the liquid months and illiquid months, respectively. This gives sample estimates for $\mathbb{E}[R_{t+1} - 1 \mid I_t]$, $\mathbb{E}[c_{t+1} \mid I_t]$, and all conditional covariances.

We then insert the conditional estimates of expected transaction costs and all covariances in (29), and estimate the model parameters using GMM on the 50 moment conditions described in Section VI of the paper. In Table IA.6 we present the parameter estimates. We see that both investor types contribute to the sharing of risk. With an R^2 of 37.6% the fit is similar to the model without regime changes in the covariances (Table III of the paper). In Figure IA.2 we graph the

---

term spread is computed as the difference between the yield on a 10-year constant maturity Treasury bond and the and the 3-month T-bill rate.

optimal holdings of the long-term investors in both regimes. As discussed in the main text, we again find a flight-to-liquidity effect.

References

Acharya, V. V. and L. H. Pedersen. "Asset pricing with liquidity risk." *Journal of Financial Economics*, 77, (2005), 375–410.

Atkins, A. B. and E. A. Dyl. "Transactions Costs and Holding Periods for Common Stocks." *The Journal of Finance*, 52, (1997), 309–325.

Bongaerts, D.; F. De Jong; and J. Driessen. "Derivative Pricing with Liquidity Risk: Theory and Evidence from the Credit Default Swap Market." *The Journal of Finance*, 66, (2011), 203–240.

De Roon, F. A.; T. E. Nijman; and B. J. Werker. "Testing for Mean-Variance Spanning with Short Sales Constraints and Transaction Costs: The Case of Emerging Markets." *The Journal of Finance*, 56, (2001), 721–742.

Hall, A. R. and D. Pelletier. "Nonnested Testing In Models Estimated Via Generalized Method of Moments." *Econometric Theory*, 27, (2011), 443–456.

Næs, R. and B. A. Ødegaard. "Liquidity and Asset Pricing: Evidence on the Role of Investor Holding Period." *Working Paper, Norwegian School of Management*, (2009).

Rivers, D. and Q. Vuong. "Model selection tests for nonlinear dynamic models." *Econometrics Journal*, 5, (2002), 1–39.

Shao, J. and D. Tu. *The Jackknife and Bootstrap*. Springer-Verlag, New York (1995).

Table IA.1: Holding Period Estimates

This table shows Atkins and Dyl (1997) holding period estimates for each portfolio in our sample. The CRSP data used are monthly data corresponding to 25 equal-weighted and value-weighted US stock portfolios sorted on illiquidity with sample period 1964–2009.

Portfolio	Holding EW (years)	Holding VW (years)
1	3.6859	5.9213
2	2.9287	3.7504
3	2.9354	3.6913
4	2.9579	3.6413
5	3.0125	4.8293
6	3.2248	5.0410
7	3.1341	4.9557
8	3.2531	5.1739
9	3.3466	5.3480
10	3.5535	7.1417
11	3.5674	7.0612
12	3.9862	10.3052
13	3.6511	7.1028
14	4.0883	10.0884
15	4.3240	11.9349
16	4.8012	14.0370
17	5.0640	16.3176
18	5.5199	21.3015
19	5.4839	15.5012
20	6.3671	26.7479
21	6.1255	26.2792
22	6.4674	20.8965
23	7.0583	30.0194
24	8.7680	39.8894
25	11.7683	89.9250

Table IA.2: GMM estimation results: robustness of R^2 to different portfolio sorts, sample periods, and horizons

This table shows the cross-sectional R^2 for various portfolio sorts (Panel A), sample periods (Panel B), and horizons (Panel C). The full sample period is 1964–2009. A value-weighted market portfolio is used. The parameters are estimated using GMM. We include four different portfolio sorts. First, monthly data corresponding to 25 value-weighted US stock portfolios sorted on the standard deviation of daily transaction costs in the past 12 months. Second, double-sorted portfolios on size and book-to-market. Third, we provide results for a 5x5 double sort, sorting on the daily covariance of the stock return and market-wide transaction costs (over the past 12 months), and on the average transaction costs. Fourth, we present results for a 5x5 double sort, sorting on the standard deviation of daily transaction costs in the past year (liquidity risk) and on the average transaction costs over the past 12 months. Panel B presents results based on the first and second half of the sample, respectively, and for the sample period of Acharya and Pedersen (2005): 1964-1999. Panel C presents results for different investor horizons. Two types of models are included. The first model is a two-horizon model (equation (6)), with $h_1 = 1$, and $h_2 = 120$, without and with constant term α (2HOR and 2HOR+α). AP indicates that the specification corresponds to a variant of the Acharya and Pedersen (2005) specification given by equation (24).

Panel A: Different portfolio sorts				
	2HOR	2HOR+α	AP	AP+α
Benchmark	72.6%	74.1%	26.6%	32.3%
σ(illiq)	74.5%	80.6%	28.6%	33.0%
B/M-by-size	46.3%	49.2%	28.4%	32.6%
Liq cov + level	59.8%	60.6%	40.6%	42.0%
Liq var + level	67.0%	67.9%	22.1%	27.4%
Panel B: Different sample periods				
	2HOR	2HOR+α	AP	AP+α
First half of sample	86.1%	88.7%	52.3%	66.9%
Second half of sample	8.7%	9.4%	-7.3%	7.3%
AP sample	70.9%	69.7%	34.8%	37.2%
Panel C: Different horizons				
	2HOR	2HOR+α		
$h_2 = 36$	74.8%	75.3%		
$h_2 = 60$	77.0%	77.2%		
$h_2 = 240$	74.8%	74.8%		
$h_1 = 3$	69.8%	69.8%		
$h_1 = 6$	53.6%	54.6%		
$h_1 = 12$	37.6%	39.2%		

Table IA.3: GMM estimation results: Robustness of liquidity risk premium to different portfolio sorts, sample periods and horizons

This table shows the model-implied liquidity risk premium for various portfolio sorts (Panel A), sample periods (Panel B), and horizons (Panel C). The full sample period is 1964–2009. A value-weighted market portfolio is used. The parameters are estimated using GMM. We include four different portfolio sorts. First, monthly data corresponding to 25 value-weighted US stock portfolios sorted on the standard deviation of daily transaction costs in the past 12 months. Second, double-sorted portfolios on size and book-to-market. Third, we provide results for a 5x5 double sort, sorting on the daily covariance of the stock return and market-wide transaction costs (over the past 12 months), and on the average transaction costs. Fourth, we present results for a 5x5 double sort, sorting on the standard deviation of daily transaction costs in the past year (liquidity risk) and on the average transaction costs over the past 12 months. Panel B presents results based on the first and second half of the sample, respectively, and for the sample period of Acharya and Pedersen (2005): 1964-1999. Panel C presents results for different investor horizons. Results are presented for the two-horizon model (equation (6)), with $h_1 = 1$, and $h_2 = 120$, without and with constant term α (2HOR and 2HOR+α).

Panel A: Different portfolio sorts		
	2HOR	2HOR+α
Benchmark	0.020%	0.016%
$\sigma(illiq)$	0.017%	0.008%
B/M-by-size	0.011%	0.008%
Liq cov + level	0.007%	0.004%
Liq var + level	0.019%	0.007%
Panel B: Different sample periods		
	2HOR	2HOR+α
First half of sample	0.008%	0.015%
Second half of sample	0.009%	0.004%
AP sample	0.017%	0.018%
Panel C: Different horizons		
	2HOR	2HOR+α
$h_2 = 36$	0.019%	0.024%
$h_2 = 60$	0.017%	0.010%
$h_2 = 240$	0.005%	0.005%
$h_1 = 3$	0.051%	0.051%
$h_1 = 6$	0.058%	0.054%
$h_1 = 12$	0.065%	0.052%

Table IA.4: GMM estimation results: number of integrated portfolios for different portfolio sorts, sample periods and horizons

This table reports how many of the 25 portfolios are held by both the short-term and long-term investors, for various portfolio sorts (Panel A), sample periods (Panel B), and horizons (Panel C). The full sample period is 1964–2009. A value-weighted market portfolio is used. The parameters are estimated using GMM. We include four different portfolio sorts. First, monthly data corresponding to 25 value-weighted US stock portfolios sorted on the standard deviation of daily transaction costs in the past 12 months. Second, double-sorted portfolios on size and book-to-market. Third, we provide results for a 5x5 double sort, sorting on the daily covariance of the stock return and market-wide transaction costs (over the past 12 months), and on the average transaction costs. Fourth, we present results for a 5x5 double sort, sorting on the standard deviation of daily transaction costs in the past year (liquidity risk) and on the average transaction costs over the past 12 months. Panel B presents results based on the first and second half of the sample, respectively, and for the sample period of Acharya and Pedersen (2005): 1964-1999. Panel C presents results for different investor horizons. Results are presented for the two-horizon model (equation (6)), with $h_1 = 1$, and $h_2 = 120$, without and with constant term α (2HOR and 2HOR+α).

Panel A: Different portfolio sorts		
	2HOR	2HOR+α
Benchmark	3	2
σ(illiq)	4	1
B/M-by-size	9	8
Liq cov + level	5	3
Liq var + level	5	2
Panel B: Different sample periods		
	2HOR	2HOR+α
First half of sample	1	3
Second half of sample	3	5
AP sample	4	3
Panel C: Different horizons		
	2HOR	2HOR+α
$h_2 = 36$	3	2
$h_2 = 60$	3	1
$h_2 = 240$	2	2
$h_1 = 3$	6	6
$h_1 = 6$	9	8
$h_1 = 12$	6	7

Table IA.5: Determinants of the regime probabilities

This table gives results of time-series regressions of the monthly probability of being in the liquid regime, as implied by the regime-switching model in Table III of the paper, on a set of financial and economic variables: the NBER recession dummy, the default spread, the volatility of stock market returns, the term spread, the short rate, and the Baker-Wurgler sentiment index. The full sample runs from January 1964 until December 2009.

	Probliq NBER	Probliq Default Sprd	Probliq Mkt Vola	Probliq Term Sprd	Probliq Riskfree	Probliq Sentiment	Probliq Full Set
NBER	-0.4230***						0.0183
	(0.0444)						(0.0738)
Default Sprd		-0.6001					19.5978***
		(2.9063)					(4.7700)
Mkt Vola			-27.8141***				-36.8513***
			(2.8210)				(4.3828)
Term Sprd				8.4181***			8.4974***
				(1.6878)			(2.1154)
Riskfree					-32.0616***		34.7615**
					(7.2171)		(17.6408)
Sentiment						-0.0126	-0.1018***
						(0.0166)	(0.0318)
Constant	0.7721***	0.7258***	1.2902***	0.5805***	0.8536***	0.6988***	0.7820***
	(0.0172)	(0.0684)	(0.0611)	(0.0358)	(0.0367)	(0.0175)	(0.1303)
R^2	0.1416	0.0001	0.1502	0.0693	0.0346	0.0011	0.2758
Observations	552	288	552	336	552	534	288

Table IA.6: Estimation results: Model with regime changes in covariances

This table shows the estimation results for the dynamic version of the model (Section VI of the paper). The estimates are based on monthly data corresponding to 25 value-weighted US stock portfolios sorted on illiquidity with sample period 1964–2009. An equal-weighted market portfolio is used. The table reports GMM estimates of the asset pricing model with two regimes for transaction costs, as given in equation (29), with a constant term for the expected returns in the illiquid regime, and where we allow both transaction costs and all return and cost covariances to differ across regimes. We set $h_1 = 1$, and $h_2 = 120$. For each portfolio, we have two moment conditions: the mean return in the illiquid regime and the mean return in the liquid regime. The parameters are estimated using GMM. For each coefficient the t-value is given in parentheses. The cross-sectional R^2 and RMSE are also reported, as well as the risk-bearing capacities ($\gamma_j h_j$).

GMM estimates of conditional asset pricing model						
γ_1	γ_2	$\alpha_{I_t=1}$	R^2	$RMSE$	$\gamma_1 h_1$	$\gamma_2 h_2$
0.7081	0.00039	-0.3143%	37.6%	0.172%	0.7081	0.0470
(1.77)	(1.39)	(-0.81)				

Figure IA.1: **Probability of illiquid regime**

The solid line of the figure gives, for each month in the 1964-2009 sample period, the probability of being in the illiquid regime, as implied by the regime-switching model in Table III of the paper: $\mathbb{P}(I_t = 1 \mid c_t, c_{t-1}, \ldots)$, conditional upon all information at time t. The dashed line contains the NBER recession dummy for each month, which is equal to 1 in case of recession and 0 otherwise.

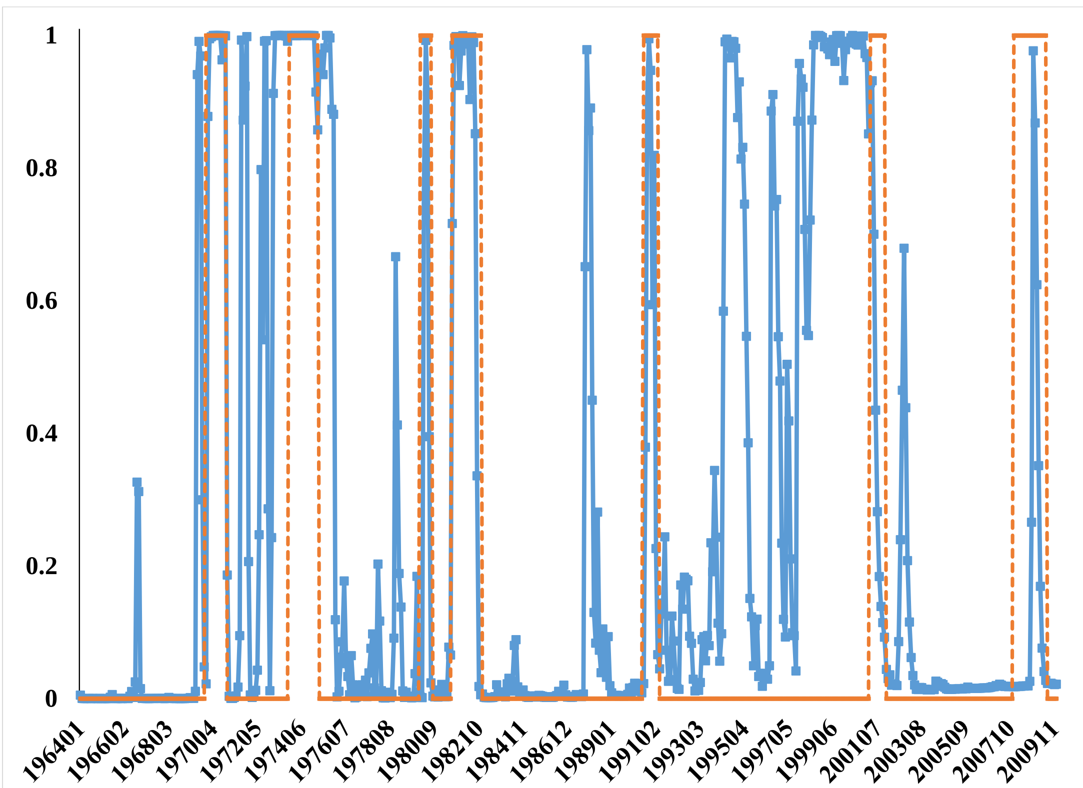

Figure IA.2: **Optimal portfolio holdings in different regimes: Regime changes in covariances** This figure gives, for each of the 25 equity portfolios sorted on transaction costs, the optimal holdings of long-term (120-month) investors for the dynamic two-horizon model in Table IA.6, both in the liquid regime and the illiquid regime, where we allow variances and covariances to differ across regimes (in addition to transaction costs). These holdings are obtained using Proposition 1, and are presented as a fraction of the total supply of the value-weighted market portfolio.