Internet Appendix for Asset Redeployability, Liquidation Value, and Endogenous Capital Structure Heterogeneity

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Internet Appendix A. Extension of Theorem 2 to the Full Parameter Space

1. Statement

In the text, we covered a limited parameter space for illustration. This appendix states the theorem and proof for the model's complete parameter space.

$$
\begin{aligned}
\text{Let } \tau_0 &= \frac{\phi(1-2\eta)}{(1+\eta)(1+\phi)+\phi(1-2\eta)}, \\
\tau_1 &= \frac{2\eta^2(\phi+1)-\phi+\eta(\phi+1)-\sqrt{(\eta+1)^2(\phi+1)(\eta^2\phi+\eta^2-2\phi-\eta\phi)}}{3\eta(\phi+1)+3\phi+2}, \\
\tau_2 &= \frac{(2\eta-1)(2\eta+\phi+2\eta\phi)}{2+3\phi} \\
&- \frac{\sqrt{(2\eta-1)^2(1+\phi)(4\eta^2(1+\phi)+\eta\phi-2(\eta+\phi))}}{2+3\phi}, \\
\tau_3 &= \frac{2\eta^2(1-\phi^2)+\eta(1+9\phi+\phi^2-5\phi^3)+2\phi+12\phi^2+7\phi^3}{2+14\phi+20\phi^2+9\phi^3+3\eta(1-\phi)(1+\phi)^2} \\
&- \frac{\sqrt{(1+\phi)(1+\eta+5\phi-\eta\phi)^2(\eta^2(1+\phi)(1+3\phi^2)-3\eta\phi^2(2+\phi)-2\phi^3)}}{2+14\phi+20\phi^2+9\phi^3+3\eta(1-\phi)(1+\phi)^2}, \\
\tau_4 &= \frac{2\eta^2(1-\phi^2)+\eta(1+9\phi+\phi^2-5\phi^3)+2\phi+12\phi^2+7\phi^3}{2+14\phi+20\phi^2+9\phi^3+3\eta(1-\phi)(1+\phi)^2} \\
&+ \frac{\sqrt{(1+\phi)(1+\eta+5\phi-\eta\phi)^2(\eta^2(1+\phi)(1+3\phi^2)-3\eta\phi^2(2+\phi)-2\phi^3)}}{2+14\phi+20\phi^2+9\phi^3+3\eta(1-\phi)(1+\phi)^2}.
\end{aligned}
$$

Region 1: $\eta \geq 1/2$ **and** $\phi < \frac{3\eta-2}{6-3\eta}$

- If $0 \leq \tau \leq \tau_1$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta-\tau}{1+\eta}$ in which all firms choose $F_L^* = \frac{1-\eta+2\tau}{1+\eta}$.

- If $\tau_1 < \tau \leq \tau_2$ there exists a unique, mixed-strategy equilibrium with price

$$\begin{aligned} P^* &= \frac{\phi\eta-(1+\phi)[\tau-\eta(1+\tau)]}{1+\phi(1+\eta)} \\ &- \frac{\sqrt{\eta(\phi+1)(2\tau(\eta\phi+(\eta-\tau)(1+\phi))+\eta\tau^2(\phi+1)-\phi(1+\tau-\eta)^2)}}{1+\phi(1+\eta)}. \end{aligned}$$

in which the proportion $1-h^*$ of firms choose $F_L^* = \frac{\tau}{\eta} + \frac{(1-\eta)}{\eta}P^*$ and the proportion h^* of firms choose $F_H^* = 1$, where

$$h^* = \frac{(1+\phi)\cdot(\eta-\tau-(1+\eta)\cdot P)}{\eta\phi+(1+\phi)(\eta-\tau)-(1+\phi(1+\eta))\cdot P}.$$

- If $\tau_2 < \tau \leq \tau_4$ there exists a unique, mixed-strategy equilibrium with price

$$\begin{aligned} P^* &= \frac{\phi(1+2\phi(1-\tau)-3\tau)+\eta(1+\phi)(1+\tau+\phi(2+\tau))-\tau}{1+(6-3\eta)\phi(1+\phi)} \\ &- \frac{\sqrt{(1+\phi)(\eta+\phi+\eta\phi)\begin{pmatrix} 3\eta^2\phi(1+\phi)-2(\phi(\tau-1)+\tau)^2 \\ +\eta(\phi(\tau-1)+\tau)(2+\phi(\tau-1)+\tau) \end{pmatrix}}}{1+(6-3\eta)\phi(1+\phi)}. \end{aligned} \tag{A-1}$$

in which the proportion $1-h^*$ of firms choose $F_L^* = \frac{\tau(1+\phi)}{(\eta+\phi+\eta\phi)} + \frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)}P^*$ and the proportion h^* of firms choose $F_H^* = 1$, where

$$h^* = \frac{(1+\phi)\cdot(\eta+\phi+\eta\phi-\tau(1+2\phi)-P\cdot(1+\eta+5\phi-\eta\phi))}{(1+2\phi)\cdot(\eta+\phi+\eta\phi-\tau(1+\phi))-P\cdot(1+5\phi-\eta\phi+5\phi^2-\eta\phi^2)}. \tag{A-2}$$

- ▹ If $\tau_4 < \tau \leq \frac{\eta+\phi+\eta\phi}{1+2\phi}$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$ in which all firms choose $F_L^* = \frac{1+2\phi+\tau+(\tau-\eta)(1+\phi)}{1+\eta+5\phi-\eta\phi}$.

Region 2: $\eta \geq 1/2$, $\phi \geq \frac{3\eta-2}{6-3\eta}$, **and** $\eta^2(1+\phi)(1+3\phi^2)-3\eta\phi^2(2+\phi)-2\phi^3 \geq 0$

- ▹ If $0 \leq \tau \leq \frac{2\eta-1}{3}$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta-\tau}{1+\eta}$ in which all firms choose $F_L^* = \frac{1-\eta+2\tau}{1+\eta}$.
- ▹ If $\frac{2\eta-1}{3} < \tau \leq \tau_3$ or if $\tau_4 < \tau \leq \frac{\eta+\phi+\eta\phi}{1+2\phi}$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$ in which all firms choose $F_L^* = \frac{1+2\phi+\tau+(\tau-\eta)(1+\phi)}{1+\eta+5\phi-\eta\phi}$.
- ▹ If $\tau_3 < \tau \leq \tau_4$ there exists a unique, mixed-strategy equilibrium with price (A-1) in which the proportion $1-h^*$ of firms choose $F_L^* = \frac{\tau(1+\phi)}{(\eta+\phi+\eta\phi)} + \frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)}P^*$ and the proportion h^* of firms choose $F_H^* = 1$, where h^* is (A-2).

Region 3: $\eta \geq 1/2$, $\phi \geq \frac{3\eta-2}{6-3\eta}$ **and** $\eta^2(1+\phi)(1+3\phi^2)-3\eta\phi^2(2+\phi)-2\phi^3 < 0$

- ▹ If $0 \leq \tau \leq \frac{2\eta-1}{3}$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta-\tau}{1+\eta}$ in which all firms choose $F_L^* = \frac{1-\eta+2\tau}{1+\eta}$.
- ▹ If $\frac{2\eta-1}{3} < \tau \leq \frac{\eta+\phi+\eta\phi}{1+2\phi}$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$ in which all firms choose $F_L^* = \frac{1+2\phi+\tau+(\tau-\eta)(1+\phi)}{1+\eta+5\phi-\eta\phi}$.

Region 4: $\eta < 1/2$ **and** $\eta^2(1+\phi)(1+3\phi^2)-3\eta\phi^2(2+\phi)-2\phi^3 \geq 0$

- ▹ If $0 < \tau \leq \tau_0$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta(1-\tau)}{1+\eta}$ in which all firms choose $F_L^* = \frac{\tau(1+\eta+\phi)+\phi\eta}{\phi(1+\eta)}$.
- ▹ If $\tau_0 < \tau \leq \tau_3$ or if $\tau_4 \leq \tau \leq \frac{\phi+\eta(1+\phi)}{1+2\phi}$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$ in which all firms choose $F_L^* = \frac{1+2\phi+\tau+(\tau-\eta)(1+\phi)}{1+\eta+5\phi-\eta\phi}$.

- ▹ If $\tau_3 < \tau \leq \tau_4$ there exists a unique, mixed-strategy equilibrium with price (A-1) in which the proportion $1-h^*$ of firms choose $F_L^* = \frac{\tau(1+\phi)}{(\eta+\phi+\eta\phi)} + \frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)}P^*$ and the proportion h^* of firms choose $F_H^* = 1$, where h^* is (A-2).

Region 5: $\eta < 1/2$ **and** $\eta^2(1+\phi)(1+3\phi^2)-3\eta\phi^2(2+\phi)-2\phi^3 < 0$

- ▹ If $0 < \tau \leq \tau_0$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta(1-\tau)}{1+\eta}$ in which all firms choose $F_L^* = \frac{\tau(1+\eta+\phi)+\phi\eta}{\phi(1+\eta)}$.
- ▹ If $\tau_0 < \tau \leq \frac{\eta+\phi+\eta\phi}{1+2\phi}$ there exists a unique, pure-strategy equilibrium with price $P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$ in which all firms choose $F_L^* = \frac{1+2\phi+\tau+(\tau-\eta)(1+\phi)}{1+\eta+5\phi-\eta\phi}$.

Region 6:

- ▹ If $\frac{\eta+\phi+\eta\phi}{1+2\phi} < \tau \leq 1$ there exists a unique, pure-strategy equilibrium with price $P^* = 0$ in which all firms choose $F_L^* = \min\left\{1, \frac{\tau(1+\phi)}{\eta(1+\phi)+\phi}\right\}$.

2. Proof

Proof when $\eta \geq 1/2$

We first consider the case $\eta \geq 1/2$. Each firm is competitive and takes the price, P, as given. For now, we assume $P > 0$. We will consider the possibility that $P = 0$ later in the proof. The marginal benefit of debt is τ for all values of debt, F. If $P > 0$ the marginal cost of debt falls into four regions:

i) For $F \in [0, P \cdot (1/\eta - 1)]$ the marginal cost of debt is 0

ii) For $F \in (P \cdot (1/\eta - 1), P]$ the marginal cost of debt is $\eta F - P \cdot (1 - \eta)$

iii) For $F \in (P, 1 - P)$ the marginal cost of debt is $\eta F - P \cdot (1 - \eta) + (F - P) \cdot \phi/(1 + \phi)$

iv) For $F \in [1 - P, 1]$ the marginal cost of debt is $(F - P) \cdot \phi/(1 + \phi)$

Importantly, the marginal cost of debt is weakly increasing and continuous in F over the first three regions but then jumps down at $F = 1 - P$ (since the financing constraint is no longer binding) after which it increases again. Therefore, for any given marginal benefit τ, there are at most two possible optimal debt choices, one where $F < 1 - P$, and one where $1 - P < F \leq 1$. Consequently, there is the possibility of both pure-strategy equilibria and mixed-strategy equilibria in which some firms choose low debt and others choose high debt. We must consider three cases: (i) $0 \leq \tau < (2\eta - 1) \cdot P$, (ii) $(2\eta - 1) \cdot P \leq \tau \leq \eta - P + (1 - 2P)\phi/(1 + \phi)$, and (iii) $\eta - P + (1 - 2P)\phi/(1 + \phi) < \tau \leq 1$.

Case 1: $0 \leq \tau < (2\eta - 1) \cdot P$

If $\tau < (2\eta - 1) \cdot P$ then firms choose either $F_L \in [P \cdot (1/\eta - 1), P)$ or $F_H \in [1 - P, 1]$ where $F_L = \tau/\eta + P \cdot (1/\eta - 1)$ and $F_H = \min\{1, P + \tau \cdot (1 + \phi)/\phi\}$.

<u>Pure-strategy equilibria</u>

There cannot exist a symmetric equilibrium with $P > 0$ in which all firms choose F_H because the aggregate demand for the risky asset would be zero but the supply is positive $[(P +$

$\phi F)/(1+\phi)]$. Therefore, if there exists a symmetric equilibrium in the case $\tau < (2\eta-1)\cdot P$, then all firms choose $F_L = \tau/\eta + P\cdot(1/\eta-1)$. The demand for the liquidated asset is then $1-P-F_L$ and the supply of the liquidated asset is P since for $F_L < P$ which implies $\tau < (2\eta-1)\cdot P$ it is optimal to liquidate the asset for all $V \leq P$. Equating supply and demand gives the unique market clearing price $P^* = (\eta-\tau)/(1+\eta)$.

Importantly, note that if $P^* = (\eta-\tau)/(1+\eta)$ then we must have $\tau < (2\eta-1)/3$ to be in a symmetric equilibrium in the case $\tau < (2\eta-1)\cdot P$.

Mixed-strategy equilibria

There is also the possibility of a mixed-strategy equilibrium (a fraction of firms choosing F_L and the remaining fraction of firms choosing F_H). Firms choosing F_L have ex ante value

$$V_L = \int_0^P PdV + \int_P^1 VdV + \int_{P+F_L}^1 (\eta V - P)dV + \tau\cdot F_L$$

and by substituting $F_L = \tau/\eta + P\cdot(1/\eta-1)$ yields

$$V_L = 0.5(\eta + (P-1)^2 + (P+\tau)^2/\eta - 2P\tau).$$

Firms choosing F_H have ex ante value

$$V_H = \int_0^\Lambda PdV + \int_\Lambda^{F_H} [V - \phi\cdot(F_H - V)]dV + \int_{F_H}^1 VdV + \tau\cdot F_H.$$

where $\Lambda = (P+\phi F)/(1+\phi)$.

If $F_H = P + \tau \cdot (1+\phi)/\phi$ then ex ante value is

$$V_H = 0.5 \cdot (P+\tau)^2 + 0.5 \cdot \tau^2/\phi + 0.5,$$

but if $F_H = 1$ then ex ante value is

$$V_1 = 0.5 \cdot (P+\phi)^2/(1+\phi) + 0.5 \cdot (1-\phi) + \tau.$$

The following result shows that in a mixed-strategy equilibrium the high-type always chooses $F_H = 1$.

Lemma 1: If $\tau < (2\eta - 1)/3$ then in a mixed-strategy equilibrium the high type chooses $F_H = 1$.

Proof: Proof by contradiction. Suppose $F_H = P + \tau \cdot (1+\phi)/\phi < 1$. Then we have:

$$G(P) \equiv V_L(P) - V_H(P) = 0.5 \cdot (\eta + (P-1)^2 + (P+\tau)^2 \cdot (1-\eta)/\eta - 2P\tau - \tau^2/\phi - 1).$$

Note, $G'(P) = P/\eta - 1 + \tau \cdot (1/\eta - 2) \leq P/\eta - 1 \leq 0$ where the first inequality follows from our assumption that $1/\eta - 1 \leq 1$ and the second from the fact that $P \leq \eta$ as the price for the asset will never exceed its maximum value. Therefore, G(P) is decreasing in P. Furthermore, in a mixed-strategy equilibrium the price is bounded above by the pure-strategy equilibrium price (i.e.

$P \leq (\eta - \tau)/(1+\eta))$ because the introduction of some high-debt firms both reduces the demand and increases the supply of the liquidated asset. Therefore,

$$\begin{aligned}
G(P) &\geq G((\eta-\tau)/(1+\eta)) \\
&= 0.5 \cdot \left[\left(\frac{1+\tau}{1+\eta} \right)^2 (1+\eta-\eta^2) + \eta - 2\tau \left(\frac{\eta-\tau}{\eta+1} \right) - \frac{\tau^2}{\phi} - 1 \right] \\
&> 0.5 \cdot \left[\left(\frac{1+\tau}{1+\eta} \right)^2 (1+\eta-\eta^2) + \eta - 2\tau \left(\frac{\eta-\tau}{\eta+1} \right) - \tau(1-\tau) - 1 \right] \\
&\geq 0 \quad \forall \tau
\end{aligned}$$

where the second inequality follows from the fact that if $F_H^* < 1$ and $P > 0$ then $\tau < \phi/(1+\phi)$ which implies $\phi > \tau/(1-\tau)$; and the third inequality is easily verified numerically. But this contradicts the optimality of F_H^*.

By Lemma 1, we must only compare V_L to V_1 to find a mixed-strategy equilibrium.

Conjecture the existence of a pure-strategy equilibrium in which $F_L = \tau/\eta + P \cdot (1/\eta - 1)$ and $P^* = (\eta-\tau)/(1+\eta)$. Substituting P^* into our expressions for V_L and V_1 implies:

$$H(\tau) \equiv V_L(\tau) - V_1(\tau) = \frac{\eta(\eta-\tau)^2(\phi+1) + \phi(\tau+1)^2 - 2(\eta+1)(\phi+1)\tau(\eta-\tau)}{2(\eta+1)^2(\phi+1)}$$

Therefore,

$$H'(\tau) = \frac{\phi - \eta(1+\phi) - 2\eta^2(1+\phi) + (2+3\phi+3\eta(1+\phi))\tau}{(1+\eta)^2(1+\phi)}$$

and

$$H''(\tau) = \frac{2+3\phi+3\eta(1+\phi)}{(1+\eta)^2(1+\phi)}$$

Note that $H'\left(\frac{2\eta-1}{3}\right) = -\frac{2}{3(1+\eta)(1+\phi)} < 0$ and $H''(\tau) > 0$ for all τ. Therefore, $H'(\tau) < 0$ for all $\tau \leq \frac{2\eta-1}{3}$.

Also, $H\left(\frac{2\eta-1}{3}\right) = \frac{2-3\eta+\phi\cdot(6-3\eta)}{18(1+\phi)} \geq 0$ if and only if $\phi \geq \frac{3\eta-2}{6-3\eta}$. Note that if $\eta < \frac{2}{3}$, then $\frac{3\eta-2}{6-3\eta} < 0$ and $H\left(\frac{2\eta-1}{3}\right) \geq 0$ for any ϕ.

Therefore, if $\phi \geq \frac{3\eta-2}{6-3\eta}$ and $\tau \leq \frac{2\eta-1}{3}$ then $H(\tau) \geq 0$ and all firms optimally choose $F_L = \tau/\eta + P\cdot(1/\eta-1)$ and the conjectured equilibrium price of $P = (\eta-\tau)/(1+\eta)$ is confirmed by the firms' debt decisions.

However, if $\phi < \frac{3\eta-2}{6-3\eta}$, then the conjectured pure-strategy equilibrium is confirmed only for $\tau \leq \tau_1$ where $H(\tau_1) \equiv 0$. For $\tau > \tau_1$ there is a mixed-strategy equilibrium in which some firms choose $F_L = \tau/\eta + P\cdot(1/\eta-1)$ and others choose $F_H = 1$ (by Lemma 1). Solving $F(\tau_1) = 0$ yields

$$\tau_1 = \frac{2\eta^2(\phi+1)-\phi+\eta(\phi+1)-\sqrt{(\eta+1)^2(\phi+1)(\eta^2\phi+\eta^2-2\phi-\eta\phi)}}{3\eta(\phi+1)+3\phi+2}$$

(Note: There is another solution to $H(\tau_1) = 0$ where $H(\tau)$ again becomes positive beyond that point. However, $H'(\tau) < 0$ for all $\tau \leq \frac{2\eta-1}{3}$ so we know $H(\tau) < 0$ for all $\tau_1 < \tau < \frac{2\eta-1}{3}$.)

For $\tau > \tau_1$ there is a unique, mixed-strategy equilibrium which is constructed by finding P that equates $V_L = V_1$, which is quadratic in P. There are two solutions, but only one where P is less than the pure-strategy price (which must be true in equilibrium as argued above) and it is

$$P^* = \frac{\phi\eta - (1+\phi)[\tau - \eta(1+\tau)]}{1+\phi(1+\eta)} - \frac{\sqrt{\eta(\phi+1)(2\tau(\eta\phi + (\eta-\tau)(1+\phi)) + \eta\tau^2(\phi+1) - \phi(1+\tau-\eta)^2)}}{1+\phi(1+\eta)},$$

Let h be the fraction of firms choosing $F = 1$. The demand for the risky asset is then $(1-h)\cdot(1-P-F_L)$ and the supply of the risky asset is $(1-h)\cdot P + h\cdot(P+\phi\cdot 1)/(1+\phi)$, therefore, market clearing requires a unique proportion of high debt firms:

$$h^* = \frac{(1+\phi)\cdot(\eta-\tau-(1+\eta)\cdot P)}{\eta\phi + (1+\phi)(\eta-\tau) - (1+\phi(1+\eta))\cdot P},$$

Finally, if there is a mixed-strategy equilibrium at the upper boundary we know that $P < (\eta-\tau)/(1+\eta)$ and therefore $\tau < (2\eta-1)/3$ at the boundary. Equating $\tau_2 = (2\eta-1)\cdot P^*(\tau_2)$ yields the upper boundary in this case:

$$\tau_2 = \frac{(2\eta-1)(2\eta+\phi+2\eta\phi)}{2+3\phi} - \frac{\sqrt{(2\eta-1)^2(1+\phi)(4\eta^2(1+\phi)+\eta\phi-2(\eta+\phi))}}{2+3\phi}$$

(Note: There is another root but it is greater than $(2\eta-1)/3$ when $\eta \geq 1/2$ so we can ignore it.)

Case 2: $(2\eta-1)\cdot P \leq \tau \leq \eta - P + (1-2P)\phi/(1+\phi)$

If $(2\eta-1)\cdot P \leq \tau \leq \eta - P + (1-2P)\phi/(1+\phi)$ then firms choose either $F_L \in [P, 1-P)$ or $F_H \in [1-P, 1]$ where $F_L = \frac{\tau(1+\phi)}{\eta+\phi+\eta\phi} + \frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)}P$ and $F_H = \min\{1, P + \tau\cdot(1+\phi)/\phi\}$.

Pure-strategy equilibria

Again, there cannot exist a pure-strategy equilibrium with $P > 0$ in which all firms choose F_H because the aggregate demand for the risky asset would be zero but the supply is positive $[(P+\phi F)/(1+\phi)]$. Therefore, in a pure-strategy equilibrium firms choose

$$F = \frac{\tau(1+\phi)}{\eta+\phi+\eta\phi} + \frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)}P.$$

The demand for the liquidated asset is $1-P-F$ and the supply of the liquidated asset is

$$\Lambda = \frac{P+\phi F}{1+\phi} = \frac{P\cdot((1-\phi)\eta+2\phi)+\phi\tau}{\eta+\phi+\eta\phi}.$$

Equating supply and demand gives the unique market clearing price

$$P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}.$$

Substituting into the expression for F yields

$$F^* = \frac{1-\eta-\eta\phi+2\phi+\tau(2+\phi)}{1+\eta+5\phi-\eta\phi}.$$

Mixed-strategy equilibria

As in Case 1, there is the possibility of a mixed-strategy equilibrium. Firms choosing F_L have ex ante value

$$V_L = \int_0^{\Lambda} P dV + \int_{\Lambda}^{F_L} [V - \phi(F_L - V)] dV + \int_{F_L}^{1} V dV + \int_{P+F_L}^{1} (\eta V - P) dV + \tau \cdot F_L$$

and by substituting $F_L = \frac{\tau(1+\phi)}{\eta+\phi+\eta\phi} + \frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)} P$ yields

$$\begin{aligned} V_L &= \left(\frac{1+\eta+6\phi-3\eta\phi}{2(\eta+\phi+\eta\phi)}\right) P^2 - \left(1 - \frac{\tau(1-\eta+2\phi-\eta\phi)}{\eta+\phi+\eta\phi}\right) P \\ &+ \left(1 + \frac{\eta(\eta-1)+\phi(\eta^2-1)+\tau^2(1+\phi)}{2(\eta+\phi+\eta\phi)}\right). \end{aligned} \tag{A-3}$$

Firms choosing F_H have ex ante value

$$V_H = \int_0^{\Lambda} P dV + \int_{\Lambda}^{F_H} [V - \phi \cdot (F_H - V)] dV + \int_{F_H}^{1} V dV + \tau \cdot F_H.$$

If $F_H = P + \tau \cdot (1+\phi)/\phi$ then ex ante value is

$$V_H = 0.5 \cdot (P+\tau)^2 + 0.5 \cdot \tau^2/\phi + 0.5 \tag{A-4}$$

but if $F_H = 1$ then ex ante value is

$$V_1 = 0.5 \cdot (P+\phi)^2/(1+\phi) + 0.5 \cdot (1-\phi) + \tau. \tag{A-5}$$

The next result shows that in a mixed-strategy equilibrium the high type always chooses $F_H = 1$.

Lemma 2: In a mixed-strategy equilibrium the high type chooses $F_H = 1$.

Proof: Proof by contradiction. Suppose $F_H^* = P + \tau \cdot (1+\phi)/\phi < 1$. Since we assume $P > 0$, this implies $\tau < \phi/(1+\phi)$. We have:

$$
\begin{aligned}
G(P) &\equiv V_L(P) - V_H(P) \\
&= \left(\frac{1+(5-4\eta)\phi}{2(\eta+\phi+\eta\phi)}\right)P^2 - \left(1 - \frac{\tau(1+\phi)(1-2\eta)}{\eta+\phi+\eta\phi}\right)P + \left(\frac{\eta}{2} - \frac{\eta\tau^2(1+\phi)^2}{2\phi(\eta+\phi+\eta\phi)}\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
G'(P) &= \left(\frac{1+(5-4\eta)\phi}{\eta+\phi+\eta\phi}\right)P - \left(1 - \frac{\tau(1+\phi)(1-2\eta)}{\eta+\phi+\eta\phi}\right) \\
&\leq \left(\frac{1+(5-4\eta)\phi}{\eta+\phi+\eta\phi}\right)\left(\frac{\eta(1+\phi)+\phi-\tau(1+2\phi)}{(5-\eta)\phi+1+\eta}\right) - \left(1 - \frac{\tau(1+\phi)(1-2\eta)}{\eta+\phi+\eta\phi}\right) \\
&\leq 0
\end{aligned}
$$

where the first inequality follows from the fact that in a mixed-strategy equilibrium the equilibrium price is less than the pure-strategy equilibrium price (i.e., $P \leq \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$) and the second inequality holds for all $\eta \geq 1/2$. Therefore,

$$
\begin{aligned}
G(P) &\geq G\left(\frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}\right) \\
&> 0 \quad \forall \tau < \phi/(1+\phi).
\end{aligned}
$$

where the last inequality is easily verified numerically. But this contradicts the optimality of F_H^*.

By Lemma 2, we must only compare V_L to V_1 to find a mixed-strategy equilibrium. We have

$$\begin{aligned} V_L - V_1 &= \frac{\left(\phi(\phi+1)(2-3\eta)+(2\phi+1)^2\right)}{2(\phi+1)(\eta\phi+\eta+\phi)}P^2 \\ &- \frac{\left((2\phi+1)(\eta+\phi+\eta\phi-\tau(1+\phi))+\eta(\phi+1)^2\tau\right)}{(\phi+1)(\eta\phi+\eta+\phi)}P \\ &+ \frac{((\eta-\tau)(1+\phi)+\phi)^2}{2(\phi+1)(\eta\phi+\eta+\phi)} \end{aligned} \tag{A-6}$$

Conjecture the existence of a pure-strategy equilibrium in which case $P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$. Substituting P^* into our expressions for V_L and V_1 implies:

$$\begin{aligned} H(\tau) = &\frac{2+14\phi+20\phi^2+9\phi^3+3\eta(1-\phi)(1+\phi)^2}{2(1+\phi)(1+\eta+5\phi-\eta\phi)^2}\tau^2 \\ &- \frac{2\eta^2(1-\phi^2)+\phi(2+12\phi+7\phi^2)+\eta(1+9\phi+\phi^2-5\phi^3)}{(1+\phi)(1+\eta+5\phi-\eta\phi)^2}\tau \\ &+ \frac{\eta^3(\phi-1)^2(1+\phi)+\phi^2(2+11\phi)+2\eta\phi(1+8\phi+\phi^2)-4\eta^2\phi(-1+2\phi+2\phi^2)}{2(1+\phi)(1+\eta+5\phi-\eta\phi)^2} \end{aligned} \tag{A-7}$$

We have two possibilities to consider. First, suppose there is a pure-strategy equilibrium at the upper boundary of Case 1, $\tau = (2\eta-1)\cdot P$. In this case, we know that $P = (\eta-\tau)/(\eta+1)$ which implies $\tau = (2\eta-1)/3$. We also know that $\phi \geq \frac{3\eta-2}{6-3\eta}$. Therefore, at the transition to this case we have $H\left(\frac{2\eta-1}{3}\right) = \frac{2-3\eta+\phi\cdot(6-3\eta)}{18(1+\phi)} \geq 0$ for $\phi \geq \frac{3\eta-2}{6-3\eta}$. Furthermore,

$$H'\left(\frac{2\eta-1}{3}\right) = -\frac{2+(10-6\eta)\phi+6(1-\eta)\phi^2}{3(1+\eta(1-\phi)+5\phi)(1+\phi)} < 0,$$

and

$$H''(\tau) = \frac{2+14\phi+20\phi^2+9\phi^3+3\eta(1-\phi)(1+\phi)^2}{(1+\phi)(1+\eta+5\phi-\eta\phi)^2} > 0.$$

We see then that $H(\tau)$ is an upward facing parabola in τ. At the lower boundary of Case 2, where $\tau = \frac{2\eta-1}{3}$, $H(\tau)$ is positive but decreasing.

Solving $H(\tau) = 0$ yields two solutions:

$$\begin{aligned}\tau_3, \tau_4 &= \frac{2\eta^2(1-\phi^2)+\eta(1+9\phi+\phi^2-5\phi^3)+2\phi+12\phi^2+7\phi^3}{2+14\phi+20\phi^2+9\phi^3+3\eta(1-\phi)(1+\phi)^2} \\ &\pm \frac{\sqrt{(1+\phi)(1+\eta+5\phi-\eta\phi)^2(\eta^2(1+\phi)(1+3\phi^2)-3\eta\phi^2(2+\phi)-2\phi^3)}}{2+14\phi+20\phi^2+9\phi^3+3\eta(1-\phi)(1+\phi)^2}\end{aligned}$$

If $\eta^2(1+\phi)(1+3\phi^2)-3\eta\phi^2(2+\phi)-2\phi^3 < 0$, the roots of the solution of $H(\tau) = 0$ are complex so $H(\tau) > 0$ for all τ. Therefore, for $\frac{2\eta-1}{3} < \tau \leq 1$, there is a pure-strategy equilibrium where all firms choose $F^* = \frac{1+2\phi+\tau+(\tau-\eta)(1+\phi)}{1+\eta+5\phi-\eta\phi}$ and $P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$. However, our free disposal assumption implies $P^* \geq 0$ which requires $\tau \leq \frac{\eta+\phi+\eta\phi}{1+2\phi}$. We thus consider pure-strategy equilibrium in the range $\frac{2\eta-1}{3} < \tau \leq \frac{\eta+\phi+\eta\phi}{1+2\phi}$.

If $\eta^2(1+\phi)(1+3\phi^2)-3\eta\phi^2(2+\phi)-2\phi^3 \geq 0$, then τ_3 and τ_4 are real. Therefore, for $\tau_3 < \tau \leq \tau_4$, we have $H(\tau) < 0$ and a mixed-strategy equilibrium where P^* equates $V_L = V_1$.

There are two solutions, but only one where P is less than the pure-strategy price (which must be true in equilibrium as argued above) and it is

$$\begin{aligned} P^* &= \frac{\phi(1+2\phi(1-\tau)-3\tau)+\eta(1+\phi)(1+\tau+\phi(2+\tau))-\tau}{1+\phi(6-3\eta)(1+\phi)} \\ &- \frac{\sqrt{(1+\phi)(\eta+\phi+\eta\phi)\begin{pmatrix} 3\eta^2\phi(1+\phi)-2(\phi(\tau-1)+\tau)^2 \\ +\eta(\phi(\tau-1)+\tau)(2+\phi(\tau-1)+\tau)) \end{pmatrix}}}{1+\phi(6-3\eta)(1+\phi)}. \end{aligned}$$

The fraction h of firms choosing $F = 1$ supporting the price P is found by equating demand for the risky asset $(1-h)\cdot(1-P-F_L)$ with supply $(1-h)\cdot(P+\phi F_L)/(1+\phi)+h\cdot(P+\phi\cdot 1)/(1+\phi)$ where $F_L = \frac{\tau(1+\phi)}{\eta+\phi+\eta\phi}+\frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)}P$. Therefore, we have the unique proportion to clear the market:

$$h^* = \frac{(1+\phi)\cdot(\eta+\phi+\eta\phi-\tau(1+2\phi)-P(1+5\phi+\eta-\eta\phi))}{(1+2\phi)(\eta+\phi+\eta\phi-\tau-\phi\tau)-P(1+5\phi(1+\phi)-\eta\phi(1+\phi))}.$$

It has been verified numerically that $\tau_4 \leq 1$ and $P^*(\tau_4) > 0$. This means that for $\tau > \tau_4$, there will be a range of pure-strategy equilibria with positive prices. Therefore, for $\frac{2\eta-1}{3} < \tau \leq \tau_3$ and $\tau_4 < \tau \leq \frac{\eta+\phi+\eta\phi}{1+2\phi}$ we have a pure-strategy equilibrium where $F^* = \frac{1+2\phi+\tau+(\tau-\eta)(1+\phi)}{1+\eta+5\phi-\eta\phi}$ and $P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$

Second, suppose there is a mixed-strategy equilibrium at the upper boundary of Case 1, i.e. $\tau_2 = (2\eta-1)\cdot P^*(\tau_2)$. Then we know that $\phi < \frac{3\eta-2}{6-3\eta}$. It can be verified that $H(\tau_2) < 0$, and, following the arguments above, $\{\tau_3, \tau_4\}$ are the same as described above. However, it can be verified that $\tau_3 < \tau_2$ if $\phi < \frac{3\eta-2}{6-3\eta}$, thus there is a mixed-strategy equilibrium for all $\tau_2 < \tau \leq \tau_4$. In the mixed-strategy equilibrium, $F_L = \frac{\tau(1+\phi)}{\eta+\phi+\eta\phi}+\frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)}P^*$ and $\{P^*, h^*\}$ are characterized

above. It is still the case that $\tau_4 \leq 1$ and $P^*(\tau_4) > 0$. Therefore, for $\tau_4 < \tau \leq \frac{\eta+\phi+\eta\phi}{1+2\phi}$ we have a pure-strategy equilibrium where $F^* = \frac{1+2\phi+\tau+(\tau-\eta)(1+\phi)}{1+\eta+5\phi-\eta\phi}$ and $P^* = \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$

We must also check that all these equilibria fall under Case 2, which requires $\tau \leq \eta - P + (1-2P)\phi/(1+\phi)$. We know that $P^* \leq \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}$, the pure-strategy equilibrium price, for all equilibria. Therefore, it is sufficient to show that:

$$\tau \leq \eta - \frac{\eta+\phi+\eta\phi-\tau(1+2\phi)}{1+\eta+5\phi-\eta\phi} + \left(1 - \frac{2(\eta+\phi+\eta\phi-\tau(1+2\phi))}{1+\eta+5\phi-\eta\phi}\right)\frac{\phi}{1+\phi}$$

which is true if and only if

$$\tau \leq \frac{\eta+2\phi-\eta\phi}{1-\phi}.$$

However, this is satisfied for all $\tau \leq \frac{\eta+\phi+\eta\phi}{1+2\phi}$. Therefore, the equilibria described above all fall within Case 2.

Case 3: $\eta - P + (1-2P)\phi/(1+\phi) < \tau \leq 1$

If $\eta - P + (1-2P)\phi/(1+\phi) < \tau \leq 1$ then all firms choose $F > 1-P$ which implies that the aggregate demand for the liquidated asset is zero. But, since supply is positive when the price is positive, this case is incompatible with an equilibrium $P^* > 0$.

Equilibrium with $P^* = 0$

We assume free disposal therefore we know $P^* \geq 0$. We now consider the possibility that $P^* = 0$ which can occur in our model because we assume limited liability (i.e., the continuation value of a firm in distress is bounded below by zero). If $P = 0$ the four regions for the marginal cost of debt collapse into one:

- ▹ For $F \in [0, 1]$ the marginal cost of debt is $\eta \cdot F + F \cdot \phi/(1+\phi)$

Consequently, if $P = 0$ there can only exist a pure-strategy equilibrium in which all firms choose $F^* = \min\left\{1, \frac{\tau(1+\phi)}{\eta(1+\phi)+\phi}\right\}$. Therefore, the aggregate demand for the asset is $1-P^*-F^* = 1-F^* \geq 0$. The aggregate supply of the asset, however, is indeterminate. In particular, because of limited liability the firm will be indifferent between liquidation at $P^* = 0$ and continuation for all $V_i \in \left[0, \frac{\phi F^*}{1+\phi}\right]$. Therefore, the price $P^* = 0$ can be supported in equilibrium if $1 - F^* \leq \frac{\phi F^*}{1+\phi}$ or $\tau \geq \frac{\eta(1+\phi)+\phi}{1+2\phi}$.

Proof when $\eta < 1/2$

We now consider the case $\eta < 1/2$. For now, assume $P > 0$. We will consider the possibility that $P = 0$ at the end of the proof. If $P > 0$ the marginal cost of debt now falls into four regions:

i) For $F \in [0, P]$ the marginal cost of debt is 0

ii) For $F \in [P, P \cdot (1/\eta - 1)]$ the marginal cost of debt is $(F - P)\phi/(1+\phi)$

iii) For $F \in [P \cdot (1/\eta - 1), 1 - P]$ the marginal cost of debt is $\eta F - P(1-\eta) + (F-P)\phi/(1+\phi)$

iv) For $F \in [1-P, 1]$ the marginal cost of debt is $(F-P)\phi/(1+\phi)$

We must consider three cases: (i) $0 \leq \tau < (P/\eta - 2P)\phi/(1+\phi)$, (ii) $(P/\eta - 2P)\phi/(1+\phi) \leq \tau \leq \eta - P + (1-2P)\phi/(1+\phi)$, and (iii) $\eta - P + (1-2P)\phi/(1+\phi < \tau \leq 1$.

Case 1: $0 \leq \tau < (P/\eta - 2P)\phi/(1+\phi)$

If $0 \leq \tau < (P/\eta - 2P)\phi/(1+\phi)$ then $F \in [P, P(1/\eta - 1)]$ and all firms equate the marginal cost of debt in this region to the marginal benefit which implies

$$F(P) = P + \frac{\tau(1+\phi)}{\phi}.$$

In this region, all firms for whom the asset is positive NPV ($\eta V \geq P$) will be able to obtain financing to purchase the asset. The demand for the liquidated asset is then $1 - P/\eta$ and the supply of the liquidated asset is $(P + \phi F)/(1+\phi) = P + \tau$. Equating supply and demand gives the equilibrium price

$$P^* = \frac{\eta(1-\tau)}{1+\eta}$$

and, therefore,

$$F^* = \frac{\tau(1+\eta+\phi) + \phi\eta}{\phi(1+\eta)}.$$

To determine the values of τ included in this case, substitute P^* into the expression

$$\tau < (P/\eta - 2P)\phi/(1+\phi) \quad \Rightarrow \quad \tau \leq \frac{\phi(1-2\eta)}{(1+\eta)(1+\phi) + \phi(1-2\eta)} \equiv \tau_0.$$

Case 2: $(P/\eta - 2P)\phi/(1+\phi) \leq \tau \leq \eta - P + (1-2P)\phi/(1+\phi)$

If $(P/\eta - 2P)\phi/(1+\phi) \leq \tau \leq \eta - P + (1-2P)\phi/(1+\phi)$ then firms choose either $F_L \in [P(1/\eta - 1), 1-P)$ or $F_H \in [1-P, 1]$ where $F_L = \frac{\tau(1+\phi)}{\eta+\phi+\eta\phi} + \frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)}P$ and $F_H = \min\{1, P + \tau \cdot (1+\phi)/\phi\}$.

Pure-strategy equilibria

There cannot exist a pure-strategy equilibrium with $P > 0$ in which all firms choose F_H because the aggregate demand for the risky asset would be zero but the supply is positive $[(P+\phi F)/(1+\phi)]$. Therefore, in a pure-strategy equilibrium, firms choose

$$F = \frac{\tau(1+\phi)}{\eta(1+\phi)+\phi} + \frac{(1-\eta)(1+\phi)+\phi}{\eta(1+\phi)+\phi}P.$$

The demand for the liquidated asset is $1-P-F$ and the supply of the liquidated asset is

$$\Lambda = \frac{P+\phi F}{1+\phi} = \frac{P \cdot [(1-\phi)\eta + 2\phi] + \phi\tau}{\eta(1+\phi)+\phi}.$$

Equating supply and demand gives the equilibrium price

$$P^* = \frac{\eta + \phi + \eta\phi - \tau(1+2\phi)}{1+\eta+5\phi-\eta\phi}.$$

Substituting into the expression for F yields

$$F^* = \frac{1 + 2\phi + \tau + (\tau - \eta)(1 + \phi)}{1 + \eta + 5\phi - \eta\phi}.$$

Mixed-strategy equilibria

Again, there is the possibility of a mixed-strategy equilibrium. The proof here follows closely the proof in Case 2 when $\eta \geq 1/2$. Firms choosing $F_L = \frac{\tau(1+\phi)}{\eta+\phi+\eta\phi} + \frac{(1-\eta)(1+\phi)+\phi}{(\eta+\phi+\eta\phi)}P$ have ex ante value V_L as described in equation (A-3), firms choosing $F_H = P + \tau \cdot (1 + \phi)/\phi$ have ex ante value V_H as described in equation (A-4), and firms choosing $F_H = 1$ have ex ante value V_1 as described in equation (A-5).

It is straightforward to show that Lemma 2 applies in the case $\eta < 1/2$ when $\tau_0 \leq \tau < \phi/(1 + \phi)$. Therefore, in a mixed-strategy equilibrium the high type always chooses $F_H = 1$. Therefore, we must only compare V_L to V_1 to find a mixed-strategy equilibrium. We also have $V_L - V_1$ as described in equation (A-6) and $H(\tau)$ as described in equation (A-7).

We know there is a pure-strategy equilibrium at the upper boundary of case 1, $\tau = \tau_0$. Therefore, at the transition to this region we have $H(\tau_0) \geq 0$. Furthermore, it can be shown numerically that for all $\phi \in [0, 1]$ and all $\eta \in [0, 1/2]$ that

$$H'(\tau_0) < 0,$$

and

$$H''(\tau) = \frac{2 + 14\phi + 20\phi^2 + 9\phi^3 + 3\eta(1-\phi)(1+\phi)^2}{(1+\phi)(1+\eta+5\phi-\eta\phi)^2} > 0.$$

We see then that $H(\tau)$ is an upward facing parabola in τ. At the lower boundary of Case 2, where $\tau = \tau_0$, $H(\tau_0)$ is positive but decreasing.

Solving $H(\tau) = 0$ yields τ_3, τ_4 as before and the remainder of the proof is identical to Case 2 when $\eta \geq 1/2$.

Case 3: $\eta - P + (1-2P)\phi/(1+\phi) < \tau \leq 1$

If $\eta - P + (1-2P)\phi/(1+\phi) < \tau \leq 1$ then all firms choose $F > 1-P$ which implies that the aggregate demand for the liquidated asset is zero. But, since supply is positive when the price is positive, this case is incompatible with an equilibrium $P^* > 0$.

Equilibrium with $P^* = 0$

Finally, as before, if $P = 0$ the four regions for the marginal cost of debt collapse into one:

- For $F \in [0,1]$ the marginal cost of debt is $\eta \cdot F + F \cdot \phi/(1+\phi)$

Consequently, if $P = 0$ there can only exist a pure-strategy equilibrium in which all firms choose $F^* = \min\left\{1, \frac{\tau(1+\phi)}{\eta(1+\phi)+\phi}\right\}$. Following the argument in the proof when $\eta \geq 1/2$, the price $P^* = 0$ can be supported in equilibrium if $\tau \geq \frac{\eta(1+\phi)+\phi}{1+2\phi}$.

Uniqueness

We've established above that the equilibrium debt choices, prices, and high-type proportion are unique functions of the exogenous parameters in each of the equilibrium types (pure strategy interior/extreme and mixed-strategy). Also, the necessary conditions for each of the equlibria form non-overlapping regions. It cannot be that a given exogenous parameter value supports multiple types of symmetric equilibria. Therefore, the equilibium is unique for any given parameter values.