Internet Appendix for "A Shadow Rate or a Quadratic Policy Rule? The Best Way to Enforce the Zero Lower Bound in the United States"

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I. Model Solutions

A. Affine Term Structure Model

In an affine term structure model (ATSM), the short rate rate r_t is an affine function in the factors $\mathbf{x}_t$ with dimension $n_x \times 1$, i.e.

$$r_t = \alpha + \boldsymbol{\beta}'\mathbf{x}_t, \tag{A–1}$$

where α is a scalar and $\boldsymbol{\beta}$ is an $n_x \times 1$ vector. No arbitrage implies that there exists an equivalent risk-neutral probability measure ($\mathbb{Q}$) such that the price of an n-period zero-coupon bond at time t ($P_{t,n}$) is equal to the expected price of a $n-1$-period bond at time $t+1$ discounted by the risk-free short rate, i.e.

$$P_{t,n} = \mathrm{E}_t^{\mathbb{Q}}\left\{\exp\left[-r_t\right] P_{n-1,t+1}\right\}. \tag{A–2}$$

The factors follow a first-order vector autoregression (VAR) process under $\mathbb{Q}$

$$\mathbf{x}_{t+1} = \left(\mathbf{I} - \boldsymbol{\Phi}\right)\mathbf{x}_t + \boldsymbol{\Phi}\boldsymbol{\mu} + \boldsymbol{\Sigma}\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}, \tag{A–3}$$

where $\boldsymbol{\Phi}$ is an $n_x \times n_x$ matrix, $\boldsymbol{\mu}$ is an $n_x \times 1$ vector, $\boldsymbol{\Sigma}$ is an $n_x \times n_x$ matrix, and $\boldsymbol{\varepsilon}_{t+1}^{\mathbb{Q}}$ $\sim$NID$(\mathbf{0}, \mathbf{I})$ is a $n_x \times 1$ vector. Here, $\mathrm{E}_t^{\mathbb{Q}}$ denotes the conditional expectation under $\mathbb{Q}$. Within this setting bond prices are affine in the factors, i.e.

$$P_{t,n} = \exp\left\{A_n + \mathbf{B}_n'\mathbf{x}_t\right\}. \tag{A–4}$$

To solve for the factor loadings, i.e. A_n and $\mathbf{B}_n$ for $n = 1, 2, ...N$, we combine equations (A–1)-(A–4) to obtain the recursive equations

$$\text{(A–5)} \qquad A_n = -\alpha + A_{n-1} + B'_{n-1}\Phi\boldsymbol{\mu} + \frac{1}{2}\mathbf{B}'_{n-1}\boldsymbol{\Sigma}\boldsymbol{\Sigma}'\mathbf{B}_{n-1}$$

$$\text{(A–6)} \qquad \mathbf{B}'_n = -\boldsymbol{\beta}' + \mathbf{B}'_{n-1}(\mathbf{I} - \boldsymbol{\Phi})$$

The time-t price of a zero-period bond must be 1, which implies that these recursions can be started with $A_0 = 0$ and $\mathbf{B}_0 = \mathbf{0}$.

B. Quadratic Term Structure Model

In a quadratic term structure model (QTSM), the short rate r_t is a quadratic function of $\mathbf{x}_t$ with dimensions $n_x \times 1$, i.e.

$$\text{(A–7)} \qquad r_t = \delta_0 + \boldsymbol{\delta}'_x\mathbf{x}_t + \mathbf{x}'_t\boldsymbol{\Delta}_{xx}\mathbf{x}_t,$$

where δ_0 is a scalar, $\boldsymbol{\delta}_x$ is an $n_x \times 1$ vector, and $\boldsymbol{\Delta}_{xx}$ is an $n_x \times n_x$ matrix. The factors follow the same VAR process as in ATSM (A–3) under the risk-neutral measure. Within this setting bond prices are quadratic in the factors, i.e.

$$\text{(A–8)} \qquad P_{t,n} = \exp\left[A_n + \mathbf{B}'_n\mathbf{x}_t + \mathbf{x}'_t\mathbf{C}_n\mathbf{x}_t\right],$$

where A_n is a scalar, $\mathbf{B}_n$ is an $n_x \times 1$ vector, and C_n is an $n_x \times n_x$ matrix. To solve for the factor loadings, i.e. A_n, $\mathbf{B}_n$, and $\mathbf{C}_n$ for $n = 1, 2, ...N$, we combine equations (A–2)-(A–4), and (A–7) to obtain the recursive equations

$$
\begin{aligned}
\text{(A–9)} \qquad A_n &= -\delta_0 + A_{n-1} + \mathbf{B}'_{n-1}\mathbf{\Phi}\boldsymbol{\mu} + (\mathbf{\Phi}\boldsymbol{\mu})'\,\mathbf{C}_{n-1}\mathbf{\Phi}\boldsymbol{\mu} \\
&\quad + \frac{1}{2}\mathbf{B}'_{n-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1}_{n-1}\mathbf{\Sigma}'\mathbf{B}_{n-1} + 2\,(\mathbf{\Phi}\boldsymbol{\mu})'\,\mathbf{C}_{n-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1}_{n-1}\mathbf{\Sigma}'\mathbf{B}_{n-1} \\
&\quad + 2\,(\mathbf{\Phi}\boldsymbol{\mu})'\,\mathbf{C}_{n-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1}_{n-1}\mathbf{\Sigma}'\mathbf{C}_{n-1}\mathbf{\Phi}\boldsymbol{\mu} - \frac{1}{2}\log\left(|\mathbf{\Gamma}_{n-1}|\right) \\
\text{(A–10)} \qquad \mathbf{B}'_n &= -\boldsymbol{\delta}'_x + \mathbf{B}'_{n-1}\left(\mathbf{I}-\mathbf{\Phi}\right) + 2\,(\mathbf{\Phi}\boldsymbol{\mu})'\,\mathbf{C}_{n-1}\left(\mathbf{I}-\mathbf{\Phi}\right) \\
&\quad + 2\mathbf{B}'_{n-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1}_{n-1}\mathbf{\Sigma}'\mathbf{C}_{n-1}\left(\mathbf{I}-\mathbf{\Phi}\right) + 4\,(\mathbf{\Phi}\boldsymbol{\mu})'\,\mathbf{C}_{n-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1}_{n-1}\mathbf{\Sigma}'\mathbf{C}_{n-1}\left(\mathbf{I}-\mathbf{\Phi}\right) \\
\text{(A–11)} \qquad \mathbf{C}_n &= -\mathbf{\Delta}_{xx} + \left(\mathbf{I}-\mathbf{\Phi}\right)'\mathbf{C}_{n-1}\left(\mathbf{I}-\mathbf{\Phi}\right) + 2\left(\mathbf{I}-\mathbf{\Phi}\right)'\mathbf{C}_{n-1}\mathbf{\Sigma}\mathbf{\Gamma}^{-1}_{n-1}\mathbf{\Sigma}'\mathbf{C}_{n-1}\left(\mathbf{I}-\mathbf{\Phi}\right)
\end{aligned}
$$

where the $n_x \times n_x$ matrix $\mathbf{\Gamma}_{n-1}$ is defined as

$$\mathbf{\Gamma}_{n-1} = \left(\mathbf{I}{-}2\mathbf{\Sigma}'\mathbf{C}_{n-1}\mathbf{\Sigma}\right).$$

The time-t price of a zero-period bond must be 1, which implies that these recursions can be started with $A_0 = 0$, $\mathbf{B}_0 = \mathbf{0}$, and $\mathbf{C}_0 = \mathbf{0}$.

C. Shadow Rate Model

This section explains how to implement the bond pricing approximation by Priebsch (2013) in a discrete-time shadow rate model (SRM). The short rate is given by:

$$\text{(A–12)} \qquad r_t = \max\left\{0, s_t\right\},$$

where the shadow rate s_t is given by

$$\text{(A–13)} \qquad s_t = \alpha + \boldsymbol{\beta}'\mathbf{x}_t,$$

where α is a scalar and $\boldsymbol{\beta}$ is an $n_x \times 1$ vector. The risk-neutral dynamics of the factors are again given by (A–3).

Bond yields are based on a second-order approximation, i.e.

$$
\begin{aligned}
\text{(A–14)} \qquad y_{t,n} &= \frac{1}{n}\mathrm{E}_t^{\mathbb{Q}}\left[\sum_{i=0}^{n-1} r_{t+i}\right] - \frac{1}{2n}\mathrm{var}_t^{\mathbb{Q}}\left[\sum_{i=0}^{n-1} r_{t+i}\right] \\
&= \frac{1}{n}\mathrm{E}_t^{\mathbb{Q}}\left[\sum_{i=0}^{n-1} \max\left\{0, s_{t+i}\right\}\right] \\
&\quad -\frac{1}{2n}\left\{\mathrm{E}_t^{\mathbb{Q}}\left[\left(\sum_{i=0}^{n-1} \max\left\{0, s_{t+i}\right\}\right)^2\right] - \left(\mathrm{E}_t^{\mathbb{Q}}\left[\sum_{i=0}^{n-1} \max\left\{0, s_{t+i}\right\}\right]\right)^2\right\}.
\end{aligned}
$$

We can compute the conditional moments of the shadow rate $\mu_{t,t+i} = \mathrm{E}_t^{\mathbb{Q}}\left[s_{t+i}\right]$, $\sigma_{t,t+i}^2 = \mathrm{var}_t^{\mathbb{Q}}\left[s_{t+i}\right]$, and $\sigma_{t,t+i,t+j} = \mathrm{cov}_t^{\mathbb{Q}}\left[s_{t+i}, s_{t+j}\right]$ using (A–3) and (A–13). Using (A–12) and standard results for the moments of a truncated Normal distribution (see Priebsch (2013)), the conditional moments of the short rate required for (A–14) are therefore given by

$$
\mathrm{E}_t^{\mathbb{Q}}\left[\max\left\{0, s_{t+i}\right\}\right] = \mu_{t,t+i}\Phi\left(\frac{\mu_{t,t+i}}{\sigma_{t,t+i}}\right) + \sigma_{t,t+i}\phi\left(\frac{\mu_{t,t+i}}{\sigma_{t,t+i}}\right)
$$

and

$$
\begin{aligned}
\mathrm{E}_t^{\mathbb{Q}}\left[\max\left\{0, s_{t+i}\right\}\max\left\{0, s_{t+j}\right\}\right] &= \left(\mu_{t,t+i}\mu_{t,t+j} + \sigma_{t,t+i,t+j}\right)\Phi_2^d\left(-\zeta_{t,t+i}, -\zeta_{t,t+j}; \chi_{t,t+i,t+j}\right) \\
&\quad +\sigma_{t,t,+j}\mu_{t,t+i}\phi\left(\zeta_{t,t+j}\right)\Phi\left(\frac{\zeta_{t,t+i} - \chi_{t,t+i,t+j}\zeta_{t,t+j}}{\sqrt{1-\chi_{t,t+i,t+j}^2}}\right) \\
&\quad +\sigma_{t,t,+i}\mu_{t,t+j}\phi\left(\zeta_1\right)\Phi\left(\frac{\zeta_{t,t+j} - \chi_{t,t+i,t+j}\zeta_{t,t+i}}{\sqrt{1-\chi_{t,t+i,t+j}^2}}\right) \\
&\quad +\sigma_{t,t,+i}\sigma_{t,t,+j}\sqrt{\frac{1-\chi_{t,t+i,t+j}^2}{2\pi}} \\
&\quad \times\phi\left(\sqrt{\frac{\zeta_{t,t+i}^2 - 2\chi_{t,t+i,t+j}\zeta_{t,t+i}\zeta_{t,t+j} + \zeta_{t,t+j}^2}{1-\chi_{t,t+i,t+j}^2}}\right)
\end{aligned}
$$

where $\zeta_{t,t+i} = \frac{\mu_{t,t+i}}{\sigma_{t,t+i}}$ and $\chi_{t,t+i,t+j} = \frac{\sigma_{t,t+i,t+j}}{\sigma_{t,t+i}\sigma_{t,t+j}}$. Here, $\phi\left(.\right)$ and $\Phi\left(.\right)$ respectively denote the probability density and cumulative distribution functions of the standard Normal distribution, and Φ_2^d denotes the decumulative bivariate Normal distribution function.

II. Extending the Sample Period

To explore the robustness of the key results in the main text of the paper, in this section we extend the sample back to June 1961. Although the start of our sample beginning in 1990 is broadly representative of many recent empirical studies using U.S. data, the period from 1990–2016 is characterized by relatively low yields with stable volatilities. In contrast, yields in the 1970s and 1980s were relatively high and much more volatile. It therefore seems possible that the non-linearities in the SRM and QTSM would have a greater effect when starting the analysis in 1961. However, we show that the ability of the SRM and QTSM to match the conditional expectations and volatilities of yields are broadly similar when the models are estimated using this longer sample.

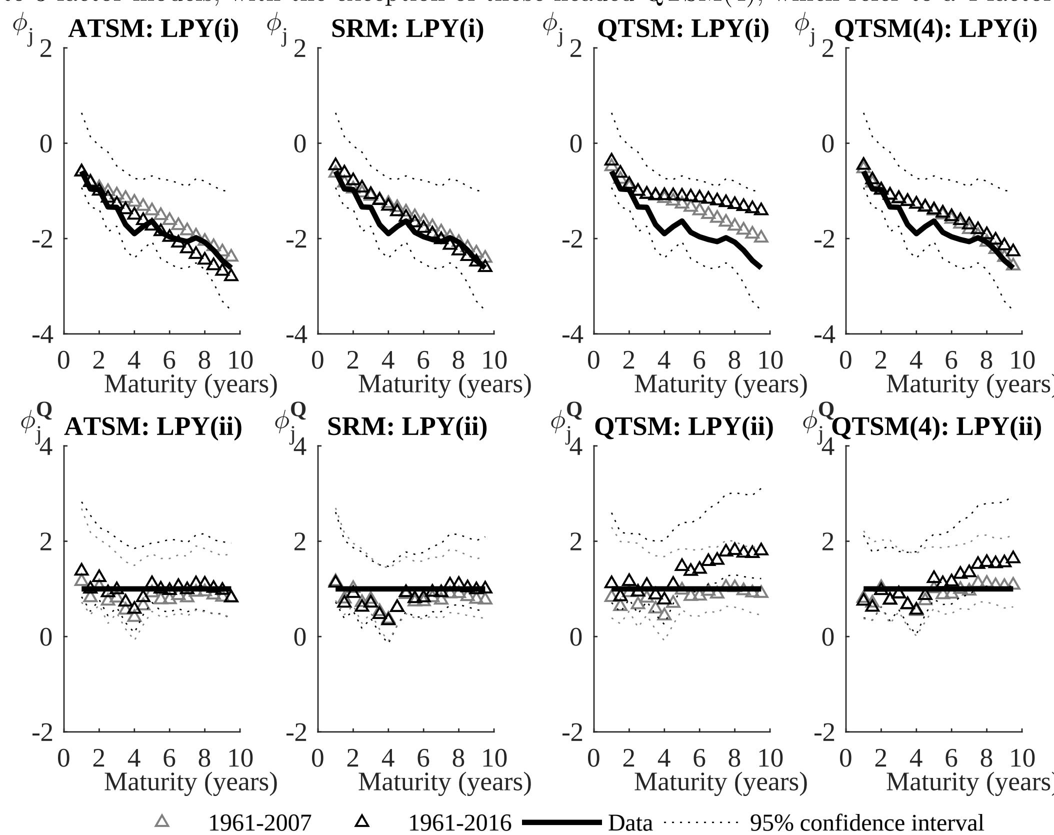
In the interests of brevity, we focus on the LPY tests to evaluate the models' ability to match conditional expectations. The procedure is essentially the same as described in the main text of the paper.[1] The top row of Figure A1 shows the ability of all 3-factor models and the 4-factor QTSM to satisfy the LPY(i) test, while the bottom row shows the equivalent results for the LPY(ii) test. For both LPY tests, the gray markers indicate model-implied loadings when the model is estimated using data from 1961–2007 (i.e. the pre-ZLB period), while the black markers show the results when the sample for estimating the model is extended to 2016.

For the LPY(i) test, the model-implied slope coefficients do not fall as much with maturity as they do in the data when the models are estimated over the 1961–2007 sample, but the differences are not statistically significant. While the deviations from the desired slope coefficients in the 3-factor QTSM are somewhat larger than in the other models, the differences are not statistically significant and the performance of the model can be improved with the addition of a fourth pricing factor, as was the case for the sample starting in 1990. Finally, the addition of the ZLB period has a much smaller effect than in the benchmark sample, which seems reasonable given that the ZLB period is now proportionally less important given the longer pre-ZLB sample.

[1] We estimate the Campbell-Shiller loadings in the data using a sample starting in Nov. 1971 to avoid problems caused by missing observations for some long-term yields prior to that date.

Figure A1: Campbell-Shiller Loadings: Sample from 1961–2016

This figure reports results from the LPY tests for the long sample. The top row of charts reports Campbell-Shiller loadings implied by models and the data. The loadings in the data are estimated using the pre-ZLB sample from Nov. 1971–Dec. 2007. The 95% confidence intervals for these estimates are computed based on a block bootstrap applied jointly to the regressand and the regressor in the Campbell-Shiller regressions in the data using a block length of 189 months and 5,000 repetitions. The model-implied loadings are the mean loadings from running 1,000 Campbell-Shiller regressions on simulated samples of 559 months, conditional on all bond yields being above 1% at all points in the simulated sample. The gray markers report results when the model parameters are estimated using data from June 1961–Dec. 2007, while the black markers report results when the model parameters are estimated on data from June 1961–Dec. 2016. The bottom row of charts reports risk-adjusted Campbell-Shiller loadings from Nov. 1971–Dec. 2007, where term premia are obtained from models estimated using data from June 1961–Dec. 2007 (gray markers) and from June 1961–Dec. 2016 (black markers). A well-specified model should return loadings equal to 1, which is highlighted using the heavy solid line. Conditional on the model estimates of term premia, the 95% confidence intervals for the risk-adjusted Campbell-Shiller loadings from the model are computed using a block bootstrap applied jointly to the regressand and the regressor in the risk-adjusted Campbell-Shiller regressions with a block length of 189 months and 5,000 repetitions. All charts refer to 3-factor models, with the exception of those headed QTSM(4), which refer to a 4-factor QTSM.

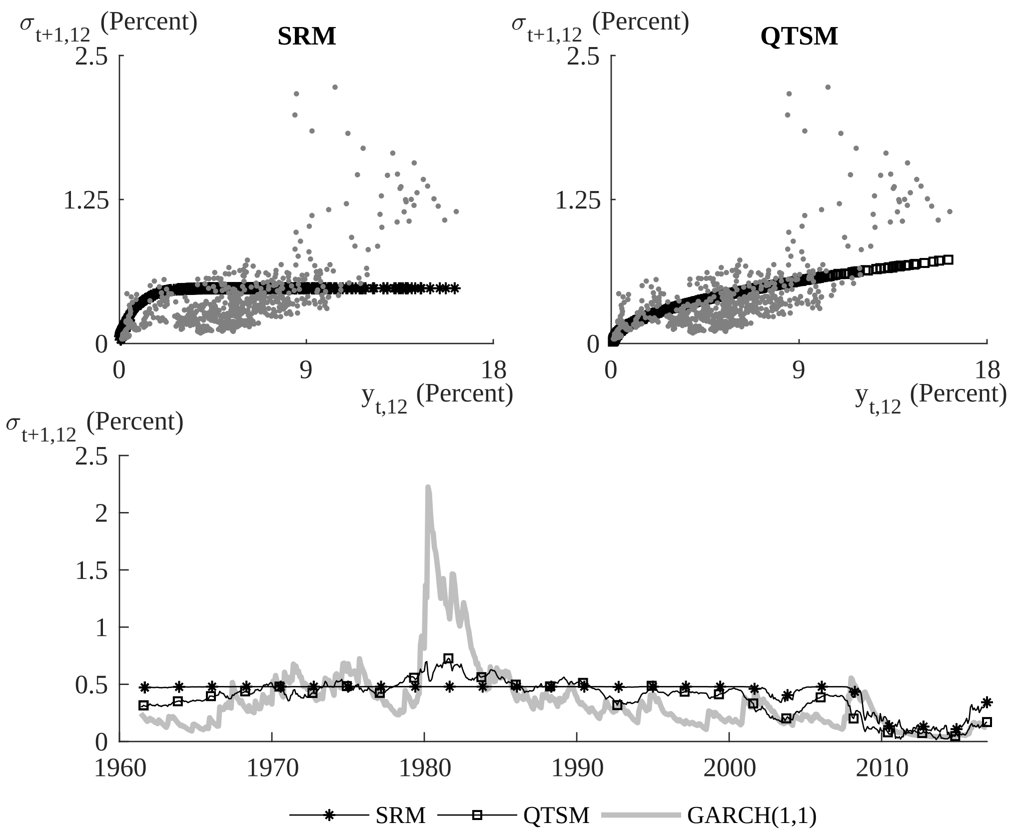


For the LPY(ii) test, the models also broadly match the desired pattern of the risk-adjusted Campbell-Shiller loadings when estimated using a sample ending in 2007, with model-implied loadings that are generally not significantly different from 1. As for the LPY(i) tests, including the ZLB period in the sample used to estimate the models has a smaller effect than in our benchmark case.

Turning to conditional volatilities, Figure A2 shows that the 3-factor models estimated over the longer sample display much the same weaknesses as in our benchmark sample period, i.e. the link between the level of yields and conditional volatilities is simply too tight to match the observed volatilities in the data when yields are away from the ZLB. Unreported results show substantively the same result for the 4-factor QTSM.

Figure A2: Conditional Volatility of Bond Yields: Sample from 1961 to 2016

This figure plots yields $y_{t+1,12}$ on the horizontal axes against 1-month-ahead conditional volatilities $\sigma_t\left(y_{t+1,12}\right)$ on the vertical axes (both in the data and from 3-factor models estimated using data from June 1961–Dec. 2016). Yields and conditional volatilities are all expressed as annualized percentages. The model-implied conditional volatilities are computed using a first-order linearization of the relationship between bond yields and the pricing factors, evaluated at the estimated factor values.



III. Estimated Model Parameters

Table A1 reports parameter estimates and estimated asymptotic standard errors for the 3-factor models estimated using data from Jan. 1990–Dec. 2016.

Table A1: Three-Factor Models: Sample from 1990 to 2016

Asymptotic standard errors (SEs) are computed using the methods provided by Andreasen and Christensen (2015). SEs for $\hat{\boldsymbol{\theta}}_{11}^{step3}$ use bandwidth parameters $w_D = 5$ and $w_T = 10$.

	ATSM		QTSM		SRM	
	Estimate	SE	Estimate	SE	Estimate	SE
α	0.0077	0.0123	-	-	0.0082	0.0010
ϕ_{11}	0.0048	0.0011	0.0035	0.0047	0.0037	0.0010
ϕ_{22}	0.0488	0.0473	0.0536	0.0211	0.0531	0.0051
ϕ_{33}	0.0502	0.0483	0.0591	0.0194	0.0571	0.0041
μ_1	-	-	0.0021	0.0094	-	-
μ_2	-	-	0.0388	0.0160	-	-
μ_3	-	-	0.0546	0.0150	-	-
$h_0\,(1,1)$	-4.28×10^{-5}	6.02×10^{-5}	-0.0020	0.0017	-5.08×10^{-5}	5.08×10^{-5}
$h_0\,(2,1)$	2.48×10^{-4}	0.0019	-0.0621	0.0521	6.15×10^{-4}	0.0015
$h_0\,(3,1)$	-3.74×10^{-4}	0.0019	0.0673	0.0520	-7.66×10^{-4}	0.0015
$h_x\,(1,1)$	0.9892	0.0127	0.9741	0.0214	0.9867	0.0106
$h_x\,(1,2)$	0.0166	0.0195	0.0110	0.0190	0.0193	0.0118
$h_x\,(1,3)$	0.0161	0.0205	0.0096	0.0196	0.0188	0.0123
$h_x\,(2,1)$	0.1727	0.3607	0.1367	0.6055	0.2403	0.2827
$h_x\,(2,2)$	1.5753	0.8182	1.6579	0.5828	1.5657	0.4028
$h_x\,(2,3)$	0.6615	0.8549	0.7867	0.6045	0.6557	0.4180
$h_x\,(3,1)$	-0.1930	0.3542	-0.1308	0.6030	-0.2626	0.2812
$h_x\,(3,2)$	-0.6569	0.8088	-0.7133	0.5828	-0.6556	0.4029
$h_x\,(3,3)$	0.2538	0.8448	0.1533	0.6047	0.2497	0.4179
σ_{11}	3.94×10^{-4}	3.39×10^{-5}	0.0035	2.25×10^{-4}	3.79×10^{-4}	2.78×10^{-5}
σ_{21}	-0.0051	0.0013	-0.0351	0.0088	-0.0042	9.36×10^{-4}
σ_{22}	0.0108	6.54×10^{-4}	0.0935	0.0055	0.0106	7.22×10^{-4}
σ_{31}	0.0047	0.0013	0.0323	0.0087	0.0039	9.28×10^{-4}
σ_{32}	-0.0108	6.52×10^{-4}	-0.0941	0.0055	-0.0108	7.67×10^{-4}
σ_{33}	1.46×10^{-4}	1.25×10^{-5}	0.0022	2.56×10^{-4}	2.84×10^{-4}	2.67×10^{-5}

References

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