New Entropy Restrictions and the Quest for Better Specified Asset Pricing Models

Internet Appendix: Not for Publication

Abstract

Section A of the Internet Appendix studies the implications of enforcing the Hansen-Jagannathan lower bound, whereas Section B sheds light on the sharpness of the lower bounds on $L[m]$. Section C presents three asset pricing models (i) difference habit, (ii) recursive utility with stochastic variance, and (iii) recursive utility with constant jump intensity. Our empirical assessment shows that each model is rejected based on the lower bounds on $L[m]$ and $L[m^2]$.

I. Internet Appendix

A. Implications of enforcing the Hansen-Jagannathan lower bound

The problem in Hansen and Jagannathan (1991, page 235) is to find the SDF with minimum variance

$$\text{(IA-1)} \qquad \min_{m \in \mathbb{S}} \left\{ \mathbb{E}\left[m^2\right] - \left(\mathbb{E}\left[m\right]\right)^2 \right\}.$$

Consider a portfolio p depicted by the return

$$\text{(IA-2)} \qquad R^p_{t,t+1} = \mathbf{a}'\mathbf{R}_{t,t+1} \qquad \text{with } \mathbf{a} = \frac{\mathbf{y}}{\mathbf{1}'\mathbf{y}} \qquad \text{and } \mathbf{y} = \Sigma^{-1}\left(\mathbf{1} - \mathbb{E}\left[q_t\right]\mathbb{E}\left[\mathbf{R}_{t,t+1}\right]\right).$$

Analogous to how we construct $m^{\bullet}$ and m^{G} in Problems 1 and 2, we use portfolio (IA-2) to construct $m^{\text{HJ}}_{t,t+1}$ consistent with the Hansen and Jagannathan (1991) lower bound. We conjecture and then verify the solution.

$$\text{(IA-3)} \qquad m^{\text{HJ}}_{t,t+1} = \beta_0 + \beta_{\text{HJ}} R^p_{t,t+1},$$

$$\text{(IA-4)} \qquad = \beta_0 + \beta_{\text{HJ}}\left(\mathbf{a}'\mathbf{R}_{t,t+1}\right),$$

where β_0 and β_{HJ} are constant parameters.

The first restriction is that the variance of $m^{\text{HJ}}_{t,t+1}$ equate to the Hansen and Jagannathan

(1991, equation (12)) minimum variance, given by $\sigma_{\mathrm{HJ}}^2 = \mathbf{y}^{'}\Sigma\mathbf{y}$,

$$\text{(IA-5)} \qquad \beta_{\mathrm{HJ}}^2 \overbrace{\operatorname{var}\left(\mathbf{a}^{'}\mathbf{R}_{t,t+1}\right)}^{\mathbf{a}^{'}\Sigma\mathbf{a}} \;=\; \overbrace{\sigma_{\mathrm{HJ}}^2}^{\mathbf{y}^{'}\Sigma\mathbf{y}}, \quad \text{implying that}$$

$$\text{(IA-6)} \qquad \beta_{\mathrm{HJ}}^2 \mathbf{y}^{'}\Sigma\mathbf{y} \;=\; \sigma_{\mathrm{HJ}}^2 (\mathbf{1}'\mathbf{y})^2.$$

Therefore, we obtain

$$\text{(IA-7)} \qquad \beta_{\mathrm{HJ}}^2 = \left(\mathbf{1}'\mathbf{y}\right)^2 \;\text{ and hence }\; \beta_{\mathrm{HJ}} = \mathbf{1}'\mathbf{y}.$$

Next, we enforce the restriction on the mean of the SDF:

$$\text{(IA-8)} \qquad \underbrace{\beta_0 + \left(\mathbf{1}'\mathbf{y}\right)\mathbb{E}\left[\mathbf{a}^{'}\mathbf{R}_{t,t+1}\right]}_{\mathbb{E}\left[m_{t,t+1}^{\mathrm{HJ}}\right]} = \mathbb{E}\left[q_t\right],$$

which yields that

$$\text{(IA-9)} \qquad \beta_0 = \mathbb{E}\left[q_t\right] - \left(\mathbf{1}'\mathbf{y}\right)\mathbb{E}[\mathbf{a}^{'}\mathbf{R}_{t,t+1}].$$

The end result is the expression for the minimum variance SDF of the type

$$\text{(IA-10)} \qquad m_{t,t+1}^{\mathrm{HJ}} \;=\; \mathbb{E}\left[q_t\right] + \left(\mathbf{1}'\mathbf{y}\right)\left(\mathbf{a}^{'}\mathbf{R}_{t,t+1} - \mathbb{E}[\mathbf{a}^{'}\mathbf{R}_{t,t+1}]\right),$$

$$\text{(IA-11)} \qquad \;=\; \mathbb{E}\left[q_t\right] + \mathbf{y}^{'}\left(\mathbf{R}_{t,t+1} - \mathbb{E}[\mathbf{R}_{t,t+1}]\right).$$

The restrictions (IA-5) and (IA-8) are used to construct m^{HJ}, and are in the flavor of how we used restrictions (C-3) and (C-10) to construct $m^{\bullet}$ (analogously, we use equations (F-2) and (F-4) to construct m^{G}).

By construction $m^{\mathrm{HJ}}_{t,t+1}$ prices correctly the generic portfolio $\mathbf{a}'\mathbf{R}_{t,t+1}$, as verified below:

$$\begin{aligned}
\mathbb{E}[m^{\mathrm{HJ}}_{t,t+1}(\mathbf{a}'\mathbf{R}_{t,t+1})] &= \frac{\mathbf{1}}{(\mathbf{1}'\mathbf{y})}\mathbb{E}[m^{\mathrm{HJ}}_{t,t+1}(\mathbf{y}'\mathbf{R}_{t,t+1})], \\
&= \frac{\mathbf{1}}{(\mathbf{1}'\mathbf{y})}\left(\mathbb{E}\left[\left(\mathbb{E}[q_t] + \left(\mathbf{y}'\mathbf{R}_{t,t+1} - \mathbb{E}\left[\mathbf{y}'\mathbf{R}_{t,t+1}\right]\right)\right)\left(\mathbf{y}'\mathbf{R}_{t,t+1}\right)\right]\right), \\
&= \frac{\mathbf{1}}{(\mathbf{1}'\mathbf{y})}\left\{\mathbb{E}[q_t]\left(\mathbf{y}'\mathbb{E}[\mathbf{R}_{t,t+1}]\right) + \mathrm{var}[\mathbf{y}'\mathbf{R}_{t,t+1}]\right\}. \qquad \text{(IA-12)}
\end{aligned}$$

Next, we note that

$$\mathrm{var}[\mathbf{y}'\mathbf{R}_{t,t+1}] = \mathbf{y}'\Sigma\mathbf{y} = (\mathbf{1} - \mathbb{E}[q_t]\,\mathbb{E}[\mathbf{R}_{t,t+1}])'\,\Sigma^{-1}(\mathbf{1} - \mathbb{E}[q_t]\,\mathbb{E}[\mathbf{R}_{t,t+1}]) \qquad \text{(IA-13)}$$

and further that

$$\begin{aligned}
\mathbf{y}'(\mathbb{E}[q_t])\,\mathbb{E}[\mathbf{R}_{t,t+1}] &= (\mathbf{1} - \mathbb{E}[q_t]\,\mathbb{E}[\mathbf{R}_{t,t+1}])'\,\Sigma^{-1}\mathbb{E}[\mathbf{R}_{t,t+1}], \qquad \text{(IA-14)} \\
&= (\mathbf{1} - \mathbb{E}[q_t]\,\mathbb{E}[\mathbf{R}_{t,t+1}])'\,\Sigma^{-1}(\mathbb{E}[q_t]\,\mathbb{E}[\mathbf{R}_{t,t+1}] - \mathbf{1} + \mathbf{1}), \\
&= -\mathbf{y}'\Sigma\mathbf{y} + (\mathbf{1} - \mathbb{E}[q_t]\,\mathbb{E}[\mathbf{R}_{t,t+1}])'\,\Sigma^{-1}\mathbf{1}, \\
&= -\mathbf{y}'\Sigma\mathbf{y} + \mathbf{y}'\mathbf{1}.
\end{aligned}$$

Equations (IA-13) and (IA-14) together imply that

$$\mathbb{E}[m^{\mathrm{HJ}}_{t,t+1}(\mathbf{a}'\mathbf{R}_{t,t+1})] = \frac{\mathbf{1}}{(\mathbf{1}'\mathbf{y})}\left\{-\mathbf{y}'\Sigma\mathbf{y} + \mathbf{y}'\mathbf{1} + \mathbf{y}'\Sigma\mathbf{y}\right\} = \frac{(\mathbf{1}'\mathbf{y})}{(\mathbf{1}'\mathbf{y})} = 1. \qquad \text{(IA-15)}$$

While the minimum variance SDF prices correctly the return of the portfolio $\mathbf{a}'\mathbf{R}_{t,t+1}$, does $m^{\text{HJ}}_{t,t+1}$ price correctly the set of return $\mathbf{R}_{t,t+1}$? To price the set of return $\mathbf{R}_{t,t+1}$, we must have (IA-16)

$$\mathbb{E}\left[m^{\text{HJ}}_{t,t+1}\mathbf{R}_{t,t+1}\right]=1 \text{ or } \mathbb{E}\left[q_t\right]\mathbb{E}\left[\mathbf{R}_{t,t+1}\right]+\text{cov}\left(\left(\mathbf{1}-\mathbb{E}\left[q_t\right]\mathbb{E}\left[\mathbf{R}_{t,t+1}\right]\right)'\Sigma^{-1}\mathbf{R}_{t,t+1},\mathbf{R}_{t,t+1}\right)=1.$$

Theoretically, this equality holds only when the dimension of $\mathbf{R}_{t,t+1}$ is one.

In summary, our theoretical results indicate a one-way implication: when the pricing restrictions in set (11) are used to construct the Hansen and Jagannathan bound, the variance of the minimum variance SDF is identical to the bound. The converse need not hold; that is, the obtained minimum variance SDF does not necessary price the $N+1$ assets employed to construct the minimum variance SDF. ■

B. Sharpness of our entropy bound on $L[m]$

How sharp is our bound on $L[m]$ compared to the bound constructed from a generic portfolio return in Backus, Chernov, and Zin (2014, Column 2 of Table I).

Table Internet Appendix-I reports our lower bounds on $L[m]$ and the associated bootstrap p-values. We consider several N (the dimensionality of $\mathbf{R}_{t,t+1}$) and draw two conclusions. First, our bounds on $L[m]$ are quantitatively sharper, implying greater hurdles on pricing models (e.g., compare bounds in Panel V versus those in Panels I through IV). Second, the bounds obtained with a portfolio are far less stringent than the corresponding bounds that rely on the SDFs correctly pricing each of the assets composing the portfolio. This can be seen by comparing the bound displayed in row (c) versus (i) and between row (d) versus (j). ■

C. *Example asset pricing models*

Our goal is to learn about the properties of $m_{t,t+1}$, and their consistency with bound restrictions. Additionally, we compare $L[m^2]$ to $4L[m]$. We focus on three models:

(i) Difference habit,

(ii) Recursive utility with stochastic variance, and

(iii) Recursive utility with constant jump intensity.

Some of the model solutions require loglinearization, whose effects are explored and elaborated in the study of Pohl, Schmedders, and Wilms (2015).

1. *Difference habit model*

The shocks in the difference habit model model are normally distributed, and the SDF is (Campbell and Cochrane (1999))

$$m_{t,t+1} = \beta g_{t+1}^{\rho-1} \left(\frac{s_{t+1}}{s_t}\right)^{\rho-1}, \tag{IA-17}$$

where g_{t+1} is consumption growth, β is the time discount parameter, and $1-\rho$ is the coefficient of relative risk aversion. Define $s_t \equiv 1-\exp(z_t)$ and $z_t \equiv \log(h_t) - \log(c_t)$, where s_t is the surplus ratio corresponding to z_t, and the habit h_{t+1} is known at t. The laws of motion for h_t and g_t are

$$\log(h_{t+1}) = \log(h) + \eta\,[B]\,\log(c_t) \quad \text{and} \quad \log(g_{t+1}) = \log(g) + \gamma[B]\,\upsilon^{\frac{1}{2}}\omega_{gt+1}, \tag{IA-18}$$

where B is the lag operator, such that $B\{s_{t+1}\} = s_t$, with backshift operators $\gamma[B] = \sum_{j=0}^{\infty} \gamma_j B^j$ and $\eta[B] = \sum_{j=0}^{\infty} \eta_j B^j$. Moreover, υ denotes the constant variance of $\log(g_t)$, and ω_{gt+1} is i.i.d. standard normal variable.

Loglinear approximation of $\log(s_t)$, in conjunction with equation (IA-18), leads to the following dynamics:

$$\text{(IA-19)} \qquad \log(s_{t+1}) - \log(s_t) = \left(\frac{s-1}{s}\right)(\eta[B]B - 1)\log(g_{t+1}).$$

Completing the model description, we define the state variable $x_t = (\gamma[B] - \gamma_0)\,\upsilon^{\frac{1}{2}}\,\omega_{gt+1}$, which governs the following dynamics of the log consumption growth:

$$\text{(IA-20)} \qquad x_t = \gamma_1 \upsilon^{\frac{1}{2}} \omega_{gt} + \varphi_g x_{t-1} \qquad \text{with} \qquad \varphi_g = \frac{\gamma_2}{\gamma_1}.$$

Models that accommodate habit have shown promise in matching salient attributes of the asset market data, including the equity premium, procyclicality of stock prices, counter-cyclicality of stock volatility, and return predictability at long horizons (e.g., see, among others, Bekaert and Engstrom (2017), Chapman (1998), Chan and Kogan (2002), and Santos and Veronesi (2010)).

2. Recursive utility models

The recursive utility models are adopted from Backus, Chernov, and Zin (2014):

$$\text{(IA-21)} \qquad U_t = \left[(1-\beta)\,c_t^{\rho} \; + \; \beta\,(\mu_t\,[U_{t+1}])^{\rho}\right]^{\frac{1}{\rho}},$$

with certainty equivalent function $\mu_t\left[U_{t+1}\right] = \left(\mathbb{E}_t\left[U_{t+1}^{\alpha}\right]\right)^{\frac{1}{\alpha}}$. Moreover, β is the time preference parameter, $\frac{1}{1-\rho}$ is the intertemporal elasticity of substitution, and $1-\alpha$ is the coefficient of relative risk aversion.

The shocks ω_{gt}, z_{gt}, and ω_{ht} are standard normal random variables, independent of each other and across time. Additionally, the jump component z_{gt} is a Poisson mixture of normals: conditional on the number of jumps j, z_{gt} is normal, with mean $j\theta$ and variance $j\delta^2$. The probability of $j \geq 0$ jumps at date t is $e^{h_{t-1}} h_{t-1}^{j}/j!$, and the jump intensity, h_{t-1}, is the mean of j.

With backshift operators characterized by $\nu\left[B\right] = \sum_{j=0}^{\infty} \nu_j B^j$ and $\psi\left[B\right] = \sum_{j=0}^{\infty} \psi_j B^j$, the state-variables in this model obey the following dynamics:

$$\text{(IA-22)} \qquad \log(g_t) = \log(g) + \gamma[B]\, \upsilon_{t-1}^{1/2} \omega_{gt} + \psi\left[B\right] z_{gt} - \psi\left[1\right] h\,\theta, \qquad h_t = h + \eta\left[B\right] \omega_{ht},$$

$$\text{(IA-23)} \qquad \upsilon_t = \upsilon \;+\; \nu\left[B\right] \omega_{\upsilon t}, \quad z_{gt} \,|\, j \sim \mathcal{N}\left(j\theta, j\delta^2\right), \quad P\left[j\right] = \exp\left(-h_{t-1}\right) \frac{\left(h_{t-1}\right)^j}{j!}.$$

A. Recursive utility model with stochastic variance. Set $h = 0$, $\eta\left[B\right] = 0$, $\psi\left[B\right] = 0$ in equations (IA-22) and (IA-23). For tractability, we consider the evolution of the transformed variable:

$$\text{(IA-24)} \qquad x_t \;=\; \varphi_g\, x_{t-1} \;+\; \gamma_1\, \upsilon_{t-1}^{1/2}\, \omega_{gt}.$$

B. Recursive utility model with constant jump intensity: In equations (IA-22) and (IA-23), set $\nu\left[B\right] = 0$.

Models that incorporate recursive preferences in conjunction with stochastic variance or jumps in the consumption growth dynamics have proved successful in explaining asset pricing quantities. We refer the reader to, among others, Epstein and Zin (1991), Bansal and Yaron (2004), Campbell and Vuolteenaho (2004), Hansen, Heaton, and Li (2008), Wachter (2013), and Zhou and Zhu (2009).

3. Empirical evidence and connection to our findings

How do the models under consideration fare when viewed from the perspective of data-based lower bounds on the entropy of m, entropy of m^2, and the volatility of m?

Our implementation of the models with difference habit (hereby DH), recursive utility with stochastic variance (hereby RU-SV), and recursive utility with constant jump intensity (hereby RU-CJI) follows the calibration procedure in Backus, Chernov, and Zin (2014, respectively, Model (4) in Table 2, Model (1) in Table 3, and Model (4) in Table 4). The corresponding model parameterizations are displayed in our Table Internet Appendix-III, which indicates that each model reasonably calibrates to consumption growth data.

Aided by our analytical representations, we generate the paths for $m_{t,t+1}$. The paths are based on the model parameters in Table Internet Appendix-III and shocks driving the fundamentals (e.g., $\omega_{\upsilon t}$ and ω_{gt} for the RU-SV). Then we obtain the sample averages of the series $\{m^2_{t,t+1}, m_{t,t+1} : t = 1, \ldots T\}$, and accordingly compute the entropies $L[m^2_{t,t+1}]$, $L[m_{t,t+1}]$, and the volatilities of $m_{t,t+1}$.

Next, we draw 50,000 paths for the shocks driving a model and, hence, obtain 50,000 paths for $m_{t,t+1}$. Panels A, B, and C of Table Internet Appendix-II report the entropies and volatilities across the models, obtained by averaging the entropies over the replications. The p-values, shown in square brackets, represent the proportion of replications for which the model-based entropy and volatility measures exceeds the corresponding lower bound obtained from the returns data in 50,000 replications of a simulation over 966 months.

How successful are the three models in generating $L[m]$ that is consistent with the data? Panel A of Table Internet Appendix-II reveals an $L[m]$ of 0.0196, 0.0217, and 0.0190, respectively, for the DH, RU-SV, and RU-CJI models. Based on our data-based performance measure, computed based on SET B, all the models are rejected (as seen by the bootstrap p-values).

Such an implication from our bound, calculated using the return properties of the risk-free bond, the equity market, and the 25 portfolios sorted by size and momentum, differ from a finding in Backus, Chernov, and Zin (2014). Specifically, the data-based lower bound in Backus, Chernov, and Zin (2014, Table 1) are generally of an order lower than the average conditional entropy $\mathbb{E}[L_t[m]]$ obtained from asset pricing models. In particular, all of the 11 $\mathbb{E}[L_t[m]]$ in Backus, Chernov, and Zin (2014, Tables II through IV) exceed the lower bound inferred from the returns on a generic portfolio taken to be the S&P 500 index.

How does one explain this discrepancy? We note that the magnitude of the lower bound on $L[m]$ in the calculations of Backus, Chernov, and Zin (2014, Table 1, row S&P 500) is 0.0040, whereas it is 0.0367, based on our lower bound and SET B. It bears emphasizing that the lower bound on $L[m]$ constructed from the returns of a (single) generic portfolio may provide an insufficient hurdle in evaluating the merits of an asset pricing model. The bounds on $L[m]$ agree in suggesting that the models are misspecified.

Elaborating further, we now argue that considering the entropy $L[m^2]$ in the model assessment can provide an important contrast to our findings based on the entropy $L[m]$. One noteworthy result is that the entropy $L[m^2]$ of the RU-CJI model is about 15-fold higher than the other two models that do not incorporate the random jump feature in the dynamics of the consumption growth. For example, the DH, RU-SV, and RU-CJI models generate $L[m^2]$ of 0.0785, 0.0869, and 1.4331, respectively (see the entries in Panel B of Table Internet Appendix-II). We further note that since the lower bound restriction implied from asset prices is 0.1956, the DH and RU-SV models are rejected at the 5% level. However, the RU-CJI model with constant jump intensity cannot be rejected at the 5% level, which is a point of departure based on the entropy $L[m]$.

Accordingly, one question emerges: Why does the RU-CJI fail to explain features of m, as reflected in asset prices when $L[m]$-based performance measure is used, while the model is successful in explaining features of m, as reflected in asset prices when the $L[m^2]$-based performance measure is used? To investigate a source of model performance, we note that the entropy measure $L[m^2]$ is substantially more sensitive to tail asymmetries and tail size of the m distribution as opposed to the entropy measure $L[m]$.

Taking such a trait of entropies into consideration, we report the moments of $m_{t,t+1}$ for each of the models in Panel D of Table Internet Appendix-II. The unexpected finding is that the RU-CJI model embeds excessive levels of skewness and kurtosis of $m_{t,t+1}$, while generating variance that is almost 90 times its DH and RU-SV model counterparts. Our contention is that the inordinate levels of the higher-order moments of $m_{t,t+1}$ give rise to the reported $L[m^2]$ of 1.4331 for the RU-CJI model.

How should one interpret a model, such as the RU-CJI, that calibrates well to the first moment, the second moment, and the autocorrelation of consumption growth but does not produce finite central moments for the distribution of $m_{t,t+1}$? This result arises because a convex transform of a random variable, which is here Poisson-distributed, increases the skewness to the right (see van Zwet (1966, page 10, Theorem 2.2.1)).

To see this analytically, we can use the density of the Poisson random variable to show that $\mathbb{E}_t[(m_{t,t+1})^k] = \mathbb{E}_t\left[e^{k\log\left(m_{t,t+1}\right)}\right] = \mathbb{E}_t\left[\mathbb{E}_t\left[e^{k\log\left(m_{t,t+1}\right)}|\, j\right]\right] = e^{G[k]}\mathbb{E}_t\left[e^{H[k]\, j}\right]$, for constants $G[k]$ and $H[k]$. Note that $e^{H[k]\, j}$ is a convex transformation of the Poisson variable J, and, for certain parameterizations, does not admit finite higher-moments of $m_{t,t+1}$. The inordinate amounts of skewness and kurtosis do not appear to be a reasonable depiction of valuation operators, which are likely to be characterized by exponential, rather than power, tails.

Finally, consider the volatility bound on m using the Hansen and Jagannathan (1991, equation (12)). As seen from Panel C of Table Internet Appendix-II, the DH and RU-SV models are rejected, but the RU-CJI model is not rejected for reasons discussed, namely, that the RU-CJI model embeds an unreasonable volatility, skewness, and kurtosis of m.

4. Details: SDF of the difference habit model

Using a loglinear approximation of $\log(s_t)$,

(IA-25) $$\log(m_{t,t+1}) = D_0 + (\rho - 1)\,\frac{1}{s}\,(1 - (1-s)\,\eta\,[B]\,B)\,\gamma\,[B]\,\upsilon^{\frac{1}{2}}\,\omega_{gt+1},$$

where $D_0 = \log(\beta) + (\rho - 1)\log(g) + (\rho - 1)\,\dfrac{(s-1)}{s}\left(\dfrac{\eta_0}{1-\varphi_h} - 1\right)\log(g).$

Given the approximation $\log(s_t) \approx 1 + \frac{(s-1)}{s} z_t$, the dynamics of the surplus consumption ratio are

$$\log(s_{t+1}) - \log(s_t) = \frac{(s-1)}{s} (\eta [B] B - 1) \log(g_{t+1}). \tag{IA-26}$$

Therefore, we may write the log SDF as

$$\begin{aligned} \log(m_{t,t+1}) &= \log(\beta) + (\rho - 1)\log(g) + (\rho - 1)\frac{(s-1)}{s}(\eta [B] B - 1)\log(g) \\ &\quad + (\rho - 1)\frac{1}{s}(1 - (1-s)\eta [B] B)\gamma [B] \upsilon^{\frac{1}{2}} \omega_{gt+1}. \end{aligned} \tag{IA-27}$$

We have the expression. ■

5. Details: SDF of the recursive utility models

Based on equations (IA-21) and (IA-23), we note that ω_{gt}, z_{gt}, and ω_{ht} are standard normal random variables, independent of each other and across time. The jump component z_{gt} is a Poisson mixture of normals: conditional on the number of jumps j, z_{gt} is normal with mean $j\theta$ and variance $j\delta^2$. The probability of $j \geq 0$ jumps at date $t+1$ is $e^{h_t} h_t^j / j!$ expands to

$$m_{t,t+1} = \exp\left(\chi_0 + a_g [B] \upsilon_t^{\frac{1}{2}} \omega_{gt+1} + a_z [B] z_{gt+1} + a_\upsilon [B] \omega_{\upsilon t+1} + a_h [B] \omega_{ht+1}\right), \tag{IA-28}$$

$$\begin{aligned} \chi_0 &= \log(\beta) + (\rho - 1)\log(g) \\ &\quad - (\alpha - \rho)(D\upsilon - Jh) - (\alpha - \rho)(\alpha/2)\left((Db_1 \nu [b_1])^2 + (Jb_1 \eta [b_1])^2\right), \end{aligned} \tag{IA-29}$$

where $a_g[B]$, $a_z[B]$, $a_\upsilon[B]$, and $a_h[B]$ are backshift operators defined as follows:

$$\text{(IA-30)} \quad a_g[B] = (\rho-1)\gamma[B] + (\alpha-\rho)\gamma[b_1], \qquad a_z[B] = (\rho-1)\psi[B] + (\alpha-\rho)\psi[b_1],$$

$$\text{(IA-31)} \quad a_\upsilon[B] = (\alpha-\rho)D(b_1\nu[b_1] - \nu[B]B), \quad a_h[B] = (\alpha-\rho)J(b_1\eta[b_1] - \eta[B]B),$$

$$\text{(IA-32)} \quad D = (\alpha/2)(\gamma[b_1])^2, \quad \text{and} \qquad J = \left(\frac{e^{\alpha\psi[b_1]\theta + (\alpha\psi[b_1]\delta)^2} - 1}{\alpha}\right).$$

The functions $\eta[b_1]$, $\nu[b_1]$, and $\gamma[b_1]$ are polynomial functions of b_1:

(IA-33)

$$\eta[b_1] = \sum_{j=0}^{\infty} b_1^j \eta_j, \quad \gamma[b_1] = \sum_{j=0}^{\infty} b_1^j \gamma_j, \quad \nu[b_1] = \sum_{j=0}^{\infty} b_1^j \nu_j, \quad \psi[b_1] = \sum_{j=0}^{\infty} b_1^j \psi_j,$$

with $\gamma_0 = 1$, where

$$\text{(IA-34)} \qquad \sum_{j=1}^{\infty} \gamma_j < \infty, \quad \sum_{j=1}^{\infty} \eta_j < \infty, \quad \sum_{j=1}^{\infty} \nu_j < \infty, \quad \sum_{j=1}^{\infty} \psi_j < \infty,$$

and

$$\text{(IA-35)} \qquad \nu[B] = \sum_{j=0}^{\infty} \nu_j B^j \qquad \text{and} \qquad \psi[B] = \sum_{j=0}^{\infty} \psi_j B^j.$$

A. Recursive utility with stochastic variance: The SDF is a special case of (IA-28) with $h = 0$, $\eta[B] = 0$, $J = 0$. The SDF takes the form

$$\text{(IA-36)} \qquad m_{t,t+1} = \exp\left(\begin{array}{c} H_0 + (\rho-1)\gamma[B]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} + (\alpha-\rho)\gamma[b_1]\upsilon_t^{\frac{1}{2}}\omega_{gt+1} \\ +(\alpha-\rho)Db_1\nu[b_1]\omega_{\upsilon t+1} - (\alpha-\rho)D\nu[B]B\omega_{\upsilon t+1} \end{array}\right),$$

with

$$\text{(IA-37)} \qquad H_0 = \log(\beta) + (\rho - 1)\log g - (\alpha - \rho)(D\upsilon) - (\alpha - \rho)(\alpha/2)\left((Db_1\nu[b_1])^2\right).$$

Now we define

$$\text{(IA-38)} \qquad x_t = (\gamma[B] - \gamma_0)\,\upsilon_t^{\frac{1}{2}}\omega_{gt+1}.$$

The state variable x_t dynamics is

$$\text{(IA-39)} \qquad x_t = \varphi_g x_{t-1} + \gamma_1 \upsilon_{t-1}^{\frac{1}{2}}\omega_{gt}, \quad \text{with} \quad \gamma_j = \varphi_g \gamma_{j-1} \text{ for } j \geq 2 \quad \text{and} \quad \varphi_g = \frac{\gamma_2}{\gamma_1}.$$

It can be shown that the dynamics of the state variable υ_t is

$$\text{(IA-40)} \qquad \upsilon_t - \upsilon = \varphi_\upsilon(\upsilon_{t-1} - \upsilon) + \nu_0\omega_{\upsilon t}, \quad \text{for } j \geq 2 \quad \text{and} \quad \varphi_\upsilon = \frac{\nu_1}{\nu_0}.$$

The SDF can be expressed as

$$\text{(IA-41)} \qquad m_{t,t+1} = \exp(H_1 + H_2 x_t + H_3 x_{t+1} + H_4 \upsilon_t + H_5 \upsilon_{t+1}),$$

where

$$H_1 = H_0 + (\alpha-\rho)D\upsilon + (\alpha-\rho)Db_1\nu[b_1]\frac{(\varphi_\upsilon - 1)}{\nu_0}\upsilon, \tag{IA-42}$$

$$H_2 = (\rho-1) - ((\alpha-\rho)\gamma[b_1] + (\rho-1))\frac{\varphi_g}{\gamma_1}, \tag{IA-43}$$

$$H_3 = \frac{(\rho-1)}{\gamma_1} + \frac{(\alpha-\rho)\gamma[b_1]}{\gamma_1}, \tag{IA-44}$$

$$H_4 = (\alpha-\rho)D\left(-b_1\nu[b_1]\frac{\varphi_\upsilon}{\nu_0} - 1\right), \quad \text{and} \tag{IA-45}$$

$$H_5 = (\alpha-\rho)Db_1\frac{\nu[b_1]}{\nu_0}. \tag{IA-46}$$

■

B. Recursive utility model with constant jump intensity: Consider the consumption growth dynamics with $\nu[B] = 0$ (in this case $\upsilon_t = \upsilon$). It can be shown that the SDF reduces to

$$m_{t,t+1} = \exp\left(\begin{array}{c} \chi_0 \\ +(\rho-1)x_t + ((\rho-1)\gamma_0 + (\alpha-\rho)\gamma[b_1])\upsilon^{\frac{1}{2}}\omega_{gt+1} \\ +(\rho-1)(\psi[B]-\psi_0)z_{gt+1} + ((\rho-1)\psi_0 + (\alpha-\rho)\psi[b_1])z_{gt+1} \\ +(\alpha-\rho)Jb_1\eta[b_1]\omega_{ht+1} - (\alpha-\rho)(h_t - h)J \end{array}\right). \tag{IA-47}$$

Now denote

$$\widetilde{x}_t = (\psi[B]-\psi_0)z_{gt+1}. \tag{IA-48}$$

The law of motion of $\widetilde{x}_t$ becomes

(IA-49) $$\widetilde{x}_t = \varphi_z \widetilde{x}_{t-1} + \psi_1 z_{gt}, \quad \text{with} \quad \varphi_z = \frac{\psi_2}{\psi_1} \quad \text{and} \quad \psi_{j+2} = \varphi_z \psi_{j+1} \ \text{ for } j \geq 1.$$

The SDF in equation (IA-47) reduces to

(IA-50)

$$m_{t,t+1} = \exp\left(G_0 + G_1 x_t + G_2 \widetilde{x}_{t-1} + G_3 z_{gt} + G_4 h_t + G_5 z_{gt+1} + G_6 \upsilon^{\frac{1}{2}} \omega_{gt+1} + G_7 \omega_{ht+1}\right),$$

with

$$\begin{aligned}
G_0 &= \chi_0 + (\alpha - \rho) hJ, & G_1 &= (\rho - 1), \\
G_2 &= (\rho - 1) \varphi_z, & G_3 &= (\rho - 1) \psi_1, \\
G_4 &= -(\alpha - \rho) J, & G_5 &= (\rho - 1) \psi_0 + (\alpha - \rho) \psi [b_1], \\
G_6 &= (\rho - 1) \gamma_0 + (\alpha - \rho) \gamma [b_1], & G_7 &= (\alpha - \rho) J b_1 \eta [b_1].
\end{aligned}$$

■

Table Internet Appendix-I

Sharpness of our entropy bounds on $m_{t,t+1}$, when SDFs correctly price each of the $N+1$ assets

Reported are the lower entropy bounds with the one-sided p-values in $\langle . \rangle$. Our lower entropy bound on $m_{t,t+1}$ is based on equation (20) and relies on the ability of the SDF to correctly price *each of the* $N+1$ assets (the risk-free bond and N risky assets). The Backus, Chernov, and Zin (2014, equation (5)) lower bound on the entropy of $m_{t,t+1}$ (denoted by BCZ) is based on the expression $\mathbb{E}[\log(R^m_{t,t+1})] - \log(R_{t+1,f})$, where $R^m_{t,t+1}$ is the return on a single risky asset or a benchmark portfolio (i.e., which we proxy by the value-weighted equity market return or equally weighted portfolio of 25 Fama-French size and book-to-market portfolios). $R_{t+1,f}$ is the gross return of the three-month Treasury bond. We employ different assets and N in the construction of the bounds. For example, in Panel I, the N risky assets are based on two data sets: SET A contains the value-weighted market returns, together with the 25 Fama-French size and book-to-market portfolios, while SET B contains the value-weighted market returns together with the 25 Fama-French size and momentum portfolios. The sample period is from July 1931 to December 2011 (966 observations). To compute these p-values, we first use the block bootstrap with a block size of 20 to generate 50,000 samples from the original data. Then we compute the lower bounds in each sample and tabulate the proportion of bootstrap samples for which the lower bound is less than zero.

	Lower bound on $m_{t,t+1}$	
	Bound	p-value
Panel I. SDF correctly prices each of the $N+1$ assets, and we set $N=26$		
(a) Set A: Market, 25 size & B/M	**0.023**	$\langle 0.000 \rangle$
(b) Set B: Market, 25 size & momentum	**0.037**	$\langle 0.003 \rangle$
Panel II. SDF correctly prices each of the $N+1$ assets, and we set $N=25$		
(c) Set C: 25 size & B/M	**0.022**	$\langle 0.000 \rangle$
(d) Set D: 25 size & momentum	**0.029**	$\langle 0.000 \rangle$
Panel III. SDF correctly prices each of the $N+1$ assets, and we set $N=11$		
(e) Set E: Market, 10 momentum	**0.020**	$\langle 0.000 \rangle$
Panel IV. SDF correctly prices each of the $N+1$ assets, and we set $N=2$		
(f) Set F: Market, Low Momentum	**0.010**	$\langle 0.000 \rangle$
(g) Set G: Market, high Momentum	**0.014**	$\langle 0.010 \rangle$
Panel V. SDF correctly prices each of the $N+1$ assets, and we set $N=1$		
	(BCZ, Eq. 5)	
(h) Set H: Market portfolio only	**0.005**	$\langle 0.005 \rangle$
(i) Set I: EWI portfolio of 25 size & B/M	**0.007**	$\langle 0.001 \rangle$
(j) Set J: EWI portfolio of 25 size & momentum	**0.007**	$\langle 0.001 \rangle$

Table Internet Appendix-II

Model comparisons using bounds

Reported are the results for bounds on the entropy of m, the entropy of m^2, and the volatility of m, for three models:

- the difference habit (denoted by DH),
- the recursive utility with stochastic variance (denoted by RU-SV),
- and the recursive utility with constant jump intensity (denoted by RU-CJI).

The one-sided p-values shown in square brackets represent the proportion of replications for which the model-based quantity (entropy or volatility) exceeds, in 50,000 replications, the lower bound computed from observed asset prices. Our lower bound on the entropy of m^2 is based on equation (12) and relies on the ability of the SDF to correctly price $N+1$ assets (the risk-free bond and N risky assets). The N risky assets are based on SET B, which contains the value-weighted market returns, together with the 25 Fama-French size and momentum portfolios. The sample period is from July 1931 to December 2011. The lower bound on the entropy of m is based on equation (20) and also relies on the ability of the SDF to correctly price $N+1$ assets. The lower bound on the volatility of m is based on Hansen and Jagannathan (1991, equation (12)). We focus on SET B, as it corresponds to the maximum lower bound on entropy measures (as in our Table Internet Appendix-I). Panel D presents the variance, skewness, and kurtosis of m, which are consistent with model parameterizations in Table Internet Appendix-III. The one-sided p-values $\langle . \rangle$, reported below the lower bounds, represent the proportion of bootstrap samples for which the lower bound is less than zero.

	Habit model	Recursive utility models		
	DH	RU-SV	RU-CJI	Lower bounds (Set B)
Panel A: Entropy of m				
$L[m]$	0.0196	0.0217	0.0190	0.0367
	[0.000]	[0.000]	[0.000]	$\langle 0.003 \rangle$
Panel B: Entropy of m^2				
$L[m^2]$	0.0785	0.0869	1.4331	0.1956
	[0.000]	[0.000]	[1.000]	$\langle 0.003 \rangle$
Panel C: Volatility bound				
Hansen and Jagannathan (1991)	0.0415	0.0444	3.344	0.1292
	[0.000]	[0.000]	[1.000]	$\langle 0.000 \rangle$
Panel D: Moments of the $m_{t,t+1}$ distribution				
Variance	0.0403	0.0444	3.3438	
Skewness	0.6041	0.6476	$+\infty$	
Kurtosis	3.6447	3.8061	$+\infty$	

Table Internet Appendix-III

Parameters employed in model implementation

Displayed in this table are the parameters that govern preferences and the dynamics of consumption growth. These parameters are adopted from Tables 2, 3, and 4 of Backus, Chernov, and Zin (2014), and likewise $\log(g)$ and η_0 are taken from their page 16. Our implementation of the models with difference habit (hereby DH), recursive utility with stochastic variance (hereby RU-SV), and recursive utility with constant jump intensity (hereby RU-CJI) follows Backus, Chernov, and Zin (2014, respectively, Model (4) in Table 2, Model (1) in Table 3, and Model (4) in Table 4). We use US annual real personal consumption expenditures as a proxy for aggregate consumption over the sample period of 1931:07 to 2011:12 (966 observations). To compare model implications with the data, we simulate a finite sample of consumption growth, c_{t+1}/c_t, over 966 months. Following convention, we then compute the annualized consumption growth as $\exp(\sum_{j=1}^{12}\log(c_{t+j}/c_{t+j-1}))$. The reported model mean, standard deviation, and auto-correlation are based on the annualized consumption growth.

Parameter	DH	RU-SV	RU-CJI	Data implied
Panel A: Preferences				
ρ	-9.0000	0.3333	0.3333	
α		-9.0000	-9.0000	
β	0.9980	0.9980	0.9980	
φ_h	0.9000			
s	0.5000			
Panel B: Consumption growth dynamics				
γ_0	1.0000	1.0000	1.0000	
$\log(g)$	0.0015	0.0015	0.0015	
η_0	0.1000			
γ_1	0.0271	0.0271	0.0281	
φ_g	0.9790	0.9790	0.9690	
$\upsilon^{1/2}$	0.0099	0.0099	0.0079	
ν_0		0.23×10^{-5}		
φ_υ		0.9870		
h			0.0008	
θ			-0.1500	
δ			0.1500	
ψ_0			1.0000	
b_1		0.9977	0.9979	
Panel C: Consumption growth				
Mean (annualized)	1.0192	1.0190	1.0189	1.0339
Std. Dev. (annualized)	0.0416	0.0415	0.0369	0.0287
Autocorrelation	0.2424	0.2433	0.1771	0.2386