Online Appendix of "Optimal Consumption and Investment under Time-Varying Liquidity Constraints" by S. Ahn, K. J. Choi, and B. H. Lim

A The Numerical Algorithm for the LTV Limits Case

The free boundary condition makes it difficult to obtain the solution in closed form. So we apply the finite difference method to get the dual value function, which satisfies the free boundary condition at every node. Since we know the value function at final time T, it is similar to an American-type option pricing. For the unconditional stability, we apply the Crank-Nikolson method and Gauss-Seidel scheme.

Let us define the value function as a CRRA utility function,

$$U(X_T) \;=\; \frac{K}{1-\gamma}X_T^{1-\gamma},$$

where the constant K is a weight on the utility at the final time.

Let us define the grid as $y = 0, \delta y, 2\delta y, \ldots, N\delta y = y_T$, and $t = 0, \delta t, 2\delta t, \ldots, M\delta t = T$. A Crank-Nicolson method is the combination of the explicit and implicit methods so the differences with respect to time and y of function $\phi(i\delta y, j\delta t) \equiv \phi_{i,j}$ are given by

$$\begin{aligned}
\phi_t &= \frac{\phi_{i,j+1}-\phi_{i,j}}{\delta t},\\
\phi_y &= \frac{1}{2}\left(\frac{\phi_{i+1,j}-\phi_{i-1,j}}{2\delta y}+\frac{\phi_{i+1,j-1}-\phi_{i-1,j-1}}{2\delta y}\right),\\
\phi_{yy} &= \frac{1}{2}\left(\frac{\phi_{i+1,j}-2\phi_{i,j}+\phi_{i-1,j}}{(\delta y)^2}+\frac{\phi_{i+1,j-1}-2\phi_{i,j-1}+\phi_{i-1,j-1}}{(\delta y)^2}\right).
\end{aligned}$$

If we substitute these differences into PDE (45) and rearrange it, for $i = 0, 1, \ldots, N-1$, and $j = 1, 2, \ldots, M-1$, we have

$$\begin{aligned}
\frac{\beta}{2}\left(\phi_{i,j} + \phi_{i,j-1}\right) &= \frac{\phi_{i,j} - \phi_{i,j-1}}{\delta t} \\
&\quad + \frac{(\beta - r) i \delta y}{4}\left(\frac{\phi_{i+1,j} - \phi_{i-1,j}}{\delta y} + \frac{\phi_{i+1,j-1} - \phi_{i-1,j-1}}{\delta y}\right) \\
&\quad + \frac{\theta^2 i^2 \delta y^2}{4}\left(\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{(\delta y)^2}\right. \\
&\qquad\qquad \left. + \frac{\phi_{i+1,j-1} - 2\phi_{i,j-1} + \phi_{i-1,j-1}}{(\delta y)^2}\right) \\
&\quad + \frac{\gamma}{1-\gamma}(i\delta y)^{1-\frac{1}{\gamma}}
\end{aligned}$$

where the boundary conditions are given by

$$\begin{aligned}
\phi_{i,M} &= \tilde{U}(i\delta y) = \frac{K\gamma}{1-\gamma}(i\delta y)^{1-\frac{1}{\gamma}}, \ i = 0, 1, \ldots, N, \\
\phi_{N,j} &= e^{-\beta\delta t}\left\{\frac{\gamma(N\delta y)^{1-\frac{1}{\gamma}}\delta t}{1-\gamma} + \phi_{N,j+1}\right\}, \ j = 0, 1, \ldots, M-1, \\
\phi_{0,j} &= 0, \ j = 0, 1, ..., M.
\end{aligned} \tag{A.1}$$

Thus, we have the following grid equation

$$\begin{aligned}
a_i \phi_{i+1,j-1} + (b_i - 1)\phi_{i,j-1} - c_i \phi_{i+1,j-1} &= -a_i \phi_{i+1,j} - (1 + b_i)\phi_{i,j} \\
&\quad + c_i \phi_{i-1,j} - d_i.
\end{aligned}$$

where the coefficients a_i, b_i, c_i, and d_i are given by

$$\begin{aligned}
a_i &= \frac{\delta t}{4}((\beta - r)i + \theta^2 i^2), \\
b_i &= -\frac{\delta t}{2}(\theta^2 i^2 + \beta), \\
c_i &= \frac{\delta t}{4}((\beta - r)i - \theta^2 i^2), \\
d_i &= \frac{\gamma \delta t}{1-\gamma}(i\delta y)^{1-\frac{1}{\gamma}}, \quad i = 1, \ldots, N-1,
\end{aligned}$$

with boundary conditions (A.1). Note that this is the system of $N-1$ linear equations with $N-1$ unknowns.

For the free boundary condition, we have to consider the backward difference because if (i, j) is the free boundary node, the value function at $(i+1, j)$ would be undefined. Thus, the free boundary condition is expressed by

$$\frac{\phi_{i,j} - \phi_{i-1,j}}{\delta y} \leq LP_0 \lambda^{\frac{\sigma_p}{\theta}} e^{\alpha(j\delta t)}(i\delta y)^{-\frac{\sigma_p}{\theta}} - m(j\delta t),$$

where the remaining human capital $m(j\delta t)$ and constant α are determined by

$$
\begin{aligned}
m(j\delta t) &= \frac{i_0}{r}\left(1 - e^{-r(M-j)\delta t}\right), \\
\alpha &= \mu_p - \frac{1}{2}\sigma_p + \frac{\sigma_p}{\theta}\left(\beta - r - \frac{\theta^2}{2}\right).
\end{aligned}
$$

The free boundary condition can be rewritten as

$$
\begin{aligned}
\phi_{i,j} &\leq \phi_{i-1,j} + LP_0 \lambda^{\frac{\sigma_p}{\theta}} e^{\alpha(j\delta t)} (i\delta y)^{-\frac{\sigma_p}{\theta}} \cdot \delta y - \delta y \cdot m(j\delta t) \\
&\triangleq \phi_{i-1,j} + g(i,j).
\end{aligned}
\tag{A.2}
$$

Therefore, for each node we have to check whether the derived value from the system with $N-1$ unknowns is less than $g(i,j)$. Notice that since there is one-to-one correspondence between wealth and parameter λ, the dual value function $\phi_{i,j}$ satisfying the inequality (A.2) is true only for a corresponding initial wealth $x = x^*(\lambda)$.

Now, we apply the Gauss-Seidel method with overrelaxation. Let us denote the system of linear equations as

$$
\mathbf{A}X = \mathbf{E}
$$

Then the Gauss-Seidel method implies that

$$
x_i^{(k+1)} = x_i^{(k)} + \frac{w}{a_{ii}}\left(e_i - \Sigma_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \Sigma_{j=i}^{N} a_{ij} x_j^{(k)}\right).
$$

Thus, we have for $i = 1$,

$$
\phi_{1,j}^{(k+1)} = \phi_{1,j}^{(k)} + \frac{w}{b_1 - 1}\left(e_1 - (b_1 - 1)\phi_{1,j}^{(k)} - a_1 \phi_{2,j}^{(k)}\right),
$$

and for $i = 2, \ldots, M-2$,

$$
\phi_{i,j}^{(k+1)} = \phi_{i,j}^{(k)} + \frac{w}{b_i - 1}\left(e_i + c_i \phi_{i-1,j}^{(k+1)} - (b_i - 1)\phi_{i,j}^{(k)} - a_i \phi_{i+1,j}^{(k)}\right),
$$

and for $i = M-1$,

$$
\begin{aligned}
\phi_{M-1,j}^{(k+1)} &= \phi_{M-1,j}^{(k)} \\
&\quad + \frac{w}{b_{M-1} - 1}\left(e_{M-1} + c_{M-1}\phi_{M-2,j}^{(k+1)} - (b_{M-1} - 1)\phi_{M-1,j}^{(k)}\right).
\end{aligned}
$$

Therefore, for each time j, the iterative scheme is obtained by

$$
\phi_{1,j}^{(k+1)} = \max\left\{g(1,j), \phi_{1,j}^{(k)} + \frac{w}{b_1 - 1}\left(e_1 - (b_1 - 1)\phi_{1,j}^{(k)} - \alpha_1 \phi_{2,j}^{(k)}\right)\right\}
$$

for $i = 2, \ldots, N-2$,

$$
\begin{aligned}
\phi_{i,j}^{(k+1)} = \max\Bigg\{ & \phi_{i-1,j}^{(k+1)} + g(i,j), \\
& \phi_{i,j}^{(k)} + \frac{w}{b_i - 1}\left(e_i + c_i \phi_{i-1,j}^{(k+1)} - (b_i - 1)\phi_{i,j}^{(k)} - \alpha_i \phi_{i+1,j}^{(k)}\right)\Bigg\},
\end{aligned}
$$

and for $i = M-1$,

$$\begin{aligned}\phi^{(k+1)}_{M-1,j} &= \max\Big\{\phi^{(k+1)}_{M-2,j} + g(M-1,j),\ \phi^{(k)}_{M-1,j} \\ &\qquad + \frac{w}{b_{M-1}-1}\left(e_{M-1} + c_{M-1}\phi^{(k+1)}_{M-2,j} - (b_{M-1}-1)\phi^{(k)}_{M-1,j}\right)\Big\}.\end{aligned}$$

As mentioned above, the derived dual value function at time 0 is true only for $x = x^*(\lambda)$. Thus, we have to repeat the procedure for each $\lambda \in (0, \bar{\lambda})$ where $\bar{\lambda}$ corresponds to $-L \cdot P_0$.

B Proof of Proposition 9

Let us define new dual variables $y_t \equiv \lambda e^{(\beta-\nu+\delta)t} H_t$ and $z_t \equiv y_t I_t^{\gamma}$. Then the dual value function can be calculated by

$$\begin{aligned} J_\nu(\lambda, q) &= \mathbb{E}\left[\int_0^\infty e^{-(\beta+\delta)t} I_t^{1-\gamma}\left(\frac{\gamma}{1-\gamma} z_t^{1-\frac{1}{\gamma}} + z_t\right) dt\right] \\ &\quad + \mathbb{E}\left[\int_0^\infty e^{-(\beta+\delta)t} \max_{X_t^{\pi_\nu}} \left\{\delta V_u\left(X_t^{\pi_\nu}, kI_t\right) - \lambda\nu e^{(\beta-\nu+\delta)t} H_t X_t^{\pi_\nu}\right\} dt\right] \\ &= q^{1-\gamma}\widetilde{\mathbb{E}}\left[\int_0^\infty e^{-(\beta+\delta)t}\left(\frac{\gamma}{1-\gamma} z_t^{1-\frac{1}{\gamma}} + z_t + \delta k^{1-\gamma}\psi_u\left(\frac{\nu}{\delta} z_t\right)\right) dt\right] \\ &\equiv q^{1-\gamma}\psi(z_t), \end{aligned}$$

where $\psi_u(z)$ is the dual value function after income shock, which is exactly the same as with the value in (20). The second equality holds from a change of measure applied in Lemma 1 and the Legendre transformation. Thus, the value function $\psi(z)$ should satisfy the following ODE:

(B.1)
$$\begin{aligned} 0 &= \frac{\sigma_z^2}{2}\psi''(z)z^2 - \left(r_I - \hat{\beta} + \nu - \delta\right)\psi'(z)z \\ &\quad - \left(\hat{\beta} - \nu + \delta\right)\psi(z) + \frac{\gamma}{1-\gamma} z^{1-\frac{1}{\gamma}} + z \\ &\quad + \delta k^{1-\gamma}\left(A_1\left(\frac{\nu}{\delta}z\right)^{\alpha_+} + \frac{\gamma}{(1-\gamma)M}\left(\frac{\nu}{\delta}z\right)^{1-\frac{1}{\gamma}} + \frac{\nu}{r_I\delta}z\right). \end{aligned}$$

Then similar to the DTI limit case, we consider the general solution, which is a sum of homogeneous and particular solutions as

$$\psi(z) = Bz^{\xi_+} + Cz^{\xi_-} + \psi_p(z),$$

with the boundary conditions,

(B.2)
$$\psi'(\overline{z}) = kD, \text{ and } \psi''(\overline{z}) = 0.$$

Notice that the dual value function satisfies the ODE (B.1) for $0 < z \leq \overline{z}$, so by the growth condition at $z = 0$, the coefficient C should be equal to 0. We conjecture the particular solution $\psi_p(z)$ as follows:

$$\psi_p(z) \quad = \quad F_1 z^{\alpha_+} + F_2 z^{1-\frac{1}{\gamma}} + F_3 z.$$

If we substitute the $\psi_p(z)$ and its derivatives in ODE (B.1), the coefficients F_1, F_2, and F_3 are obtained.

Moreover, the free boundary $\overline{z}$ and the coefficient B are determined by the boundary conditions in (B.2). Specifically, we have

$$\begin{aligned} kD &= B\xi_+ \overline{z}^{\xi_+ - 1} + F_1 \alpha_+ \overline{z}^{\alpha_+ - 1} + F_2 \left(1 - \frac{1}{\gamma}\right) \overline{z}^{-\frac{1}{\gamma}} + F_3, \\ 0 &= B\xi_+(\xi_+ - 1)\overline{z}^{\xi_+ - 2} + F_1 \alpha_+ (\alpha_+ - 1)\overline{z}^{\alpha_+ - 2} + F_2 \left(\frac{1}{\gamma} - 1\right) \frac{1}{\gamma} \overline{z}^{-\frac{1}{\gamma} - 1}. \end{aligned}$$

Finally, for a given initial wealth x, income level q, and ν, the parameter $\lambda^*(\nu)$ in (52) should satisfy the following relationship:

$$-x \quad = \quad \frac{\partial J_\nu(\lambda^*(\nu), q)}{\partial \lambda} = q\psi'(\lambda^*(\nu) q^\gamma),$$

that is, we need to solve

$$-x \quad = \quad q \left(B\xi_+ z^{\xi_+ - 1} + F_1 \alpha_+ z^{\alpha_+ - 1} + F_2 \left(1 - \frac{1}{\gamma}\right) z^{-\frac{1}{\gamma}} + F_3 \right).$$