Internet Appendix for Equilibrium Price Dynamics of Emission Permits

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A Calculating Equilibrium Prices: Global Problem and PDE

We provide a strategy to calculate equilibrium permit prices within our model as the solution of a system of partial differential equations (PDEs). Permit prices are determined by cumulative economy-wide emissions x_t according to Proposition 2, but it is not simply possible to evaluate the expectation in expression (7) since the dynamics of this nontradable underlying is driven by the companies' endogenous abatement measures. More specifically, economy-wide abatement ξ_t results as the sum of the companies' abatement strategies that solve the individual optimization problems (4). We can show, however, that the equilibrium economy-wide abatement strategy ξ_t can be obtained by solving a global problem on aggregate emission volumes. In this global problem, a central planner optimizes economy-wide abatement in light of possible penalties for exceeding emissions, while the gains and losses from emissions trading cancel out on aggregate.

Proposition 6 (Global Problem)**.** *For a competitive equilibrium with optimal individual abatement strategies ξ^i, $i \in I$, the aggregate abatement strategy ξ with $\xi_t = \sum_{i \in I} \xi_t^i$ solves the global problem*

$$\text{(A-1)} \qquad \min_{\xi} \mathbb{E}_0 \left\{ \int_0^{T_n} e^{-rt} C(\xi_t) dt + \sum_{j=1}^{n} e^{-rT_j} p_j (x_{T_j} - q_j)^+ \right\},$$

with aggregate abatement cost function C as defined in the proof and economy-wide emissions x_t following the dynamics (8)*. The permit price of the ongoing compliance period is equal to the instantaneous marginal abatement costs of the economy,*

$$S_k(t) = \frac{\partial C}{\partial \xi}(\xi_t), \quad t \in [T_{k-1}, T_k]. \tag{A-2}$$

The other way round, for a solution ξ of the global problem (A-1)*, the $S_1, \ldots, S_n$ defined by equation* (7) *are equilibrium permit price processes.*

The first part of the Proposition implies, together with Proposition 2, that the equilibrium price processes $S_1, \ldots, S_n$ are unique, as they are determined by the aggregate abatement strategy of the economy according to equation (7), which is the (unique) solution of the global problem (A-1). The existence of the equilibrium is formally established by the second part of the Proposition, and we formally construct the corresponding trading and abatement strategies that solve the individual optimization problems (4) in the proof in Appendix C.

We derive the solution of the global problem (A-1) in terms of a system of PDEs. For that we follow a backward induction approach, starting at the last compliance period $[T_{n-1}, T_n]$ and proceeding to the periods $[T_{n-2}, T_{n-1}], \ldots, [0, T_1]$. For each compliance period k, we include the period $k+1$ solution into the terminal condition and settle the problem by dynamic programming. We state the resulting system of PDEs in here for the case of business-as-usual emissions following an arithmetic Brownian motion (17) and a quadratic abatement cost function (18) as used in the calibration, and refer to the proof in Appendix C for the derivation in the general case.

Proposition 7 (PDEs)**.** *For the global problem* (A-1)*, optimal abatement* ξ_t *at time* $t \in [T_{k-1}, T_k]$ *is given by*

$$\text{(A-3)} \qquad \xi_t = \frac{1}{\gamma} e^{r(t-T_{k-1})} \frac{\partial V_k}{\partial x}(t, x_t, y_t),$$

where V_k *is the time-*T_{k-1} *expected value of an optimal strategy starting at* T_{k-1}*.* V_k *solves the characteristic PDE*

$$\text{(A-4)} \qquad \frac{\partial V_k}{\partial t} = -y_t \frac{\partial V_k}{\partial x} + \frac{1}{2\gamma} e^{r(t-T_{k-1})} \left(\frac{\partial V_k}{\partial x} \right)^2 - \frac{\partial V_k}{\partial y} \mu_y - \frac{1}{2} \frac{\partial^2 V_k}{\partial x^2} \sigma_e^2 - \frac{1}{2} \frac{\partial^2 V_k}{\partial y^2} \sigma_y^2$$

with boundary condition

$$\text{(A-5)} \qquad V_k(T_k, x_{T_k}, y_{T_k}) = e^{-r(T_k - T_{k-1})} (p_k (x_{T_k} - q_k)^+ + V_{k+1}(T_k, x_{T_k}, y_{T_k}))$$

and $V_{n+1} = 0$.

As shown by Proposition 6, the solution for optimal economy-wide abatement at time t directly implies the equilibrium permit price of the ongoing compliance period through equation (A-2). Thus, permit prices can be computed by numerically solving the system of PDEs (A-4), (A-5), starting from period n and proceeding backwards.

B Effects of Risk Aversion on Emission Permit Prices

In this section, we depart from the assumption of risk-neutral agents in our model and outline the effects of risk aversion on emission permit prices. The existing literature (Seifert, Uhrig-Homburg, and Wagner (2008), Carmona, Delarue, Espinosa, and Touzi (2013)) provides several technical results for the case of risk-averse agents in one-period emission trading systems, but refrains from a detailed economic discussion of related effects.

While the risk-neutral (linear utility) case allows us to consider the cash-flows from emissions trading separately from the state of the general economy, this is not possible anymore under standard specifications of risk aversion, such as CRRA preferences or, more general, recursive preferences as proposed by Epstein and Zin (1991). Rather, it becomes necessary to account for the overall consumption level of agents in different states of the world. For our discussion, we do so by introducing an exogenously specified pricing kernel Λ_t into the company's optimization problem (4), yielding

$$\text{(A-6)} \quad \min_{(\theta^i,\xi^i)} \mathbb{E}_0 \left\{ \int_0^{T_n} \Lambda_t C^i(\xi_t^i)dt + \sum_{j=1}^n \int_0^{T_j} \Lambda_t S_j(t)\theta_{j,t}^i dt + \sum_{j=1}^n \Lambda_{T_j} p_j (x_{T_j}^i - Q_j^i)^+ \right\},$$

and solve for the equilibrium along the lines of Section II.B. In particular, this generalizes the characterization of emission permit prices in Proposition 2 to

$$\text{(A-7)} \qquad S_k(t) \;=\; \sum_{j=k}^n \mathbb{E}_t \left\{ \frac{\Lambda_{T_j}}{\Lambda_t} 1_{\{x_{T_j} > q_j\}} \right\} p_j,$$

replacing discounting with the risk-free rate by the stochastic discount factor.

To understand the effects of risk aversion on emission permit prices, we need to analyze the correlation of Λ_t with the emissions of the economy, x_t. At first sight, it seems natural to assume that a state of high emissions x_t means that the level of production and consumption in the economy is high, translating to low marginal utility. In this case, Λ_t and x_t would be negatively correlated, leading to a stronger discounting of possible penalties p_k and therefore lower emission permit prices today. In other words, agents would have to pay penalties for exceeding emissions when they can easily afford them. Consequently, this would lead to lower levels of emission abatement as well, as marginal abatement costs and emission permit prices are equated in equilibrium also for the case of risk aversion.

However, recall that the original motivation for introducing an emission trading system at all is that high emissions are associated with the risk of climate change, which brings along significant growth risks for the economy (see Pindyck (2012)). This effect works in the opposite direction, and it is very well possible that it outweighs the aspect of higher productivity levels, implying that high emissions stand for a "bad state" of the economy. If that is the case, Λ_t and x_t would be positively correlated, turning the aforementioned effects around: Emission permit prices are higher than in the risk-neutral case, and the amount of abatement actions implemented increases as well.

Finally, let us discuss how these effects work over different horizons, i.e., how risk aversion changes the contribution of compliance periods in the near and remote future to the value of today's emission permit prices. If we assume that the growth risks related to high emissions dominate other effects, it follows that a scenario of excessive emissions for a *remote* compliance period is particularly harmful to the economy, as it means that emissions have been high over an extended period of time. In contrast, a scenario where emissions are high only in the short run is associated with limited growth risks. Consequently, possible penalties for remote compliance periods would be discounted much less strongly relative to the risk-neutral case, and the value components attributable to those future periods become more important for today's emission permit prices. This effect is amplified if agents put a heavier weight on longer-term risks, as in case of a preference for early resolution of uncertainty. In other words, agents' risk aversion together with emissions-related growth risks would induce a fat right tail of the risk-neutral distribution, which would be even more pronounced over longer horizons.

C Proofs of Theoretical Results

Proof of Proposition 1. We characterize the optimal trading and abatement strategies (θ^i, ξ^i) of the individual companies for given permit price processes $S_1(t), \dots, S_n(t)$. The resulting optimality conditions relate the company's marginal abatement costs and the expected penalty payments to the given permit prices. We first decompose the individual optimization problem (4) into a recursive system of n simpler problems, one for each compliance period of the emission trading system, including the value function V_k^i of the period k problem into the terminal condition of the period $k-1$ problem, with $V_{n+1}^i = 0$:

$$\begin{aligned} & V_k^i(t, x_t^i, y_t^i, Q_{k,t}^i, \dots, Q_{n,t}^i) \\ \text{(A-8)} \qquad & = \min_{(\theta_k^i, \xi_k^i)} \mathbb{E}_{T_{k-1}} \bigg\{ \int_t^{T_k} e^{-r(s-T_{k-1})} C^i(\xi_s^i) ds + \sum_{j=k}^{n} \int_t^{T_k} e^{-r(s-T_{k-1})} S_j(s) \theta_{j,s}^i ds \\ & \quad + e^{-r(T_k - T_{k-1})} (p_k (x_{T_k}^i - Q_{k,T_k}^i)^+ + V_{k+1}^i(T_k, x_{T_k}^i, y_{T_k}^i, Q_{k+1,T_k}^i, \dots, Q_{n,T_k}^i)) \bigg\}, \end{aligned}$$

for $t \in [T_{k-1}, T_k]$ and $k = 1, \dots, n$, where (θ_k^i, ξ_k^i) is the restriction of (θ^i, ξ^i) to the time interval $[T_{k-1}, T_k]$ and we introduce the additional state variables $Q_{k,t}^i = \sum_{j=1}^{k} \left(a_j^i + \int_0^{\min\{t, T_j\}} \theta_{j,s}^i ds \right)$ in generalization of $Q_k^i = Q_{k,T_k}^i$. According to the dynamic programming principle (see Bertsekas (1976)), an optimal solution (θ^i, ξ^i) of the original problem is also a solution of the decomposed problem (A-8), and V_1^i is identical to the value function of the original problem for $t \in [0, T_1]$. The dynamics of the state variables follow from equations (3), (1), and the definition of $Q_{k,t}^i$ as

$$\begin{aligned} & dx_t^i = (y_t^i - \xi_t^i) dt + \sigma_e^i dW_t^i, \\ \text{(A-9)} \qquad & dy_t^i = \mu_y^i(t) dt + \sigma_y^i(t) dZ_t^i, \end{aligned}$$

$$dQ^i_{l,t} = \sum_{j=k}^{l} \theta^i_{l,t} dt, \quad l = k, \ldots, n.$$

Note that y^i_t is a Markovian diffusion processes with deterministic coefficients. We derive optimality conditions for the trading and abatement strategy (θ^i, ξ^i) by applying the stochastic maximum principle to the problems (A-8), proceeding recursively from $k = n$ to $k = 1$.[1] A strategy (θ^i_n, ξ^i_n) for period n that *min*imizes the costs according to objective (A-8) *max*imizes the Hamiltonian

$$\text{(A-10)} \quad H_n(t, x^i, y^i, \theta^i, \xi^i, \rho_n) = \rho_{n,x^i}(t) \cdot (y^i_t - \xi^i_t) + \rho_{n,y^i}(t) \cdot \mu^i_y(t) + \rho_{n,Q^i_n}(t) \cdot \theta^i_{n,t} - e^{-r(t-T_{n-1})}(C^i(\xi^i_t) + S_n(t)\theta^i_{n,t}).$$

at every point in time $t \in [T_{n-1}, T_n]$, where $(\rho_{n,x^i}, \rho_{n,y^i}, \rho_{n,Q^i_n})$ are the adjoint processes corresponding to the state variables (x^i, y^i, Q^i_n). Differentiating the Hamiltonian (A-10) with respect to the control variables and setting the derivatives to zero yields the optimality conditions

$$\text{(A-11)} \quad \begin{aligned} \frac{\partial H_n}{\partial \xi^i} &= -\rho_{n,x^i}(t) - e^{-r(t-T_{n-1})}\frac{\partial C^i}{\partial \xi^i}(\xi^i_t) &= 0, \\ \frac{\partial H_n}{\partial \theta^i_n} &= \rho_{n,Q^i_n}(t) - e^{-r(t-T_{n-1})} S_n(t) &= 0. \end{aligned}$$

It remains to derive the adjoint processes ρ_{n,x^i} and ρ_{n,Q^i_n}, which are defined by the stochastic differential equations

$$\text{(A-12)} \quad \begin{aligned} d\rho_{n,x^i}(t) &= \omega_{n,x^i}(t) dW^i_t + \zeta_{n,x^i}(t) dZ^i_t, \\ d\rho_{n,Q^i_n}(t) &= \omega_{n,Q^i_n}(t) dW^i_t + \zeta_{n,Q^i_n}(t) dZ^i_t, \end{aligned}$$

[1] See Yong and Zhou (1999), Chapter 3 for a comprehensive introduction of the stochastic maximum principle for optimal control problems. For our problem, we apply the stochastic maximum principle for the case of a nonsmooth terminal condition, see Chighoub, Djehiche, and Mezerdi (2009).

with stochastic processes $(\omega_{n,x^i}, \omega_{n,Q_n^i}, \zeta_{n,x^i}, \zeta_{n,Q_n^i})$ and terminal conditions

$$\text{(A-13)} \qquad \begin{aligned} \rho_{n,x^i}(T_n) &= -e^{-r(T_n - T_{n-1})} 1_{\{x^i_{T_n} > Q^i_{n,T_n}\}} p_n, \\ \rho_{n,Q_n^i}(T_n) &= e^{-r(T_n - T_{n-1})} 1_{\{x^i_{T_n} > Q^i_{n,T_n}\}} p_n. \end{aligned}$$

We can directly identify the solution[2]

$$\text{(A-14)} \qquad \begin{aligned} \rho_{n,x^i}(t) &= -e^{-r(T_n - T_{n-1})} \mathbb{P}_t \left\{ x^i_{T_n} > Q^i_{n,T_n} \right\} p_n, \\ \rho_{n,Q_n^i}(t) &= e^{-r(T_n - T_{n-1})} \mathbb{P}_t \left\{ x^i_{T_n} > Q^i_{n,T_n} \right\} p_n. \end{aligned}$$

The adjoint processes can be interpreted as the shadow price of the corresponding state variable. For example, ρ_{n,Q_n^i} is the value that can be attributed to having one marginal unit of period-n permits more. Here, this is the discounted penalty weighted by the probability of penalties to accrue, which makes perfect economic sense.

Inserting the adjoint processes into equations (A-11), we arrive at the condition

$$\text{(A-15)} \qquad \frac{\partial C^i}{\partial \xi^i}(\xi^i_t) = e^{-r(T_n - t)} \mathbb{P}_t \left\{ x^i_{T_n} > Q^i_{n,T_n} \right\} p_n = S_n(t),$$

which proves Proposition 1 for $t \in [T_{n-1}, T_n]$ and $k = n$.

Before proceeding to $k = n - 1$, note that under suitable regularity conditions the negative of the adjoint process for a state variable equals the first derivative of the value function with respect to the same variable (see Clarke and Vinter (1987)), that is

$$\text{(A-16)} \qquad -\rho_{n,x^i}(t) = \frac{\partial V_n^i}{\partial x^i}, \qquad -\rho_{n,y^i}(t) = \frac{\partial V_n^i}{\partial y^i}, \qquad -\rho_{n,Q_n^i}(t) = \frac{\partial V_n^i}{\partial Q_n^i}$$

for $t \in [T_{n-1}, T_n)$.

Now consider the optimal control problem (A-8) for $k = n - 1$. In this case the Hamiltonian

[2] For the existence of a regular solution we refer to Carmona et al. (2013).

is given by

$$
\text{(A-17)}\quad H_{n-1}(t, x^i, y^i, \theta^i, \xi^i, \rho_{n-1}) = \rho_{n-1,x^i}(t) \cdot (y_t^i - \xi_t^i) + \rho_{n-1,y^i}(t) \cdot \mu_y^i(t)
$$
$$
+ \rho_{n-1,Q_{n-1}^i}(t) \cdot \theta_{n-1,t}^i + \rho_{n-1,Q_n^i}(t) \cdot (\theta_{n-1,t}^i + \theta_{n,t}^i)
$$
$$
- e^{-r(t-T_{n-2})}(C^i(\xi_t^i) + S_{n-1}(t)\theta_{n-1,t}^i + S_n(t)\theta_{n,t}^i),
$$

and as before we obtain the optimum by differentiating with respect to the control variables and setting the derivatives to zero:

$$
\text{(A-18)}\quad \begin{aligned}
\frac{\partial H_{n-1}}{\partial \xi^i} &= -\rho_{n-1,x^i}(t) - e^{-r(t-T_{n-2})}\frac{\partial C^i}{\partial \xi^i}(\xi_t^i) = 0,\\
\frac{\partial H_{n-1}}{\partial \theta_{n-1}^i} &= \rho_{n-1,Q_{n-1}^i}(t) + \rho_{n-1,Q_n^i}(t) - e^{-r(t-T_{n-2})}S_{n-1}(t) = 0,\\
\frac{\partial H_{n-1}}{\partial \theta_n^i} &= \rho_{n-1,Q_n^i}(t) - e^{-r(t-T_{n-2})}S_n(t) = 0.
\end{aligned}
$$

It is left to insert the adjoint processes $(\rho_{n-1,x^i}, \rho_{n-1,Q_{n-1}^i}, \rho_{n-1,Q_n^i})$ solving the equations

$$
\text{(A-19)}\quad \begin{aligned}
d\rho_{n-1,x^i}(t) &= \omega_{n-1,x^i}(t)dW_t^i + \zeta_{n-1,x^i}(t)dZ_t^i,\\
d\rho_{n-1,Q_{n-1}^i}(t) &= \omega_{n-1,Q_{n-1}^i}(t)dW_t^i + \zeta_{n-1,Q_{n-1}^i}(t)dZ_t^i,\\
d\rho_{n-1,Q_n^i}(t) &= \omega_{n-1,Q_n^i}(t)dW_t^i + \zeta_{n-1,Q_n^i}(t)dZ_t^i,
\end{aligned}
$$

with terminal conditions

$$
\text{(A-20)}\quad \begin{aligned}
\rho_{n-1,x^i}(T_{n-1}) &= -e^{-r(T_{n-1}-T_{n-2})}(1_{\{x_{T_{n-1}}^i > Q_{n-1,T_{n-1}}^i\}}p_{n-1}\\
&\qquad + \frac{\partial V_n^i}{\partial x^i}(T_{n-1}, x_{T_{n-1}}^i, y_{T_{n-1}}^i, Q_{n,T_{n-1}}^i)),\\
\rho_{n-1,Q_{n-1}^i}(T_{n-1}) &= e^{-r(T_{n-1}-T_{n-2})}1_{\{x_{T_{n-1}}^i > Q_{n-1,T_{n-1}}^i\}}p_{n-1},\\
\rho_{n-1,Q_n^i}(T_{n-1}) &= -e^{-r(T_{n-1}-T_{n-2})}\frac{\partial V_n^i}{\partial Q_n^i}(T_{n-1}, x_{T_{n-1}}^i, y_{T_{n-1}}^i, Q_{n,T_{n-1}}^i),
\end{aligned}
$$

After inserting the derivatives of V_n^i according to equations (A-14) and (A-16), we identify the solution

$$
\begin{aligned}
\rho_{n-1,x^i}(t) &= -\sum_{j=n-1}^{n} e^{-r(T_j-T_{n-2})}\mathbb{P}_t\left\{x^i_{T_j} > Q^i_{j,T_j}\right\}p_j,\\
\rho_{n-1,Q^i_{n-1}}(t) &= e^{-r(T_{n-1}-T_{n-2})}\mathbb{P}_t\left\{x^i_{T_{n-1}} > Q^i_{n-1,T_{n-1}}\right\}p_{n-1},\\
\rho_{n-1,Q^i_n}(t) &= e^{-r(T_n-T_{n-2})}\mathbb{P}_t\left\{x^i_{T_n} > Q^i_{n,T_n}\right\}p_n.
\end{aligned} \tag{A-21}
$$

Using the adjoint processes in the first order conditions (A-18) finally yields the optimality conditions

$$
\begin{aligned}
\frac{\partial C^i}{\partial \xi^i}(\xi^i_t) = \sum_{j=n-1}^{n} e^{-r(T_j-t)}\mathbb{P}_t\left\{x^i_{T_j} > Q^i_{j,T_j}\right\}p_j &= S_{n-1}(t),\\
e^{-r(T_n-t)}\mathbb{P}_t\left\{x^i_{T_n} > Q^i_{n,T_n}\right\}p_n &= S_n(t),
\end{aligned} \tag{A-22}
$$

which proves Proposition 1 for $t \in [T_{n-2}, T_{n-1}]$ and $k = n-1$. Proceeding along the same lines for $k = n-2$ to $k = 1$ completes the proof of Proposition 1.

Proof of Proposition 2. As the optimality condition (6) holds for all companies $i_1, i_2 \in I$, we have

$$
\sum_{j=k}^{n} e^{-r(T_j-t)}\mathbb{P}_t\left\{x^{i_1}_{T_j} > Q^{i_1}_j\right\}p_j = \sum_{j=k}^{n} e^{-r(T_j-t)}\mathbb{P}_t\left\{x^{i_2}_{T_j} > Q^{i_2}_j\right\}p_j, \quad t \in [0, T_k], \tag{A-23}
$$

for $k = 1, \ldots, n$, which especially implies $1_{\{x^{i_1}_{T_k} > Q^{i_1}_k\}} = 1_{\{x^{i_2}_{T_k} > Q^{i_2}_k\}}$. Under this condition, $x^i_{T_k} > Q^i_k$ for *one* $i \in I$ is equivalent to $x^i_{T_k} > Q^i_k$ for *all* $i \in I$, and also equivalent to $\sum_{i\in I} x^i_{T_k} > \sum_{i \in I} Q^i_k$. Consequently, Proposition 2 follows from condition (6), as $x_{T_k} = \sum_{i\in I} x^i_{T_k}$ and $q_k = \sum_{i\in I} Q^i_k$.

Proof of Proposition 3. We write the period-k emission permit price S_k as a function of time t and of the state variables x_t and y_t with dynamics (8) and (9). Applying Itô's Lemma yields

$$\text{(A-24)} \qquad \begin{aligned} dS_k &= \frac{\partial S_k}{\partial t}dt + \frac{\partial S_k}{\partial x}dx + \frac{\partial S_k}{\partial y}dy + \frac{1}{2}\frac{\partial^2 S_k}{\partial x^2}dx^2 + \frac{\partial^2 S_k}{\partial x \partial y}dxdy + \frac{1}{2}\frac{\partial^2 S_k}{\partial y^2}dy^2 \\ &= \mu_{S_k} S_k dt + \frac{\sqrt{\left(\frac{\partial S_k}{\partial x}\sigma_e\right)^2 + \left(\frac{\partial S_k}{\partial y}\sigma_y(t)\right)^2}}{S_k} S_k dB_t, \end{aligned}$$

with $\mu_{S_k} = \frac{\frac{\partial S_k}{\partial t} + \frac{\partial S_k}{\partial x}(y_t - \xi_t) + \frac{\partial S_k}{\partial y}\mu_y(t) + \frac{1}{2}\frac{\partial^2 S_k}{\partial x^2}\sigma_e^2 + \frac{1}{2}\frac{\partial^2 S_k}{\partial y^2}\sigma_y^2(t)}{S_k}$ and $B_t = W_t + Z_t$. Therefore, the relative volatility of S_k is given by expression (12), which proves the Proposition.

Proof of Proposition 4. The backwardation of a forward contract with maturity $\bar{t}$ is defined as $B(t, \bar{t}) = S_1(t) - e^{-r(\bar{t}-t)}F(t, \bar{t})$, where the case $B(t, \bar{t}) \leq 0$ is called contango, and $B(t, \bar{t}) > 0$ describes a backwardation. A backwardation is called strong if $F(t, \bar{t}) < S_1(t)$, and weak otherwise. From equation (13), it directly follows that $B(t, \bar{t}) = 0$ for forwards with maturity in the same compliance period, $\bar{t} \in [t, T_1)$, which proves Proposition 4a). On the other hand, it follows from equation (14) that $B(t, \bar{t}) = e^{-r(T_1-t)}\mathbb{P}_t\{x_{T_1} > q_1\}p_1$ for forwards maturing in the following compliance period, $\bar{t} \in (T_1, T_2)$. Therefore, inter-period forwards are in contango if $\mathbb{P}_t\{x_{T_1} > q_1\} = 0$, and in backwardation otherwise. Finally, it follows that the condition for strong backwardation, $F(t, \bar{t}) < S_1(t)$, is fulfilled if and only if $\mathbb{P}_t\{x_{T_1} > q_1\} > (e^{r(T_1-t)} - e^{-r(\bar{t}-T_1)})\frac{S_1(t)}{p_1}$, which completes the proof of Proposition 4b).

Proof of Proposition 5. According to equation (16) and the definition of the time-aggregated convenience yield $D_t(\bar{t})$, we have

$$\text{(A-25)} \qquad F(t, \bar{t}) = e^{r(\bar{t}-t)}\mathbb{E}_t\left\{e^{-D_t(\bar{t})}\right\}S_1(t).$$

From equation (13) it follows that $\mathbb{E}_t\left\{e^{-D_t(\bar{t})}\right\} = 1$ for all intra-period forwards $\bar{t} \in [t, T_1)$, which is equivalent to $D_t(\bar{t}) = 0$ for all $\bar{t} \in [t, T_1)$, proving Proposition 5a). On the other hand, we directly obtain Proposition 5b) for inter-period forwards $\bar{t} \in (T_1, T_2)$ from equations (14) and (A-25).

Proof of Proposition 6. To prove the Proposition, we show first that a competitive equilibrium of the model yields a solution of the global problem on aggregate volumes, (A-1). Second, we prove that a solution of the global problem (A-1) gives rise to permit price processes and individual trading and abatement strategies that characterize a competitive market equilibrium.[3]

For the first step, let us assume that a competitive equilibrium is given in form permit price processes $S_1, \ldots, S_n$ as well as individual trading and abatement strategies (θ^i, ξ^i) that solve the individual optimization problems (4). It is immediately clear that the combination of the optimal individual strategies, $(\Theta, \Xi) = (\theta^i, \xi^i)_{i \in I}$, solves any linear combination of the individual problems, especially the sum, which is

$$\text{(A-26)} \qquad \min_{(\Theta,\Xi)} \mathbb{E}_0 \left\{ \sum_{i \in I} \left(\int_0^{T_n} e^{-rt} C^i(\xi_t^i) dt + \sum_{j=1}^{n} e^{-rT_j} p_j (x_{T_j}^i - Q_j^i)^+ \right) \right\}.$$

Note that this is the joint cost problem of a central planner, and the fact that the equilibrium trading and abatement strategies solve this problem implies that a competitive equilibrium

[3]Our approach partly builds on Seifert et al. (2008) and Carmona, Fehr, and Hinz (2009). In the appendix of Seifert et al. (2008) it is shown that the global problem acting on aggregate volumes is equivalent to the sum of all companies' individual solutions, under the crucial assumption that all companies' emissions are driven by the same Wiener process. Without this assumption, Carmona et al. (2009) prove for a discrete-time framework with one compliance period that the solution of the global problem is optimal also for the individual companies.

of our model is also Pareto efficient. In the joint cost problem, gains and losses from trading emission permits cancel out due to the market clearing condition.

We can now reformulate the joint cost problem (A-26) to the simplified aggregate problem (A-1). For that, recall that a competitive equilibrium equates the probability of penalties across companies according to equation (6) in Proposition 1, which implies

$$\sum_{i\in I}(x^i_{T_k} - Q^i_k)^+ = (x_{T_k} - q_k)^+ \tag{A-27}$$

for $k = 1, \ldots, n$. Further, define the aggregate abatement cost function C as

$$C(\xi_t) = \sum_{i\in I} C^i(t, {c^i}^{-1}(c^{\widehat{i}}(\xi^{\widehat{i}}_t))), \tag{A-28}$$

where $\widehat{i} \in I$ is one arbitrarily chosen single company, $c^i = \frac{\partial C^i}{\partial \xi^i}$ is the first derivative of C^i, ${c^i}^{-1}$ is its inverse function, and $\xi^{\widehat{i}}_t$ is implicitly defined through $\xi_t = \sum_{i\in I} {c^i}^{-1}(c^{\widehat{i}}(\xi^{\widehat{i}}_t))$. C is well-defined because, first, $c^i(\xi^i_t)$ is equal for all companies $i \in I$ for a competitive equilibrium according to condition (5) of Proposition 1, and second, c^i is strictly increasing and thus invertible due to the strict convexity of C^i. Together with the differentiability of C_i, it follows that C is strictly convex and differentiable with respect to ξ_t and it is

$$\sum_{i\in I} C^i(\xi^i_t) = C(\xi_t). \tag{A-29}$$

By using these aggregate functions in the objective (A-26), it follows that the aggregate abatement strategy of a competitive model equilibrium solves the global problem (A-1) on aggregate volumes. Equation (A-2) then follows directly from condition (5) of the competitive equilibrium together with the fact that we have $\frac{\partial C}{\partial \xi}(\xi_t) = \frac{\partial C^i}{\partial \xi^i}(\xi^i_t)$ by definition of the aggregate abatement function C.

Note that this first part of the Proposition directly implies the uniqueness of the equilibrium price processes $S_1, \ldots, S_n$, which are determined by the aggregate abatement strategy ξ of a competitive equilibrium according to Proposition 2, and therefore by the (unique) solution of problem (A-1). The equilibrium is also unique with respect to the *individual* abatement strategies ξ^i, which follow directly from aggregate abatement due to the strict convexity of the abatement cost functions. On the other hand, for the individual trading strategies θ^i we only have the restriction

$$1_{\{x^i_{T_k} > Q^i_k\}} = 1_{\{x_{T_k} > q_k\}} \tag{A-30}$$

with $k = 1, \ldots, n$ for all companies $i \in I$, which results from condition (6) in Proposition 1 and Proposition 2, and we will show that indeed all sets of trading strategies fulfilling this condition support the equilibrium. Intuitively, equilibrium trading strategies allocate the available permits in such way that a single company can only be long (short) of permits at the end of a compliance period if the whole economy is long (short) of permits, while it is irrelevant how permits are distributed among companies beyond this condition and especially within a compliance period.

Now we establish the existence of the competitive model equilibrium by proving the second part of the Proposition. For that, assume that an abatement strategy ξ^* solving the global problem (A-1) is given, and construct permit price processes $S_1, \ldots, S_n$ according to Proposition 2, equation (7), as

$$S^*_k(t) = \sum_{j=k}^{n} e^{-r(T_j - t)} \mathbb{P}_t \left\{ x^*_{T_j} > q_j \right\} p_j, \quad t \in [0, T_k]. \tag{A-31}$$

The asterisks indicate that economy-wide realized emissions x^*_t follow the dynamics (8) with abatements ξ chosen according to ξ^*.

It is straightforward to recover a solution (Θ^*, Ξ^*) of the joint cost problem (A-26) from ξ^* by inverting the process from before, i.e., we define abatement strategies ξ^{i^*} for the single companies with $\xi^*_t = \sum_{i\in I} \xi^{i^*}_t$ according to the optimality conditions

$$\frac{\partial C^i}{\partial \xi^i}(\xi^{i^*}_t) = \sum_{j=k}^{n} e^{-r(T_j - t)} \mathbb{P}_t \left\{ x^*_{T_j} > q_j \right\} p_j, \quad t \in [T_{k-1}, T_k], \tag{A-32}$$

and choose a market-clearing trading strategy Θ^* that fulfills condition (A-30) for $k = 1, \ldots, n$ for all companies $i \in I$.

We show that given the permit price processes (A-31), the solution (Θ^*, Ξ^*) of the joint cost problem (A-26) is also optimal for the individual problems (4), establishing the existence of a competitive market equilibrium with equilibrium permit prices $S^*_1, \ldots, S^*_n$. For that, expand the expected value in objective (4) by adding and subtracting the term $\sum_{j=1}^{n} \int_{T_{j-1}}^{T_j} e^{-rt} S^*_j(t) \xi^i_t dt$ and split it in two parts according to

$$\begin{aligned}
&\mathbb{E}_0 \Big\{ \int_0^{T_n} e^{-rt} C^i(\xi^i_t) dt + \sum_{j=1}^{n} \int_0^{T_j} e^{-rt} S^*_j(t) \theta^i_{j,t} dt + \sum_{j=1}^{n} e^{-rT_j} p_j (x^i_{T_j} - Q^i_j)^+ \Big\} \\
&\quad = \mathbb{E}_0 \Big\{ \int_0^{T_n} e^{-rt} C^i(\xi^i_t) dt - \sum_{j=1}^{n} \int_{T_{j-1}}^{T_j} e^{-rt} S^*_j(t) \xi^i_t dt \Big\} \\
&\qquad + \mathbb{E}_0 \Big\{ \sum_{j=1}^{n} \Big(\int_0^{T_j} e^{-rt} S^*_j(t) \theta^i_{j,t} dt + \int_{T_{j-1}}^{T_j} e^{-rt} S^*_j(t) \xi^i_t dt + e^{-rT_j} p_j (x^i_{T_j} - Q^i_j)^+ \Big) \Big\}.
\end{aligned} \tag{A-33}$$

We rewrite the first expectation value as

$$\mathbb{E}_0 \Big\{ \sum_{j=1}^{n} \int_{T_{j-1}}^{T_j} e^{-rt} (C^i(\xi^i_t) - S^*_j(t) \xi^i_t) dt \Big\} \tag{A-34}$$

by subdividing the first integral. Obviously, this term is minimized by all abatement strategies ξ^i fulfilling

$$\frac{\partial C^i}{\partial \xi^i}(\xi_t^i) = S_k^*(t), \quad t \in [T_{k-1}, T_k]. \tag{A-35}$$

Since both an individually optimal strategy and (Θ^*, Ξ^*) fulfill this condition (see equations (5) and (A-31), (A-32)), the resulting value is the same for both strategies.

To transform the second expectation value in expression (A-33), note that for an individually optimal strategy we have

$$\begin{aligned}
& e^{-rT_k} p_k (x_{T_k}^i - Q_k^i)^+ \\
=& e^{-rT_k} p_k 1_{\{x_{T_k}^i > Q_k^i\}} (x_{T_k}^i - Q_k^i) \\
=& e^{-rT_k} (S_k^*(T_k) - S_{k+1}^*(T_k)) \left(\int_0^{T_k} (y_t^i + e_t^i - \xi_t^i) dt - \sum_{j=1}^{k} (a_j^i + \int_0^{T_j} \theta_{j,t}^i dt) \right)
\end{aligned} \tag{A-36}$$

due to condition (6) and we further insert the definitions of $x_{T_k}^i$ and Q_k^i. Using this in the second term of expression (A-33), reordering sums and integrals shows that the control variables cancel out for an individually optimal strategy and the resulting value is

$$\mathbb{E}_0 \left\{ \sum_{j=1}^{n} e^{-rT_j} (S_j^*(T_j) - S_{j+1}^*(T_j)) (\int_0^{T_j} (y_t^i + e_t^i) dt - q_j^i) \right\}. \tag{A-37}$$

Since the optimal strategy of the global problem also fulfills condition (6), it results in the same value. Overall, this means that (Θ^*, Ξ^*) solves the individual optimization problems (4), and $S_1^*, \ldots, S_n^*$ are equilibrium permit price processes, which completes the proof the Proposition.

Proof of Proposition 7. We decompose the stochastic optimal control problem (A-1) into n simpler problems as in the proof of Proposition 1, that is we consider the system

(A-38) $$V_k(t,x_t,y_t) = \min_{\xi_k} \mathbb{E}_{T_{k-1}}\Bigg\{\int_t^{T_k} e^{-r(s-T_{k-1})}C(\xi_s)ds + e^{-r(T_k-T_{k-1})}(p_k(x_{T_k}-q_k)^+ + V_{k+1}(T_k,x_{T_k},y_{T_k}))\Bigg\},$$

for $t \in [T_{k-1}, T_k]$ and $k = 1, \ldots, n$, where V_k is the value function of the period k problem, $V_{n+1} = 0$, and ξ_k is the restriction of the strategy ξ to $[T_{k-1}, T_k]$.

Each of the single optimization problems in equation (A-38) can be settled by the standard dynamic programming approach along the lines of Sethi and Thompson (2006).[4] The principle of optimality yields

(A-39) $$V_k(t,x_t,y_t) = \min_{\xi_t} \mathbb{E}_{T_{k-1}}\Big\{e^{-r(t-T_{k-1})}C(\xi_t)dt + V_k(t+dt, x_t+dx_t, y_t+dy_t)\Big\}.$$

On the other hand, by applying Itô's Lemma to $V_k(t,x_t,y_t)$ with dynamics of x_t and y_t as given in equations (8) and (9) we get

(A-40) $$\mathbb{E}_{T_{k-1}}\{dV_k\} = \left(\frac{\partial V_k}{\partial t} + \frac{\partial V_k}{\partial x}(y_t-\xi_t) + \frac{\partial V_k}{\partial y}\mu_y(t) + \frac{1}{2}\frac{\partial^2 V_k}{\partial x^2}\sigma_e^2 + \frac{1}{2}\frac{\partial^2 V_k}{\partial y^2}\sigma_y^2(t)\right)dt.$$

Using this in condition (A-39) leads to the Hamilton-Jacobi-Bellman (HJB) equation

(A-41) $$0 = \min_{\xi_t}\Bigg\{e^{-r(t-T_{k-1})}C(\xi_t) + \frac{\partial V_k}{\partial t} + \frac{\partial V_k}{\partial x}(y_t-\xi_t) + \frac{\partial V_k}{\partial y}\mu_y(t) + \frac{1}{2}\frac{\partial^2 V_k}{\partial x^2}\sigma_e^2 + \frac{1}{2}\frac{\partial^2 V_k}{\partial y^2}\sigma_y^2(t)\Bigg\}.$$

By differentiating the right-hand side with respect to ξ_t and setting the derivative to zero we obtain the solution

(A-42) $$\xi_t = c^{-1}(e^{r(t-T_{k-1})}\frac{\partial V_k}{\partial x}),$$

[4]See also Seifert et al. (2008). Recall that Section 3.3 of Carmona et al. (2013) applies to our problem, and we impose standard regularity conditions on V_k.

where c stands for $\frac{\partial C}{\partial \xi_t}$. By inserting the solution (A-42) into equation (A-41), we finally arrive at the characteristic PDE

$$\text{(A-43)}\quad \begin{aligned} \frac{\partial V_k}{\partial t} &= -e^{-r(t-T_{k-1})} C(c^{-1}(e^{r(t-T_{k-1})} \frac{\partial V_k}{\partial x})) \\ &\quad - \frac{\partial V_k}{\partial x}(y_t - c^{-1}(e^{r(t-T_{k-1})} \frac{\partial V_k}{\partial x})) - \frac{\partial V_k}{\partial y}\mu_y(t) - \frac{1}{2}\frac{\partial^2 V_k}{\partial x^2}\sigma_e^2 - \frac{1}{2}\frac{\partial^2 V_k}{\partial y^2}\sigma_y^2(t), \end{aligned}$$

and the boundary condition

$$\text{(A-44)}\qquad V_k(T_k, x_{T_k}, y_{T_k}) = e^{-r(T_k - T_{k-1})}(p_k(x_{T_k} - q_k)^+ + V_{k+1}(T_k, x_{T_k}, y_{T_k}))$$

follows from equation (A-38). Inserting the specific case of a quadratic abatement cost function according to equation (18) into the equations (A-42) and (A-43) yields the Proposition.

D Simulation of Volatility Smiles

To characterize the volatility smile in emissions markets within our calibrated model, we perform an extensive simulation study of permit option prices for a number of different time points, emissions scenarios, and option maturities. To be precise, emission permit options are typically written on permit futures with the same maturity[5] rather than on the spot permit itself, and the price of a European call option with strike K and maturity $\bar{t}$ is given by the discounted expected payoff in our risk-neutral setting, that is

$$\text{(A-45)}\qquad C(t, \bar{t}, K) = e^{-r(\bar{t}-t)}\mathbb{E}_t\left\{\left(F(\bar{t}, \bar{t}) - K\right)^+\right\}.$$

Given t, $\bar{t}$, x_t, and y_t, we calculate call option prices (A-45) by Monte–Carlo simulation

[5] We abstract from the fact that there are usually a few days between the option's expiry and the maturity date of the futures contract.

and compute the related Black (1976) implied volatility IV_{ATM} at-the-money, i.e., for strike $K = F(t, \bar{t})$, as well as for one strike price above and one below the current futures price, given by $K^{\pm} = F(t, \bar{t})(1 \pm IV_{ATM}\sqrt{\bar{t} - t})$. For each scenario considered, we capture the slope of the volatility smile by $\frac{IV_{K^+} - IV_{K^-}}{K^+ - K^-}$, such that positive (negative) values stand for an upward-(downward-)sloping volatility smile, and values close to zero imply that the smile is almost symmetric.

Given this procedure, we compute the slope of the volatility smile for 150 different scenarios of realized and prevailing emissions, time points, and option maturities. For the sake of brevity we report results only for the setting of two compliance periods and low abatement costs, but we obtain qualitatively similar results for all other cases. As Figure 3 illustrates, the volatility smile in emissions markets is downward-sloping or almost symmetric for the vast majority of emissions scenarios, while there are no cases with a clearly upward-sloping smile. In fact, the slope is negative for 141 of the 150 scenarios. The strongest upward-slope is 0.002 which is, compared to the strongest downward-slope of -0.0166, very close to a symmetric smile. Further, the downward-slope of the volatility smile is strongest for scenarios of very low emissions. These results are in accordance with the negative relation of emission permit prices and volatilities and reveal how this pattern translates to option prices, which we summarize by Prediction 3.

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