Internet Appendix for

Staying on Top of the Curve: A Cascade Model of Term Structure Dynamics

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A. Proofs of Section II (Cascade Models)

Throughout the appendix, we make explicit the dependence of the response function with respect to the number of factors:

(IA1) $$a_{j,n}(\tau) = \frac{K_j * \cdots * K_n(\tau)}{\kappa_j} = \sum_{i=j}^{n} \alpha_{i,j,n} K_i(\tau)$$

for all $\tau \in \mathbb{R}_+$ and $j \leq n$.

Proof of Proposition 1

We prove by induction that for all $n \geq 1$,

(IA2) $$x_{n,t} \quad = \quad \theta_r + \sum_{j=1}^{n} a_{j,n}(t)(x_{j,0} - \theta_r) + \sum_{j=1}^{n} \sigma_j \int_0^t a_{j,n}(t-s) dW_{j,s},$$

where $a_{j,n}(\cdot)$ is given by (IA1).

Consider the one-factor case ($n = 1$). We infer from Assumption 1 and Ito's lemma that

$$d(e^{\kappa_1 t} x_{1,t}) = e^{\kappa_1 t}(\kappa_1 \theta_r \, dt + \sigma_1 dW_{1,t}).$$

Integrating both sides and then dividing by $e^{\kappa_1 t}$, we obtain that property (IA2) holds for $n = 1$.

We now assume that property (IA2) holds for an n-factor structure. Ito's lemma implies $d(e^{\kappa_{n+1} t} x_{n+1,t}) = e^{\kappa_{n+1} t}(\kappa_{n+1} x_{n,t} dt + \sigma_{n+1} dW_{n+1,t})$. We integrate both sides and divide them by

$e^{\kappa_{n+1}t}$, which yields:

$$x_{n+1,t} = e^{-\kappa_{n+1}t}x_{n+1,0} + \int_0^t \kappa_{n+1}e^{-\kappa_{n+1}(t-s)}x_{n,s}\,ds + \sigma_{n+1}\int_0^t e^{-\kappa_{n+1}(t-s)}dW_{n+1,s}.$$

Substitute out $x_{n,s}$ according to equation (IA2),

$$\begin{aligned}\int_0^t \kappa_{n+1}e^{-\kappa_{n+1}(t-s)}x_{n,s}\,ds &= \theta_r(1-e^{-\kappa_{n+1}t}) + \sum_{j=1}^{n}(x_{j,0}-\theta_r)a_{j,n+1}(t) \\ &\quad + \sum_{j=1}^{n}\sigma_j\int_0^t \kappa_{n+1}e^{-\kappa_{n+1}(t-s)}\left[\int_0^s a_{j,n}(s-u)dW_{j,u}\right]ds.\end{aligned}$$

We apply Fubini's theorem for stochastic integrals to the last terms, and conclude that property (IA2) holds for the $n+1$-factor structure.

Proof of Proposition 2

Lemma IA1. If $\tau > 0$ is a local optimum of the response function $a_{j,n}$, then $a''_{j,n}(\tau) = \kappa_{j+1}a'_{j+1,n}(\tau)$.

Proof. We know from (IA1) that

$$a_{j,n}(\tau) = \frac{\kappa_{j+1}}{\kappa_j}\int_0^\tau K_j(\tau-s)a_{j+1,n}(s)ds.$$

We differentiate this relation with respect to τ and obtain $a'_{j,n}(\tau) = \kappa_{j+1}a_{j+1,n}(\tau) - \kappa_j a_{j,n}(\tau)$ for all τ, which in turn implies $a''_{j,n}(\tau) = \kappa_{j+1}a'_{j+1,n}(\tau) - \kappa_j a'_{j,n}(\tau)$ for all τ. An interior local optimum of $a_{j,n}$ therefore satisfies $a''_{j,n}(\tau) = \kappa_{j+1}a'_{j+1,n}(\tau)$. □

Single-peakedness. We now show by backward induction that for all $j = n-1, \cdots, 1$, the function

$a_{j,n}(\cdot)$ is single peaked and reaches a maximum at $\bar{\tau}_{j,n}$. Furthermore, $\bar{\tau}_{1,n} \geq \cdots \geq \bar{\tau}_{n,n}$.

The property holds for $j = n-1$, as is shown in the main text. Assume now that the property holds for $j+1 \leq n$. Let $\bar{\tau}_{j,n}$ denote the smallest local maximum of $a_{j,n}$. We know that $a''_{j,n}(\bar{\tau}_{j,n}) \leq 0$, and that $a''_{j,n}(\bar{\tau}_{j,n}) = \kappa_{j+1} a'_{j+1,n}(\bar{\tau}_{j,n})$. Hence $a'_{j+1,n}(\bar{\tau}_{j,n}) \leq 0$, which implies that $\bar{\tau}_{j,n} \geq \bar{\tau}_{j+1,n}$. If the function $a_{j,n}(\cdot)$ is nonmonotonic on $[\bar{\tau}_{j,n}, +\infty)$, there exists a local minimum $\tau > \bar{\tau}_{j,n}$. Since $\tau > \bar{\tau}_{j,n} \geq \bar{\tau}_{j+1,n}$, we know that $a''_{j,n}(\tau) = \kappa_{j+1} a'_{j+1,n}(\tau) < 0$, which is a contradiction. We conclude that $a_{j,n}(\cdot)$ is single peaked and reaches a maximum at $\bar{\tau}_{j,n} \geq \bar{\tau}_{j+1,n}$.

Closed-form expressions and upper bound. The analytical solutions and proofs for the convolutions of exponential density functions are given, among other places, in Akkouchi (2008). The inequality $\sum_{j=1}^{n} a_{j,n}(\tau) \leq 1$ can be proved for all n by a forward recursion. Starting at $n = 1$, the inequality holds since $a_{1,1}(t) = e^{-\kappa_1 t} \leq 1$ for $t \geq 0$. We now assume that the inequality holds for an $(n-1)$-factor structure. We infer that

$$\sum_{j=1}^{n} a_{j,n}(t) = e^{-\kappa_n t} + \int_0^t \kappa_n e^{-\kappa_n (t-s)} \sum_{j=1}^{n-1} a_{j,n-1}(s) ds.$$

The inequality $\sum_{j=1}^{n-1} a_{j,n-1}(s) \leq 1$ for all s implies that $\sum_{j=1}^{n} a_{j,n}(t) \leq 1$.

Proof of Proposition 3

Under the affine specification, the exponential-affine bond pricing coefficients satisfy the following system of ordinary differential equations (Duffie and Kan (1996)):

$$\text{(IA3)} \qquad b'(\tau) = \mathbf{e}_n - \kappa^{\mathbb{Q}\top} b(\tau),$$

$$\text{(IA4)} \qquad c'(\tau) = b(\tau)^\top \kappa^{\mathbb{Q}} \theta^{\mathbb{Q}} - \frac{1}{2} b(\tau)^\top \Sigma b(\tau),$$

with initial conditions $b(\tau) = 0$ and $c(\tau) = 0$, where $\mathbf{e}_n$ denotes a vector with the value one in the nth position, and 0 otherwise. We now derive the solutions $b(\tau)$ and $c(\tau)$ when risk premia are constant ($\Lambda = 0$).

Price loadings. The matrices κ and $\kappa^{\mathbb{Q}}$ coincide and, by equation (IA3), the function $b(\tau) = [b_{1,n}(\tau), \cdots, b_{n,n}(\tau)]^\top$ satisfies

$$\text{(IA5)} \qquad b'_{j,n}(\tau) = -\kappa_j b_{j,n}(\tau) + \kappa_{j+1} b_{j+1,n}(\tau), \quad j = 1, ..., n-1,$$

$$\text{(IA6)} \qquad b'_{n,n}(\tau) = 1 - \kappa_n b_{n,n}(\tau).$$

Equation (IA6) implies that $b_{n,n}(\tau) = (1 - e^{-\kappa_n \tau})/\kappa_n = \int_0^\tau a_{n,n}(s) ds$. We infer from (IA5) that for all $j \in \{1, \cdots, n-1\}$,

$$\frac{d}{d\tau}\left[e^{\kappa_j \tau} b_{j,n}(\tau)\right] = e^{\kappa_j \tau} \kappa_{j+1} b_{j+1,n}(\tau),$$

and thus $b_{j,n} = (\kappa_{j+1}/\kappa_j) K_j * b_{j+1,n}$. By a simple recursion, the function $b_{j,n}$ is equal to $(\kappa_n/\kappa_j) K_j *$

$\ldots * K_{n-1} * b_{n,n}$, and its derivative is therefore

$$b'_{j,n} = \kappa_j^{-1} K_j * \ldots * K_{n-1} * K_n = a_{j,n}.$$

We conclude that

$$\text{(IA7)} \qquad b_{j,n}(\tau) = \int_0^\tau a_{j,n}(\tau')d\tau' = \sum_{i=j}^{n} \alpha_{i,j,n}\left(1 - e^{-\kappa_i \tau}\right),$$

for all $j \in \{1, ..., n-1\}$.

Intercept. By (IA4), the function $c_n(\tau)$ satisfies

$$c_n(\tau) = \theta_r \kappa_1 \int_0^\tau b_{1,n}(\tau)\, d\tau - \sum_{j=1}^{n} \gamma_j \sigma_j^2 \int_0^\tau b_{j,n}(s)\, ds - \frac{1}{2}\sum_{j=1}^{n} \sigma_j^2 \int_0^\tau b_{j,n}^2(s)\, ds.$$

Since by (IA7)

$$\int_0^\tau b_{j,n}(s)\, ds = \sum_{i=j}^{n} \alpha_{i,j,n}\left(\tau - \frac{1 - e^{-\kappa_i \tau}}{\kappa_i}\right),$$
$$\int_0^\tau b_{j,n}^2(s) ds = \sum_{i=j}^{n}\sum_{k=j}^{n} \alpha_{i,j,n}\alpha_{k,j,n}\left[\tau - \frac{1 - e^{-\kappa_i \tau}}{\kappa_i} - \frac{1 - e^{-\kappa_k \tau}}{\kappa_k} + \frac{1 - e^{-(\kappa_i + \kappa_k)\tau}}{\kappa_i + \kappa_k}\right],$$

the expression provided for $c_n(\tau)$ holds.

Long-run level of the state vector under $\mathbb{Q}$. As we explain in Section II.C of the main text, the log-run mean of the state vector under $\mathbb{Q}$ is given by

$$\text{(IA8)} \qquad \theta^{\mathbb{Q}} = (\kappa^{\mathbb{Q}})^{-1}(\kappa_1 \theta_r - \gamma_1 \sigma_1^2, -\gamma_2 \sigma_2^2, \ldots, -\gamma_n \sigma_n^2)^\top$$

under any affine risk premium specification. When risk premia are constant, the matrix κ^{-1} is lower triangular matrix with elements $(\kappa^{-1})_{i,j} = \kappa_j^{-1}$ if $j \leq i$. We then easily verify that equation (IA8) reduces to

$$\theta^{\mathbb{Q}} = \left(\theta_r - \gamma_1\sigma_1^2/\kappa_1, \theta_r - \gamma_1\sigma_1^2/\kappa_1 - \gamma_2\sigma_2^2/\kappa_2, ..., \theta_r - \sum_{i=1}^{n} \gamma_i\sigma_i^2/\kappa_i\right)^{\top},$$

and we conclude that the proposition holds.

Long-Run Yield under Constant Risk Premia

We infer from Proposition 3 that

$$y_\infty = \lim_{\tau\to+\infty} \frac{c(\tau)}{\tau} = \theta_r\kappa_1 \sum_{i=1}^{n} \alpha_{i,1,n} - \sum_{j=1}^{n} \gamma_j\sigma_j^2 \sum_{i=j}^{n} \alpha_{i,j,n} - \sum_{j=1}^{n} \frac{\sigma_j^2}{2} \sum_{i=j}^{n}\sum_{k=j}^{n} \alpha_{i,j,n}\alpha_{k,j,n}.$$

We integrate (IA1) and obtain:

$$\text{(IA9)} \qquad \int_0^{+\infty} a_{j,n}(\tau)d\tau = \frac{1}{\kappa_j} = \sum_{i=j}^{n} \alpha_{i,j,n}.$$

The long-run yield therefore satisfies

$$y_\infty \quad = \quad \theta_r - \sum_{j=1}^{n} \frac{\sigma_j^2}{\kappa_j^2}\left(\gamma_j\kappa_j + \frac{1}{2}\right).$$

B. Proofs of Section III (Convergence)

Proof of Proposition 4

To develop intuition, we begin by showing pointwise convergence of the response functions. Let $\{E_\ell\}_{\ell=1}^{\infty}$ denote a sequence of independent random variables, where E_ℓ has probability density function (p.d.f.) K_ℓ for each ℓ. For a given j, the characteristic function of $E_j + \ldots + E_n$ $(n \geq j)$ is

$$\psi_{j,n}(\xi) = \mathbb{E}e^{i\xi(E_j+\ldots+E_n)} = \left[\prod_{\ell=j}^{n}\left(1 - \frac{i\xi}{\kappa_\ell}\right)\right]^{-1}.$$

Since $\sum_{\ell=j}^{\infty} \kappa_\ell^{-1} < \infty$, the sequence of characteristic functions $\psi_{j,n}(\xi)$ converges pointwise to

$$\psi_{j,\infty}(\xi) = \left[\prod_{\ell=j}^{\infty}\left(1 - \frac{i\xi}{\kappa_\ell}\right)\right]^{-1}.$$

The limiting function $\psi_{j,\infty}(\xi)$ is continuous at $\xi = 0$ and is therefore the characteristic function of a probability distribution P_∞ on the real line. Since $\psi_{j,\infty}(\xi)$ is also integrable, the distribution P_∞ is absolutely continuous with respect to Lebesgue measure, and we denote by g_∞ its p.d.f. By Lebesgue's dominated convergence theorem, the sequence of p.d.f.'s $K_j * \cdots * K_n$ converges pointwise to g_∞. The sequence of response functions $\{a_{j,n}(\tau)\}_n$ therefore has a pointwise limit as $n \to \infty$ for every $\tau \in \mathbb{R}_+$.

Uniform Convergence of the Response Functions

The geometric progression of the adjustment speeds allows us to derive an analytical expression for the limiting response function, as we now show. By Proposition 2, the coefficients $\alpha_{i,j,n}$ $(1 \leq j \leq$

$i \leq n)$ satisfy:

$$\text{(IA10)} \qquad \alpha_{i,j,n} = \frac{1}{\kappa_j} \frac{(-1)^{i-j} b^{-(i-j)(i-j+1)/2}}{F(n-i)F(i-j)},$$

where $F(0) = 1$ and $F(k) = (1-b^{-1})\ldots(1-b^{-k})$ for all $k \geq 1$. The sequence $\{F(k)\}$ is decreasing, converges to a strictly positive limit F_∞, and satisfies

$$\text{(IA11)} \qquad \frac{F(k) - F_\infty}{F(k)} = 1 - \prod_{i=k+1}^{+\infty} \left(1 - b^{-i}\right) \leq \sum_{i=k+1}^{+\infty} b^{-i} = \frac{b^{-k}}{b-1}$$

for all k. As n goes to infinity, the coefficient $\alpha_{i,j,n}$ therefore converges to

$$\text{(IA12)} \qquad \bar{\alpha}_{i,j} = \frac{1}{\kappa_j} \frac{(-1)^{i-j} b^{-(i-j)(i-j+1)/2}}{F_\infty F(i-j)},$$

for every i and j, $1 \leq j \leq i$. Furthermore, we obtain from (IA11) the useful inequalities

$$\text{(IA13)} \qquad |\alpha_{i,j,n} - \bar{\alpha}_{i,j}| \leq \frac{b^{-(i-j)(i-j-1)/2}}{(1-b^{-1})\,\kappa_1 F_\infty^2} b^{-n},$$

$$\text{(IA14)} \qquad \sum_{i=j}^{n} \left|\alpha_{i,j,n} - \bar{\alpha}_{i,j}\right| \leq \frac{C_0}{(1-b^{-1})\,\kappa_1} b^{-n},$$

$$\text{(IA15)} \qquad \sum_{i=j}^{\infty} \kappa_i |\bar{\alpha}_{i,j}| \leq C_0,,$$

where $C_0 = F_\infty^{-2} \sum_{\ell=0}^{\infty} b^{-\ell(\ell-1)/2}$.

We now show that the sequence of response functions of factor j, $\{a_{j,n}(\tau)\}_{n=j}^{\infty}$, converges uniformly on $\mathbb{R}_+$ to

$$\bar{a}_j(\tau) = \sum_{i=j}^{+\infty} \bar{\alpha}_{i,j} K_i(\tau). \tag{IA16}$$

By (IA15), the function $\bar{a}_j(\tau)$ is well-defined for every τ. Since

$$a_{j,n}(\tau) - \bar{a}_j(\tau) = \sum_{i=j}^{n} (\alpha_{i,j,n} - \bar{\alpha}_{i,j}) K_i(\tau) - \sum_{i=n+1}^{\infty} \bar{\alpha}_{i,j} K_i(\tau), \tag{IA17}$$

the inequality $\left|a_{j,n}(\tau) - \bar{a}_j(\tau)\right| \leq \sum_{i=j}^{n} \kappa_i |\alpha_{i,j,n} - \bar{\alpha}_{i,j}| + \sum_{i=n+1}^{\infty} \kappa_i |\bar{\alpha}_{i,j}|$ holds for all $\tau \in \mathbb{R}_+$. We infer from (IA12) and (IA13) that

$$\left|a_{j,n}(\tau) - \bar{a}_j(\tau)\right| \leq C_1 b^{j-n}, \tag{IA18}$$

where $C_1 = \left(b + \sum_{\ell=0}^{\infty} b^{-\ell(\ell-3)/2}\right) / [(b-1)F_\infty^2]$. We conclude that convergence is uniform on $\mathbb{R}_+$. Furthermore, since each function $a_{j,n}$ is continuous, the limiting response function is continuous.

L^2 Convergence of the Response Functions

Lemma IA2. There exists $C_2 \in \mathbb{R}_+$ such that

$$\sum_{i=\ell}^{+\infty} \left|\bar{\alpha}_{i,j}\right| \|K_i\|_2 \leq C_2\, b^{j/2-\ell} \tag{IA19}$$

for all j and ℓ, $1 \leq j \leq \ell$. The limiting response function of factor j is therefore square-integrable and satisfies $\left\|\bar{a}_j\right\|_2 \leq C_2\, b^{-j/2}$.

Proof. Since $\|K_i\|_2 = \sqrt{\kappa_i/2}$, the series $\sum_{i=\ell}^{+\infty} \left|\bar{\alpha}_{i,j}\right| \|K_i\|_2$ is bounded above by

$$\frac{1}{F_\infty^2\sqrt{\kappa_j}}\sum_{i=\ell}^{+\infty} b^{-(i-j)^2/2} \leq \frac{1}{F_\infty^2\sqrt{\kappa_j}}\sum_{i=\ell}^{+\infty} b^{-(i-j)+1/2} = \frac{b}{F_\infty^2\sqrt{\kappa_1}b^{j/2}}\frac{b^{-(\ell-j)}}{1-b^{-1}}.$$

We conclude that the lemma holds with $C_2 = b^2/[(b-1)\sqrt{\kappa_1}F_\infty^2]$. □

We also show:

Lemma IA3. There exists $C_3 \in \mathbb{R}_+$ such that $\|a_{j,n} - \bar{a}_j\|_2 \leq C_3\, b^{j/2-n}$ for all j and n, $1 \leq j \leq n$.

Proof. Equation (IA17) and Lemma IA2 imply that $\|a_{j,n} - \bar{a}_j\|_2 \leq \sum_{i=j}^{n} |\alpha_{i,j,n} - \bar{\alpha}_{i,j}|\sqrt{\kappa_i} + C_2\, b^{j/2-n}$. We infer from (IA13) that

$$\text{(IA20)} \qquad \sum_{i=j}^{n} |\alpha_{i,j,n} - \bar{\alpha}_{i,j}|\sqrt{\kappa_i} \leq C_3'\, b^{j/2-n},$$

where $C_3' = \sum_{\ell=0}^{\infty} b^{-(\ell-1)^2/2} / \left[(1-b^{-1})\sqrt{\kappa_1}F_\infty^2\right]$. Letting $C_3 = C_3' + C_2$, we conclude that the inequality $\|a_{j,n} - \bar{a}_j\|_2 \leq C_3\, b^{j/2-n}$ holds. □

The sequence of response functions $\{a_{j,n}\}_{n=j}^{\infty}$ therefore converges to $\bar{a}_j$ in L^2.

Scaling of the Limiting Response Functions

The exponential densities satisfy: $K_{i+1}(\tau) = bK_i(b\tau)$ for all $i \geq 1$. Furthermore by (IA10), the coefficients of the response functions satisfy $\alpha_{i+1,j+1,n+1} = \alpha_{i,j,n}/b$ for all i, j, n. Hence

$$a_{j+1,n+1}(\tau) = \sum_{i=j}^{n} \alpha_{i+1,j+1,n+1} K_{i+1}(\tau) = \sum_{i=j}^{n} b^{-1}\alpha_{i,j,n}\, bK_i(b\tau) = a_{j,n}(b\tau).$$

We let n go to infinity and conclude that $\bar{a}_{j+1}(\tau) = \bar{a}_j(b\tau)$.

Proof of Proposition 5

L^2 Convergence of the Short Rate Process

The processes

$$y_{\infty,t} = \sum_{j=1}^{+\infty} \sigma_j \int_0^t \bar{a}_j(t-s)dW_{j,s} \quad \text{and} \quad z_{\infty,t} = \theta_r + \sum_{j=1}^{\infty} \bar{a}_j(t)(x_{j,0} - \theta_r)$$

are well-defined under Conditions 1 and 2. Indeed, Lemma IA2 implies that $\sum_{j=1}^{+\infty} \sigma_j^2 \|\bar{a}_j\|_2^2 \leq C_2^2 \sum_{j=1}^{+\infty} b^{-j}\sigma_j^2 < \infty$ and $\sum_{j=1}^{\infty} \|\bar{a}_j\|_2 \, |x_{j,0} - \theta_r| \leq C_2 \sum_{j=1}^{\infty} b^{-j/2}|x_{j,0} - \theta_r| < \infty$. Consider

$$y_{n,t} = \sum_{j=1}^{n} \sigma_j \int_0^t a_{j,n}(t-s)dW_{j,s} \quad \text{and} \quad z_{n,t} = \theta_r + \sum_{j=1}^{n} a_{j,n}(t)(x_{j,0} - \theta_r).$$

We now verify that the sequences y_n and z_n respectively converge to y_∞ and z_∞ in $L^2(\Omega \times [0,T])$.

Convergence of the stochastic component y_n. Since

$$y_{n,t} - y_{\infty,t} = \sum_{j=1}^{n} \sigma_j \int_0^t (a_{j,n} - \bar{a}_j)(t-s)dW_{j,s} - \sum_{j=n+1}^{+\infty} \sigma_j \int_0^t \bar{a}_j(t-s)dW_{j,s},$$

we know that

$$\mathbb{E}^{\mathbb{P}}\left[(y_{n,t} - y_{\infty,t})^2\right] \leq \sum_{j=1}^{n} \sigma_j^2 \|a_{j,n} - \bar{a}_j\|_2^2 + \sum_{j=n+1}^{+\infty} \sigma_j^2 \|\bar{a}_j\|_2^2,$$

and infer from Lemmas IA2 and IA3 that:

$$\mathbb{E}^{\mathbb{P}}\left[(y_{n,t}-y_{\infty,t})^2\right] \leq C_3^2 \sum_{j=1}^{n} b^{j-2n}\sigma_j^2 + C_2^2 \sum_{j=n+1}^{+\infty} b^{-j}\sigma_j^2.$$

We observe that $\sum_{j=1}^{n} b^{j-2n}\sigma_j^2 \leq b^{-n}\left(\sum_{j=1}^{\infty} b^{-j}\sigma_j^2\right) + \sum_{j=\lfloor n/2\rfloor+1}^{n} b^{-j}\sigma_j^2$, where $\lfloor\cdot\rfloor$ denotes the integer part of a real number. We conclude that under Condition 1, $\|y_n - y_\infty\|_{L^2(\Omega\times[0,T])}$ converges to 0.

Convergence of the deterministic component z_n. We observe that

$$\|z_n - z_\infty\|_{L^2(\Omega\times[0,T])} \leq \sum_{j=1}^{n} \|a_{j,n}-\bar{a}_j\|_2\,|x_{j,0}-\theta_r| + \sum_{j=n+1}^{\infty} \|\bar{a}_j\|_2\,|x_{j,0}-\theta_r|.$$

Lemmas IA2 and IA3 imply that

$$\|z_n - z_\infty\|_{L^2(\Omega\times[0,T])} \leq C_3 \sum_{j=1}^{n} b^{j/2-n}|x_{j,0}-\theta_r| + C_2 \sum_{j=n+1}^{\infty} b^{-j/2}|x_{j,0}-\theta_r|.$$

The inequalities $\sum_{j=1}^{\lfloor n/2\rfloor} b^{j/2-n}|x_{j,0}-\theta_r| \leq b^{-n/2}\sum_{j=1}^{\infty} b^{-j/2}|x_{j,0}-\theta_r|$ and $\sum_{j=\lfloor n/2\rfloor+1}^{n} b^{j/2-n}|x_{j,0}-\theta_r| \leq \sum_{j=\lfloor n/2\rfloor}^{n} b^{-j/2}|x_{j,0}-\theta_r|$, combined with Condition 2, imply that $\|z_n - z_\infty\|_{L^2(\Omega\times[0,T])}$ converges to 0. We conclude that the process $x_n = y_n + z_n$ converges to x_∞ in $L^2(\Omega\times[0,T])$.

Continuity of the Sample Paths

We verify that the stochastic component y_∞ and the deterministic component z_∞ are both continuous.

Stochastic component y_∞. We seek to show that y_∞ satisfies the Kolmogorov continuity condition.

The increment $y_{\infty,t+h} - y_{\infty,t}$ has variance

$$\mathbb{E}^{\mathbb{P}}\left[\left(y_{\infty,t+h} - y_{\infty,t}\right)^2\right] = \sum_{j=1}^{+\infty} \sigma_j^2 \left\{ \int_0^h [\bar{a}_j(s)]^2 ds + \int_0^t [\bar{a}_j(s+h) - \bar{a}_j(s)]^2 ds \right\}.$$

By (IA16), $\int_0^h [\bar{a}_j(s)]^2 ds = \sum_{i=j}^{\infty} \sum_{i'=j}^{\infty} \bar{\alpha}_{i,j} \bar{\alpha}_{i',j} (\kappa_i + \kappa_{i'})^{-1} \kappa_i \kappa_{i'} [1 - e^{-(\kappa_i + \kappa_{i'})h}]$. Note that $1 - e^{-x} \leq x^{\zeta}$ for all $x \geq 0$ and $0 < \zeta < 1$. We assume without loss of generality that $0 < \varepsilon < 1/2$ and let $\zeta = 1 - \varepsilon$. We infer that

$$\text{(IA21)} \qquad \int_0^h [\bar{a}_j(s)]^2 ds \leq h^{1-\varepsilon} \sum_{i=j}^{\infty} \sum_{i'=j}^{\infty} \frac{\kappa_i |\bar{\alpha}_{i,j}| \kappa_{i'} |\bar{\alpha}_{i',j}|}{(\kappa_i + \kappa_{i'})^{\varepsilon}} \leq C_0^2 \kappa_j^{-\varepsilon} h^{1-\varepsilon},$$

where the last inequality builds on (IA15) and $\kappa_i + \kappa_{i'} \geq \kappa_j$.

Since equation (IA16) implies that $\bar{a}_j(t+h-s) - \bar{a}_j(t-s) = \sum_{i=j}^{\infty} \bar{\alpha}_{i,j} \kappa_i e^{-\kappa_i (t-s)} (e^{-\kappa_i h} - 1)$, we know that

$$\int_0^{+\infty} [\bar{a}_j(s+h) - \bar{a}_j(s)]^2 ds = \sum_{i=j}^{\infty} \sum_{i'=j}^{\infty} \kappa_i \bar{\alpha}_{i,j} \kappa_{i'} \bar{\alpha}_{i',j} \frac{(1 - e^{-\kappa_i h})(1 - e^{-\kappa_{i'} h})}{\kappa_i + \kappa_{i'}}.$$

Since $1 - e^{-\kappa_i h} \leq (\kappa_i h)^{\frac{1-\varepsilon}{2}}$, $1 - e^{-\kappa_{i'} h} \leq (\kappa_{i'} h)^{\frac{1-\varepsilon}{2}}$, and $(\kappa_i \kappa_{i'})^{1/2} \leq \kappa_i + \kappa_{i'}$, we infer: $(\kappa_i + \kappa_{i'})^{-1}(1 - e^{-\kappa_i h})(1 - e^{-\kappa_{i'} h}) \leq (\kappa_i + \kappa_{i'})^{-\varepsilon} h^{1-\varepsilon} \leq \kappa_j^{-\varepsilon} h^{1-\varepsilon}$. Hence:

$$\text{(IA22)} \qquad \int_0^{+\infty} [\bar{a}_j(s+h) - \bar{a}_j(s)]^2 ds \leq C_0^2 \kappa_j^{-\varepsilon} h^{1-\varepsilon}.$$

Combining (IA21) and (IA22), we infer that $\mathbb{E}^{\mathbb{P}}\left[\left(y_{\infty,t+h} - y_{\infty,t}\right)^2\right] \leq 2C_0^2 h^{1-\varepsilon} \sum_{j=1}^{\infty} \kappa_j^{-\varepsilon} \sigma_j^2$. Since

$y_{\infty,t+h} - y_{\infty,t}$ is Gaussian, its fourth moment satisfies

$$\mathbb{E}^{\mathbb{P}}\left[\left(y_{\infty,t+h} - y_{\infty,t}\right)^4\right] = 3\left\{\mathbb{E}^{\mathbb{P}}\left[\left(y_{\infty,t+h} - y_{\infty,t}\right)^2\right]\right\}^2,$$

and therefore

$$\mathbb{E}^{\mathbb{P}}\left[\left(y_{\infty,t+h} - y_{\infty,t}\right)^4\right] \leq 12 C_0^4 h^{2-2\varepsilon}\left(\sum_{j=1}^{\infty} \kappa_j^{-\varepsilon}\sigma_j^2\right)^2. \tag{IA23}$$

Since $\varepsilon < 1/2$ by assumption, the process y_∞ satisfies the Kolmogorov continuity condition.

Deterministic component z_∞. We observe that for any $t \geq 0$,

$$|z_{n,t} - z_{\infty,t}| \leq \sum_{j=1}^{n} |a_{j,n}(t) - \bar{a}_j(t)|\,|x_{j,0} - \theta_r| + \sum_{j=n+1}^{\infty} |\bar{a}_j(t)|\,|x_{j,0} - \theta_r|. \tag{IA24}$$

Let $C_x = \sup_{1\leq j<\infty}(b^{\eta j}|x_{j,0} - \theta_r|)$. The inequalities $|\bar{a}_j(t)| \leq C_0$ and (IA18) imply

$$|z_{n,t} - z_{\infty,t}| \leq C_x C_1 b^{-n} \sum_{j=1}^{n} b^{(1-\eta)j} + C_x C_0 \sum_{j=n+1}^{\infty} b^{-\eta j}$$

and therefore

$$|z_{n,t} - z_{\infty,t}| \leq C_x\left(C_1 n b^{-\eta n} + \frac{C_0 b^{-\eta n}}{b^{\eta} - 1}\right)$$

The sequence z_n converges uniformly to z_∞ on the time interval $[0,T]$. Since each function z_n is continuous, we conclude that the function z_∞ is continuous on $[0,T]$.

Proof of Proposition 6

We begin by showing three useful lemmas.

Lemma IA4. The price loadings $b_{j,n}(\tau)$, $j \leq n$, of the cascade DTSM with n factors satisfy

$$|b_{j,n}(\tau) - \bar{b}_j(\tau)| \leq C_4 b^{-n}, \tag{IA25}$$

for every $\tau \in \mathbb{R}_+$, where $C_4 = \kappa_1^{-1} F_\infty^{-2} (1-b^{-1})^{-1} \left[1 + \sum_{\ell=0}^{\infty} b^{-\ell(\ell-1)/2}\right]$. For every $j \geq 1$, the sequence $\{b_{j,n}(\cdot)\}_{n=j}^{\infty}$ therefore converges uniformly to $\bar{b}_j(\cdot)$ on $\mathbb{R}_+$.

Proof. Proposition 3 and equation (IA17) imply that

$$|b_{j,n}(\tau) - \bar{b}_j(\tau)| \leq \int_0^\tau |a_{j,n}(\tau') - \bar{a}_j(\tau')| d\tau' \leq \sum_{i=j}^{n} |\alpha_{i,j,n} - \bar{\alpha}_{i,j}| + \sum_{i=n+1}^{\infty} |\bar{\alpha}_{i,j}|.$$

Inequality (IA12) implies that

$$\sum_{i=n+1}^{\infty} |\bar{\alpha}_{i,j}| \leq \frac{1}{\kappa_j F_\infty^2} \sum_{i=n+1}^{\infty} b^{-(i-j)(i-j+1)/2} \leq \frac{b^{-n}}{(1-b^{-1})\, \kappa_1 F_\infty^2}. \tag{IA26}$$

We infer from (IA14) and (IA26) that (IA25) holds. Hence $\{b_{j,n}(\cdot)\}_{n=j}^{\infty}$ converges uniformly to $\bar{b}_j(\cdot)$ on $\mathbb{R}_+$. □

Lemma IA5. The limiting function $\bar{b}_j(\tau)$ is nonnegative and monotonically increases from 0 to $1/\kappa_j$ as τ varies from 0 to $+\infty$. Furthermore,

$$\sum_{j=n+1}^{\infty} |\bar{b}_j(\tau)| \leq \frac{b^{-n}}{(1-b^{-1})\, \kappa_1}$$

for every $\tau \in \mathbb{R}_+$ *and* $n \geq 0$.

Proof. By Proposition 1, Proposition 3, and Lemma IA4, the limiting function $\bar{b}_j(\tau) = \int_0^\tau \bar{a}_j(\tau) d\tau$ is nonnegative and increases from 0 to $1/\kappa_j$. Hence $\sum_{j=n+1}^\infty |\bar{b}_j(\tau)| \leq \sum_{j=n+1}^\infty \kappa_j^{-1}$ and we conclude that the lemma holds. $\square$

We next turn to the limit of the intercept $c_n(\tau)$ defined in Proposition 3. We infer from (IA9) that

$$\begin{aligned} c_n(\tau) &= \left[\theta_r - \sum_{j=1}^n \frac{\sigma_j^2}{\kappa_j^2}\left(\gamma_j \kappa_j + \frac{1}{2}\right)\right]\tau + \sum_{i=1}^n c_{i,n} \frac{1 - e^{-\kappa_i \tau}}{\kappa_i} \\ &\quad - \sum_{j=1}^n \frac{\sigma_j^2}{2} \sum_{i=j}^n \sum_{k=j}^n \alpha_{i,j,n} \alpha_{k,j,n} \left[\frac{1 - e^{-(\kappa_i + \kappa_k)\tau}}{\kappa_i + \kappa_k}\right], \end{aligned}$$

where

$$c_{i,n} = -\theta_r \kappa_1 \alpha_{i,1,n} + \sum_{j=1}^i \sigma_j^2 \left(\gamma_j + \frac{1}{\kappa_j}\right) \alpha_{i,j,n}$$

for every $i \in \{1, \ldots, n\}$. Let $\bar{c}_i = -\theta_r \kappa_1 \bar{\alpha}_{i,1} + \sum_{j=1}^i \sigma_j^2 \left(\gamma_j + 1/\kappa_j\right) \bar{\alpha}_{i,j}$.

Lemma IA6. Under Conditions 1 and 3, the following properties hold:

$$\sum_{i=1}^\infty |\bar{c}_i| < \infty \quad \textit{and} \quad \sum_{j=1}^\infty \sigma_j^2 \sum_{i=j}^\infty \sum_{k=j}^\infty \frac{|\bar{\alpha}_{i,j}|\,|\bar{\alpha}_{k,j}|}{\kappa_i + \kappa_k} < \infty.$$

Furthermore, $\sum_{i=1}^n |c_{i,n} - \bar{c}_i|/\kappa_i$ *and* $\sum_{j=1}^n \sigma_j^2 \sum_{i=j}^n \sum_{k=j}^n |\alpha_{i,j,n}\,\alpha_{k,j,n} - \bar{\alpha}_{i,j}\,\bar{\alpha}_{k,j}|/(\kappa_i + \kappa_k)$ *converge to*

0 as n goes to infinity. Let

$$\bar{c}(\tau)=\left[\theta_r-\sum_{j=1}^{\infty}\frac{\sigma_j^2}{\kappa_j^2}\left(\gamma_j\kappa_j+\frac{1}{2}\right)\right]\tau+\sum_{i=1}^{\infty}\bar{c}_i\frac{1-e^{-\kappa_i\tau}}{\kappa_i}-\sum_{j=1}^{\infty}\frac{\sigma_j^2}{2}\sum_{i=j}^{\infty}\sum_{k=j}^{\infty}\bar{\alpha}_{i,j}\bar{\alpha}_{k,j}\frac{1-e^{-(\kappa_i+\kappa_k)\tau}}{\kappa_i+\kappa_k}.$$

There exists a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ *converging to 0 such that*

$$|c_n(\tau)-\bar{c}(\tau)| \quad \leq \quad \varepsilon_n$$

for every $n\geq 1$ *and* $\tau\in[0,T]$, *so that the sequence* $c_n(\cdot)$ *converges uniformly to* $\bar{c}(\cdot)$ *on* $[0,T]$.

Proof. Inequality (IA15) implies that $\sum_{i=j}^{\infty}|\bar{\alpha}_{i,j}|\leq C_0/\kappa_j$, and therefore

$$\begin{aligned}\sum_{i=1}^{\infty}|\bar{c}_i| &\leq |\theta_r|\kappa_1\sum_{i=1}^{\infty}|\bar{\alpha}_{i,1}|+\sum_{j=1}^{\infty}\sigma_j^2\left(|\gamma_j|+\frac{1}{\kappa_j}\right)\sum_{i=j}^{\infty}|\bar{\alpha}_{i,j}| \\ &\leq C_0|\theta_r|+C_0\sum_{j=1}^{\infty}\sigma_j^2\left(\frac{|\gamma_j|}{\kappa_j}+\frac{1}{\kappa_j^2}\right).\end{aligned}$$

We infer that $\sum_{i=1}^{\infty}|\bar{c}_i|<\infty$ under Conditions 1 and 3. Similarly,

$$\sum_{j=1}^{\infty}\sigma_j^2\sum_{i=j}^{\infty}\sum_{k=j}^{\infty}\frac{|\bar{\alpha}_{i,j}|\,|\bar{\alpha}_{k,j}|}{\kappa_i+\kappa_k}\leq\sum_{j=1}^{\infty}\sigma_j^2\sum_{i=j}^{\infty}\sum_{k=j}^{\infty}\frac{|\bar{\alpha}_{i,j}|\,|\bar{\alpha}_{k,j}|}{\kappa_1}\leq\frac{C_0^2}{\kappa_1}\sum_{j=1}^{\infty}\frac{\sigma_j^2}{\kappa_j^2}$$

is finite under Condition 1.

Since $|c_{i,n}-\bar{c}_i|\leq\kappa_1|\theta_r|\,|\alpha_{i,1,n}-\bar{\alpha}_{i,1}|+\sum_{j=1}^{i}\sigma_j^2\left(|\gamma_j|+1/\kappa_j\right)|\alpha_{i,j,n}-\bar{\alpha}_{i,j}|$, we infer from (IA13) that

$$\frac{|c_{i,n}-\bar{c}_i|}{\kappa_i}\leq C_5\,b^{-n},$$

where $C_5 = [|\theta_r| + \sum_{j=1}^{\infty}(\sigma_j^2/\kappa_j)(|\gamma_j| + 1/\kappa_j)] \,/\, [(1-b^{-1})\,\kappa_1\, F_\infty^2]$. Hence $\sum_{i=1}^{n} |c_{i,n} - \bar{c}_i|/\kappa_i$ converges to 0 as n goes to infinity.

The inequality $|\alpha_{i,j,n}\,\alpha_{k,j,n} - \bar{\alpha}_{i,j}\,\bar{\alpha}_{k,j}| \le |\alpha_{i,j,n}|\,|\alpha_{k,j,n} - \bar{\alpha}_{k,j}| + |\bar{\alpha}_{k,j}|\,|\alpha_{i,j,n} - \bar{\alpha}_{i,j}|$ implies that

$$\sum_{i=j}^{n}\sum_{k=j}^{n} |\alpha_{i,j,n}\,\alpha_{k,j,n} - \bar{\alpha}_{i,j}\,\bar{\alpha}_{k,j}| \le \sum_{i=j}^{n}\sum_{k=j}^{n} |\alpha_{i,j,n}|\,|\alpha_{k,j,n} - \bar{\alpha}_{k,j}| + \sum_{i=j}^{n}\sum_{k=j}^{n} |\bar{\alpha}_{k,j}|\,|\alpha_{i,j,n} - \bar{\alpha}_{i,j}|.$$

We infer from (IA10), (IA14), and (IA15) that

$$\sum_{i=j}^{n}\sum_{k=j}^{n} |\alpha_{i,j,n}\,\alpha_{k,j,n} - \bar{\alpha}_{i,j}\,\bar{\alpha}_{k,j}| \le \frac{2C_0}{\kappa_j}\frac{C_0\, b^{-n}}{(1-b^{-1})\kappa_1} = 2C_6 \frac{b^{-n}}{\kappa_j},$$

where $C_6 = C_0^2/[(1-b^{-1})\kappa_1]$. Hence,

$$\sum_{j=1}^{n} \frac{\sigma_j^2}{2} \sum_{i=j}^{n}\sum_{k=j}^{n} \frac{|\alpha_{i,j,n}\,\alpha_{k,j,n} - \bar{\alpha}_{i,j}\,\bar{\alpha}_{k,j}|}{\kappa_i + \kappa_k} \le C_6\, b^{-n} \sum_{j=1}^{n} \frac{\sigma_j^2}{\kappa_j^2}$$

converges to 0 as n goes to infinity.

Since

$$\begin{aligned}
|c_n(\tau) - \bar{c}(\tau)| \quad \le \quad & \left[\sum_{j=n+1}^{\infty} \frac{\sigma_j^2}{\kappa_j^2}\left(|\gamma_j|\kappa_j + \frac{1}{2}\right)\right]\tau + \sum_{i=1}^{n} \frac{|c_{i,n} - \bar{c}_i|}{\kappa_i} + \sum_{i=n+1}^{\infty} \frac{|\bar{c}_i|}{\kappa_i} \\
& + \sum_{j=1}^{n} \frac{\sigma_j^2}{2} \sum_{i=j}^{n}\sum_{k=j}^{n} \frac{|\alpha_{i,j,n}\alpha_{k,j,n} - \bar{\alpha}_{i,j}\bar{\alpha}_{k,j}|}{\kappa_i + \kappa_k} \\
& + \sum_{j=1}^{\infty} \frac{\sigma_j^2}{2} \sum_{i=j}^{\infty}\sum_{k=j}^{\infty} \frac{|\bar{\alpha}_{i,j}\bar{\alpha}_{k,j}|}{\kappa_i + \kappa_k} - \sum_{j=1}^{n} \frac{\sigma_j^2}{2} \sum_{i=j}^{n}\sum_{k=j}^{n} \frac{|\bar{\alpha}_{i,j}\bar{\alpha}_{k,j}|}{\kappa_i + \kappa_k},
\end{aligned}$$

we note that $|c_n(\tau) - \bar{c}(\tau)| \leq \varepsilon_n$, where

$$\begin{aligned}\varepsilon_n &= \sum_{j=n+1}^{\infty} \frac{\sigma_j^2}{\kappa_j^2}\left(|\gamma_j|\kappa_j + \frac{1}{2}\right) T + C_5\, n\, b^{-n} + \sum_{i=n+1}^{\infty} \frac{|\bar{c}_i|}{\kappa_i} + C_6 b^{-n} \sum_{j=1}^{\infty} \frac{\sigma_j^2}{\kappa_j^2} \\ &\quad + \sum_{j=1}^{\infty} \frac{\sigma_j^2}{2} \sum_{i=j}^{\infty} \sum_{k=j}^{\infty} \frac{|\bar{\alpha}_{i,j}\bar{\alpha}_{k,j}|}{\kappa_i + \kappa_k} - \sum_{j=1}^{n} \frac{\sigma_j^2}{2} \sum_{i=j}^{n} \sum_{k=j}^{n} \frac{|\bar{\alpha}_{i,j}\bar{\alpha}_{k,j}|}{\kappa_i + \kappa_k}.\end{aligned}$$

We conclude that $c_n(\cdot)$ converges uniformly to $\bar{c}(\cdot)$ on $[0,T]$. □

We now show that the yield on a zero-coupon bond with maturity $\tau > 0$,

(IA27) $$y_n(X_t, \tau) = \tau^{-1}\left[\sum_{j=1}^{n} b_{j,n}(\tau) x_{j,t} + c_n(\tau)\right]$$

converges to $y_\infty(X_t, \tau) = [\bar{b}(\tau)X_t + \bar{c}(\tau)]/\tau$ as $n \to \infty$. Indeed, we observe that

$$\|y_n(\cdot,\tau) - y_\infty(\cdot,\tau)\|_{L^2(\Omega\times[0,T])} = \left(\mathbb{E}^{\mathbb{P}}\left\{\int_0^T [y_n(X_t,\tau) - y_\infty(X_t,\tau)]^2\, dt\right\}\right)^{1/2}$$

is bounded above by

$$\frac{1}{\tau}\left[\sum_{j=1}^{n}\left|b_{j,n}(\tau) - \bar{b}_j(\tau)\right| + \sum_{j=n+1}^{\infty}\left|\bar{b}_j(\tau)\right|\right] \sup_{j\geq 1}\left\|x_{j,\cdot}\right\|_{L^2(\Omega\times[0,T])} + \frac{|c_n(\tau) - \bar{c}(\tau)|\sqrt{T}}{\tau}.$$

We infer from Lemmas IA4 to IA6 that

$$\|y_n(\cdot,\tau) - y_\infty(\cdot,\tau)\|_{L^2(\Omega\times[0,T])} \leq \frac{1}{\tau}\left\{\left[C_4 n b^{-n} + \frac{b^{-n}}{(1-b^{-1})\,\kappa_1}\right] \sup_{j\geq 1}\left\|x_{j,\cdot}\right\|_{L^2(\Omega\times[0,T])} + \varepsilon_n\sqrt{T}\right\}.$$

As Proposition 5 shows, the sequence $\{x_{j,\cdot}\}_{j=1}^{\infty}$ has a limit in $L^2(\Omega\times[0,T])$ and is therefore bounded in $L^2(\Omega \times [0,T])$, so that $\sup_{j\geq 1} \left\|x_{j,\cdot}\right\|_{L^2(\Omega\times[0,T])} < \infty$. We conclude that the yield of maturity τ converges to $y_\infty(X_t, \tau)$ in $L^2(\Omega \times [0,T])$.

Limiting Economy

The Radon Nikodým derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ in the cascade DTSM with n factors is conveniently denoted by:

$$M_{n,t} = \exp\left(-\sum_{j=1}^{n} \gamma_j \sigma_j W_{j,t} - \frac{t}{2}\sum_{j=1}^{n} \gamma_j^2 \sigma_j^2\right). \tag{IA28}$$

The limiting behavior of $M_{n,t}$ as n goes to infinity is driven by the limiting behavior of the series $\sum_{j=1}^{n} \gamma_j^2 \sigma_j^2$. If $\sum_{j=1}^{\infty} \gamma_j^2 \sigma_j^2 = \infty$, then $M_{n,t}$ converges to 0 almost surely. On the other hand if $\sum_{j=1}^{\infty} \gamma_j^2 \sigma_j^2 < \infty$, the martingale sequence $\{M_{n,t}\}_n$ converges to a limiting distribution $M_{\infty,t}$. We can then easily check that there exists a limiting process $M_{\infty,t}$. The measure $\mathbb{Q}$ with Radon-Nikodým derivative $M_{\infty,t}$ is therefore well-defined.

C. Robustness and Extensions

In this section, we verify the robustness of key empirical results and develop several extensions of the baseline cascade DTSM.

Scaling Specification

The baseline specification considers that adjustment speeds progress geometrically (Assumption 4). To verify the validity of this scaling rule, we estimate an extension of the ten-factor model in which each κ_j is a free parameter. The total number of parameters of the cascade term structure model increases by 8, from the original 6 (including the error variance) to 14. In Figure IA.1, we plot the logarithm of the κ_j estimates in circles, and as a solid line the linear relation implied by Assumption 4 with $\kappa_1 = 0.0388$ and $b = 1.869$. The estimates for the free parameters κ_j vary around the solid line, which suggests that adjustment speeds are approximately geometric.

The baseline version of the cascade further assumes that volatilities and risk premia are identical across components (Assumptions 5 and 6). To assess the validity of these assumptions, we estimate a different extension of the ten-factor cascade, in which volatilities and risk premia satisfy:

$$\sigma_j^2 = \sigma_1^2 b^{(j-1)s_\sigma}, \tag{IA29}$$

$$\gamma_j \sigma_j = \gamma_1 \sigma_1 b^{(j-1)s_\gamma}, \tag{IA30}$$

for all $j \in \{1, \ldots, n\}$. This relaxed specification adds two new scaling parameters, s_σ and s_γ, in order to capture the frequency scaling of volatilities and risk premia.

Table IA.1 reports parameter estimates. The scaling coefficients s_σ and s_γ are both negative, suggesting that volatilities and risk premia are smaller for higher frequency factors. Nevertheless, the estimates are small in magnitude, implying only modest variation in volatilities and risk premia between low-frequency and high-frequency factors. Overall, the variances and premia of the factors do not vary nearly as much across frequencies as the mean reversion speed. Assumptions 5 and 6 can be viewed as convenient simplifications that offer the benefits of parsimony and robust identification but can be relaxed in particular applications.

Link with Risk Premium Regressions

The baseline version of the cascade assumes a simple, constant market price of risk. Indeed, the state-space estimation framework is generally well suited to identify the risk-neutral dynamics from the cross-sectional behavior of the interest rate term structure, but has more difficulties with the statistical dynamics and the cross-section of bond returns.

The slightly more general affine risk premium specification (Assumption 3) has important implications for expected bond returns. Consider a zero-coupon bond of maturity τ held at t. The following proposition links the risk premium sensitivity matrix Λ to the linear regression coefficients of excess bond returns on the factors.

Proposition IA.1 (Expected bond returns under the cascade DTSM) *Under Assumptions 1 to 3, the instantaneous risk premium at time t on a bond of maturity τ is given by*

(IA31) $$RP(X_t,\tau) = -[\Lambda^\top \Sigma b(\tau)]^\top X_t + c'(\tau) - b(\tau)^\top \kappa\theta$$

for all τ *and* X_t.

Proof. We infer from Ito's lemma that $\mathbb{E}^{\mathbb{P}}[d\ln P(X_t;\tau)] = [b'(\tau)^\top X_t + c'(\tau)]dt + b(\tau)^\top \kappa(X_t - \theta)dt$. The instantaneous risk premium is therefore

$$RP(X_t,\tau) = [b'(\tau) + \kappa^\top b(\tau)]^\top X_t + c'(\tau) - b(\tau)^\top \kappa\theta - e_n^\top X_t.$$

We infer from equation (IA3) that $b'(\tau) + \kappa^\top b(\tau) - e_n = -(\Sigma\Lambda)^\top b(\tau)$ and conclude that the instantaneous risk premium satisfies (IA31) for all τ and X_t. $\square$

A large body of empirical research relates bond excess returns to a small set of yields or forward rates. For instance, early studies report regressions of excess returns on a measure of the term-structure slope, often as a test of the expectation hypothesis (Campbell and Shiller (1991), Fama and Bliss (1987), Ilmanen (1995), Stambaugh (1988)). More recently, Cochrane and Piazzesi (2005) propose to explain bond excess returns with a portfolio of forward rates and find that such multivariate regressions generate much higher R^2 coefficients than earlier studies. To link our approach to this empirical tradition, we note that forward rates are affine functions of the state vector under our model. Consider n instantaneous forwards of maturities $\tau_1, ..., \tau_n$. Let $f_t = [f(X_t,\tau_1), \ldots, f(X_t,\tau_n)]^\top$ denote the corresponding n–dimensional vector of rates. Since

$$f(X_t,\tau) \quad \equiv \quad -\frac{\partial \ln P}{\partial \tau}(X_t,\tau) = b'(\tau)^\top X_t + c'(\tau),$$

there is a one-to-one affine relationship between the state vector and the vector of forward rates:

$$f_t = B^\top X_t + f^C,$$

where $f^C = [c'(\tau_1), \ldots, c'(\tau_n)]^\top$ is a constant vector, and $B = [b'(\tau_1), ..., b'(\tau_n)]$ is a constant $n \times n$ matrix. Any finding from regressions on forward rates can therefore be immediately mapped in terms of the state vector X_t and the sensitivity matrix Λ, and conversely.

The empirical findings of Cochrane and Piazzesi (2005, 2008) can be accommodated in our model if we assume that the rank of the sensitivity matrix Λ is equal to one: $\Lambda = \Sigma^{-1} l \lambda^\top$. Under this assumption, a single factor, $\lambda^\top X_t$, explains the cross-section of instantaneous risk premia:

$$RP(X_t, \tau) = -[l^\top b(\tau)](\lambda^\top X_t) + c'(\tau) - b(\tau)^\top \kappa\theta,$$

for every τ.

As an illustration, we estimate a version of the ten-factor model with identical, time-varying market prices of risk:

$$g_{j,t} = \gamma + \lambda^\top X_t. \tag{IA32}$$

We maintain dimension-invariant assumptions on adjustment speeds (Assumption 4) and component volatilities (Assumption 5). Figure IA.2 plots the estimated loading coefficients, λ_j, on each frequency component. Similar to Cochrane and Piazzesi's observations, the loadings show a tent-shaped pattern, where the coefficients are positive for intermediate frequencies but negative for both high and low frequencies. It is thus straightforward to adapt the cascade to permit time-varying risk premia in a manner that captures known features of the data.

Stochastic Volatility and Interest Rate Option Pricing

Prior literature shows that interest-rate volatility movements are largely "unspanned" by the interest-rate term structure.[1] Therefore, the specification of stochastic volatility should not have a large impact on the term structure. However, when pricing interest-rate options, the specification of stochastic volatility becomes critical. We propose to allow all components of the state vector X_t to be driven by a common stochastic variance, which is itself generated by a cascade structure. Specifically, we consider m variance factors $v_{1,t}, \ldots, v_{m,t}$, satisfying:

$$\begin{aligned} d(v_{i,t}) &= \kappa_i^v (v_{i-1,t} - v_{i,t})dt + \omega \sqrt{v_{m,t}} \left(\rho \, dW_{i,t} + \sqrt{1-\rho^2} \, dZ_{i,t} \right), \\ \kappa_i^v &= \beta^{i-1} \kappa_1^v, \end{aligned}$$

for every $i \in \{1,...,m\}$, where $\rho \in [-1,1]$, $\beta \in (1,+\infty)$, $\theta_v \in (0,+\infty)$, $v_{0,t} = \theta_v$ for all t, and $Z_t = (Z_{1,t}, \ldots, Z_{m,t})$ is a Wiener process independent of W_t. All state variables have $x_{j,t}$ stochastic volatility $\sigma_{j,t}^2 = v_{m,t}$. The stochastic volatility cascade implies that each variance component mean reverts at a geometrically increasing rate toward the next lower frequency component. In addition to the geometric progression of mean-reversion speeds, the specification achieves dimension-invariance by assuming a constant and identical coefficient ω describing the volatility of volatility, and identical correlations between the interest rate and variance innovations.

[1] See the evidence in Collin-Dufresne and Goldstein (2002), Fan, Gupta, and Ritchken (2003), Heidari and Wu (2003), and Li and Zhao (2006).

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TABLE IA.1
Scaling of Volatilities and Risk Premia

This table reports the maximum likelihood estimates and standard errors (in parentheses) of the model parameters that govern the scaling of volatilities and risk premia across components in a 10-factor cascade.

	Estimates	Standard Errors
κ_1	0.0390	(0.0008)
σ_1	0.0194	(0.0002)
θ_r	0.0000	(0.0000)
γ	-0.3471	(0.0045)
b	1.8546	(0.0098)
s_γ	-0.1665	(0.0079)
s_σ	-0.1577	(0.0055)

FIGURE IA.1
Scaling of Adjustment Speeds

The circles are estimated as free parameters. The solid line is generated from the benchmark model with geometrically distributed adjustment speeds $\kappa_j = \kappa_1 b^{j-1}$.

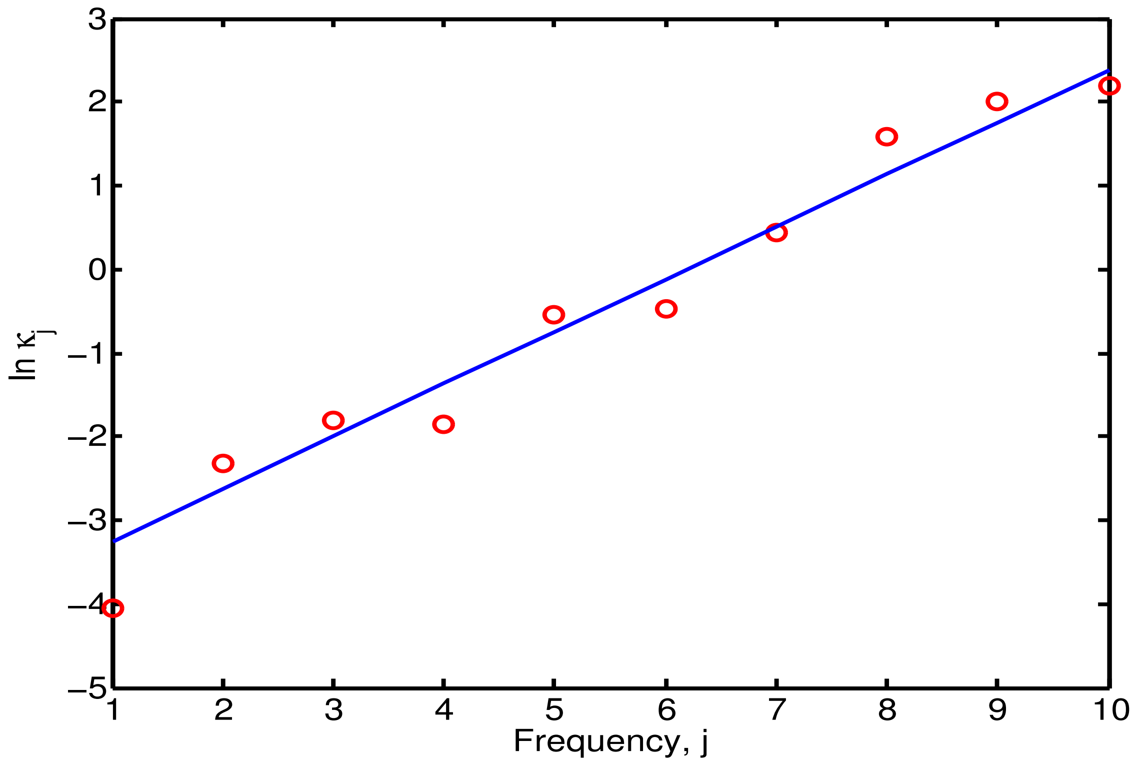

FIGURE IA.2
Factor Loading in the Single Market Price of Risk Factor

Circles denote the estimated loading coefficients on each frequency component for the single market price of risk factor on a ten-factor model. The model maintains dimension-invariant assumptions on risk $\sigma_j = \sigma_1$ and mean-reversion speeds $\kappa_j = \kappa_1 b^{j-1}$.