Internet Appendix

to

Pitfalls in the Use of Systemic Risk Measures

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A. Sensitivities

In this section we explore the sensitivities of systemic risk measures (SRMs) summarized in Table 1 in a systematic way. We duplicate the table here with references to the analyses provided in this section. As in the main text, a superscript ^{n} marks cases where the opposite sensitivity would appear only under very unusual conditions.

Parameter	Effect type	ΔCoVaR	Exp. ΔCoVaR	MES	BETA
σ_i	Direct	$\pm$; A.1	$+$; A.5	$+$; A.5	$+$; A.4
	Relative	$\pm$; A.11	$+$; A.12	$+^n$; A.12	$+$; A.12
β_i	Direct	$+$; A.2	$+$; A.7	$+^n$; A.8	$\pm$; A.6
	Relative	$\pm$; A.14	$\pm$; A.13	$\pm$; A.13	$\pm$; A.13
w_i	Direct	$\pm$; A.3	$+$; A.10	$+$; A.10	$\pm$; A.9
	Relative	$+^n$; A.16	$+^n$; A.15	$+^n$; A.15	$+^n$; A.15

We start the analysis with a number of definitions and auxiliary formulas. First we set $w_j^* \equiv (1-w_i)^{-1} w_j$, $j \neq i$, to define bank weights within the sub-system excluding bank i. Corresponding averages of the exposure to systematic risk are $\beta_* \equiv \sum_{k\neq i} w_k^* \beta_k$ and

$$\bar{\beta} \equiv \sum_{k=1}^{N} w_k \beta_k = w_i \beta_i + (1-w_i)\beta_* .$$

Setting $\varepsilon_* \equiv \sum_{j\neq i} w_j^* \varepsilon_j$ and

$$\bar{\varepsilon} \equiv \sum_{k=1}^{N} w_k \varepsilon_k = w_i \varepsilon_i + (1-w_i)\varepsilon_*$$

for aggregate idiosyncratic risks, the index return then reads

$$R_S = \bar{\beta} F + \bar{\varepsilon} = w_i R_i + (1-w_i)(\beta_* F + \varepsilon_*),$$

while its variance, using $\sigma_* \equiv \sigma(\varepsilon_*)$, can be written in different ways:

$$\text{(A-1)} \qquad \sigma^2(R_S) = \bar{\beta}^2 \sigma_F^2 + \bar{\sigma}^2 = (w_i \beta_i + (1-w_i)\beta_*)^2 \sigma_F^2 + w_i^2 \sigma_i^2 + (1-w_i)^2 \sigma_*^2 .$$

The covariances $\operatorname{cov}(R_j, R_S)$ play a central role; we denote them by $c_{j,S}$ and obtain the following representation:

$$\text{(A-2)} \qquad c_{j,S} \equiv \operatorname{cov}(R_j, R_S) = \beta_j \bar{\beta} \sigma_F^2 + w_j \sigma_j^2 .$$

As we assumed all β_j to be positive, all $c_{j,S}$ are positive, too. For bank i we will also use the form

$$c_{i,S} = w_i \sigma^2(R_i) + (1-w_i)\beta_i \beta_* \sigma_F^2 .$$

A.1. Direct Effect of Idiosyncratic Risk on ΔCoVaR

The new notation turns equation (3) into

$$\begin{aligned} \Delta\text{CoVaR}_{\alpha}^{S,i} &= \frac{c_{i,S}}{\sigma\left(R_i\right)}\Phi^{-1}(1-\alpha) \\ &= \left[w_i\sigma(R_i) + (1-w_i)\,\beta_i\beta_*\frac{\sigma_F^2}{\sigma(R_i)}\right]\Phi^{-1}(1-\alpha)\,. \end{aligned} \tag{A-3}$$

Applying $\partial\sigma\left(R_i\right)/\partial\sigma_i = \sigma_i/\sigma\left(R_i\right)$, we obtain

$$\begin{aligned} \frac{\partial\Delta\text{CoVaR}}{\partial\sigma_i} &= \left[w_i\frac{\sigma_i}{\sigma\left(R_i\right)} - (1-w_i)\,\beta_i\beta_*\frac{\sigma_F^2}{\sigma^2(R_i)}\frac{\sigma_i}{\sigma\left(R_i\right)}\right]\Phi^{-1}\left(1-\alpha\right) \\ &\propto \frac{w_i}{1-w_i} - \beta_i\beta_*\frac{\sigma_F^2}{\sigma^2(R_i)} = \frac{w_i}{1-w_i} - \frac{\beta_*}{\beta_i}\left(1+\frac{\sigma_i^2}{\beta_i^2\sigma_F^2}\right)^{-1}, \end{aligned}$$

which is the formula that gives equation (10) in the special case $\beta_k = 1$ for all k.

A.2. Direct Effect of Systematic Risk on ΔCoVaR

The partial derivative to systematic risk is always positive:

$$\begin{aligned} &\frac{\partial}{\partial\beta_i}\left[\Delta\text{CoVaR}_{\alpha}^{S,i}\right] \propto \frac{\partial}{\partial\beta_i}\left[\frac{c_{i,S}}{\sigma\left(R_i\right)}\right] \propto \sigma^2\left(R_i\right)\frac{\partial c_{i,S}}{\partial\beta_i} - c_{i,S}\sigma\left(R_i\right)\frac{\partial\sigma\left(R_i\right)}{\partial\beta_i} \\ &= \sigma^2\left(R_i\right)\frac{\partial c_{i,S}}{\partial\beta_i} - c_{i,S}\frac{1}{2}\frac{\partial\sigma^2\left(R_i\right)}{\partial\beta_i} = \left(\beta_i^2\sigma_F^2+\sigma_i^2\right)\sigma_F^2\left(w_i\beta_i+\bar{\beta}\right) - \left(\beta_i\bar{\beta}\sigma_F^2 + w_i\sigma_i^2\right)\beta_i\sigma_F^2 \\ &\propto \left(\beta_i^2\sigma_F^2+\sigma_i^2\right)\left(w_i\beta_i+\bar{\beta}\right) - \left(\beta_i\bar{\beta}\sigma_F^2+w_i\sigma_i^2\right)\beta_i = w_i\beta_i^3\sigma_F^2 + \bar{\beta}\sigma_i^2 > 0\,. \end{aligned}$$

A.3. Direct Effect of Size on ΔCoVaR

We assume that the size of bank i changes while the other banks' size is kept constant. This means that all w_j^* remain constant (and so β_*), which is why equation (A-3) leads to the simple formula:

$$\begin{aligned} \frac{\partial\Delta\text{CoVaR}_{\alpha}^{S,i}}{\partial w_i} &= \left[\sigma\left(R_i\right) - \beta_i\beta_*\frac{\sigma_F^2}{\sigma\left(R_i\right)}\right]\Phi^{-1}\left(1-\alpha\right) \\ &\propto -\beta_i\beta_*\sigma_F^2 + \sigma^2\left(R_i\right) = \beta_i\,\Delta_\beta\,\sigma_F^2 + \sigma_i^2\,, \end{aligned} \tag{A-4}$$

where $\Delta_\beta \equiv \beta_i - \beta_*$. We observe that a bank's weight in the system has an ambiguous effect. Generally, we would expect the ΔCoVaR to increase with size. That is true in many cases, e.g., if $\beta_i \geq \beta_*$. Formula (A-4) shows that the partial derivative is then positive.

If, on the contrary, a bank has a rather low exposure to the systematic risk factor and also comparably low idiosyncratic risk, the derivative can have the opposite sign, for instance, if $\sigma_i = 8\%$, $\sigma_F = 20\%$ (both p.a.), $\beta_i = 0.5$, and $\beta_* = 1$.

In Section III.B of the article it has already become clear that the effect of size on a SRM must be discussed together with its effect on the system risk $\sigma\left(R_S\right)$. Indeed, a negative size sensitivity is not necessarily a problem because the system return would become less volatile if such a bank gaines weight, as we now show. Assume the partial derivative in equation (A-4) is negative. It implies $\sigma_i^2 + \Delta_\beta \beta_i \sigma_F^2 < 0$ and the weaker condition $\Delta_\beta < 0$. With $\partial\bar{\beta}/\partial w_i = \bar{\beta}\,\Delta_\beta$, we obtain from equation (13) (which is a direct implication of equation (A-1)):

$$
\begin{aligned}
\text{(A-5)} \qquad \frac{\partial \sigma^2 (R_S)}{\partial w_i} &\propto \bar{\beta}\,\Delta_\beta\,\sigma_F^2 + w_i \sigma_i^2 - (1 - w_i)\,\sigma_*^2 \\
&< (w_i \beta_i + (1 - w_i)\,\beta_*)\,\Delta_\beta\,\sigma_F^2 + w_i \sigma_i^2 \\
\text{(A-6)} \qquad &< w_i \underbrace{\left(\beta_i \Delta_\beta\,\sigma_F^2 + \sigma_i^2\right)}_{<0} + (1 - w_i)\,\beta_* \underbrace{\Delta_\beta}_{<0}\,\sigma_F^2 < 0\,.
\end{aligned}
$$

In short form, we conclude

$$
\frac{\partial \Delta \mathrm{CoVaR}_\alpha^{S,i}}{\partial w_i} < 0 \qquad \Rightarrow \qquad \frac{\partial \sigma^2 (R_S)}{\partial w_i} < 0.
$$

The ΔCoVaR thus sets a correct incentive insofar as a bank is rewarded for growth through a ΔCoVaR-based systemic risk charge only if this lowers the volatility of the system return.

However, the implication cannot be reversed. There are realistic conditions under which $\partial \Delta \mathrm{CoVaR}_\alpha^{S,i}/\partial w_i$ is positive while $\partial \sigma^2 (R_S)/\partial w_i$ is not, for instance if a small bank's Δ_β is moderately negative but idiosyncratic risks are considerable. This holds all the more if the other banks' idiosyncratic risks are not well diversified.

A.4. Direct Effect of Idiosyncratic Risk on BETA

Using the equations (A-1) and (A-2), standard calculus shows

$$
\begin{aligned}
\frac{\partial\, \mathrm{BETA}_i}{\partial \sigma_i} &\propto \frac{\partial\, \mathrm{BETA}_i}{\partial \sigma_i^2} = \frac{\partial}{\partial \sigma_i^2}\left[\frac{c_{i,S}}{\sigma^2 (R_S)}\right] \propto \sigma^2 (R_S)\,\frac{\partial c_{i,S}}{\partial \sigma_i^2} - c_{i,S}\frac{\partial \sigma^2 (R_S)}{\partial \sigma_i^2} \\
&\propto \sigma^2 (R_S) - c_{i,S} w_i = \sum_{k=1}^{N} w_k c_{k,S} - c_{i,S} w_i = \sum_{k \neq i} w_k c_{k,S} > 0.
\end{aligned}
$$

A.5. Direct Effect of Idiosyncratic Risk on Exposure ΔCoVaR and MES

Inspecting equation (A-1), the standard deviation $\sigma(R_S)$ is obviously a growing function of σ_i^2, and so is BETA_i, as shown in Section A.4. The exposure ΔCoVaR equals $\Phi^{-1}(1-\alpha) \times \mathrm{BETA}_i \times \sigma(R_S)$, which is a product of two monotonic functions of σ_i^2 and a constant. Hence, the exposure ΔCoVaR is increasing in σ_i.

As the MES differs from the exposure ΔCoVaR only by a constant factor and an offset that does not depend on σ_i, it shows the same monotonicity w.r.t. idiosyncratic risk.

A.6. Direct Effect of Systematic Risk on BETA

The partial derivative of BETA is proportional to the following expression:

$$
\begin{aligned}
\frac{\partial\, \mathrm{BETA}_i}{\partial \beta_i} &= \frac{\partial}{\partial \beta_i}\left[\frac{c_{i,S}}{\sigma^2 (R_S)}\right] \propto \sigma^2 (R_S)\,\frac{\partial c_{i,S}}{\partial \beta_i} - c_{i,S}\frac{\partial \sigma^2 (R_S)}{\partial \beta_i} \\
&= \left(\bar{\beta}^2 \sigma_F^2 + \bar{\sigma}^2\right)\left[\bar{\beta} + w_i \beta_i\right] \sigma_F^2 - \left(\beta_i \bar{\beta} \sigma_F^2 + w_i \sigma_i^2\right) 2\bar{\beta} w_i \sigma_F^2\,.
\end{aligned}
$$

We divide by σ_F^2 and estimate the outcome from below by removing the second addend of $\bar{\sigma}^2 = w_i^2\sigma_i^2 + (1-w_i)^2\sigma_*^2$ so that we obtain:

$$\begin{aligned}
\frac{\partial\,\mathrm{BETA}_i}{\partial\beta_i} &\propto \ldots \geq \left(\bar{\beta}^2\sigma_F^2 + w_i^2\sigma_i^2\right)\left[\bar{\beta} + w_i\beta_i\right] - 2\bar{\beta}w_i\beta_i\bar{\beta}\sigma_F^2 - 2\bar{\beta}w_iw_i\sigma_i^2 \\
&= \bar{\beta}^3\sigma_F^2 + w_i\beta_i\bar{\beta}^2\sigma_F^2 + w_i^2\bar{\beta}\sigma_i^2 + w_i^3\beta_i\sigma_i^2 - 2w_i\beta_i\bar{\beta}^2\sigma_F^2 - 2w_i^2\bar{\beta}\sigma_i^2 \\
&= \bar{\beta}^3\sigma_F^2 + w_i^3\beta_i\sigma_i^2 - w_i\beta_i\bar{\beta}^2\sigma_F^2 - w_i^2\bar{\beta}\sigma_i^2 \\
&= \bar{\beta}^2\sigma_F^2\bar{\beta} + w_i^3\beta_i\sigma_i^2 - w_i\beta_i\bar{\beta}^2\sigma_F^2 - w_i\beta_iw_i^2\sigma_i^2 - (1-w_i)\,\beta_*w_i^2\sigma_i^2 \\
&= (1-w_i)\,\beta_*\left(\bar{\beta}^2\sigma_F^2 - w_i^2\sigma_i^2\right) \propto \bar{\beta}^2\sigma_F^2 - w_i^2\sigma_i^2\ .
\end{aligned}$$

Hence, we can state:

$$w_i < \frac{\bar{\beta}\sigma_F}{\sigma_i} \quad \Rightarrow \quad \frac{\partial\,\mathrm{BETA}_i}{\partial\beta_i} > 0\,.$$

The condition is fulfilled unless a bank is very dominant in the system and/or has substantially more idiosyncratic risk than the other banks in the aggregate (so that σ_*^2 is small). Nevertheless, the case $w_i\sigma_i > \bar{\beta}\sigma_F$ is possible; it is the same under which equation (12) becomes negative, with the implication that the relative effect on exposure ΔCoVaR, MES and BETA is negative.

We therefore state that β_i can have a negative direct effect on BETA, at least qualitatively. Numerical tests suggest that the effect is weak, as can be illustrated by the scenario of Figure 3, where $\sigma_i = 0.4$, $\sigma_F = 0.1$ (both p.a.); $N = 50$, $w_i = 0.45$; $w_j = 0.65/49$ and $\beta_j = 1$ for $j \neq i$. Although these conditions are set to strengthen the effect, an increase of β_i from 0.5 to 2 diminishes BETA by 4.1% only, from 1.963 to 1.882.

A.7. Direct Effect of Systematic Risk on Exposure ΔCoVaR

The exposure ΔCoVaR is monotonic in β_i, which can be shown directly:

$$\begin{aligned}
\frac{\partial\,\Delta\mathrm{CoVaR}_\alpha^{i,S}}{\partial\beta_i} &\propto \frac{\partial}{\partial\beta_i}\left[\frac{c_{i,S}}{\sigma(R_S)}\right] \propto \sigma(R_S)\frac{\partial c_{i,S}}{\partial\beta_i} - c_{i,S}\frac{\partial\sigma(R_S)}{\partial\beta_i} \\
&\propto \sigma^2(R_S)\frac{\partial c_{i,S}}{\partial\beta_i} - \frac{1}{2}c_{i,S}\frac{\partial\sigma^2(R_S)}{\partial\beta_i} \\
&= \left(\bar{\beta}^2\sigma_F^2 + \bar{\sigma}^2\right)\left[\bar{\beta} + w_i\beta_i\right]\sigma_F^2 - \left(\beta_i\bar{\beta}\sigma_F^2 + w_i\sigma_i^2\right)\bar{\beta}w_i\sigma_F^2 \\
&\propto \left(\bar{\beta}^2\sigma_F^2 + \bar{\sigma}^2\right)\left[\bar{\beta} + w_i\beta_i\right] - \left(\beta_i\bar{\beta}\sigma_F^2 + w_i\sigma_i^2\right)\bar{\beta}w_i \\
&= \bar{\beta}^3\sigma_F^2 + \bar{\sigma}^2\bar{\beta} + \bar{\sigma}^2w_i\beta_i - w_i^2\sigma_i^2\bar{\beta} \\
&= \bar{\beta}^3\sigma_F^2 + \left(w_i^2\sigma_i^2 + (1-w_i)^2\sigma_*^2\right)\bar{\beta} + \bar{\sigma}^2w_i\beta_i - w_i^2\sigma_i^2\bar{\beta} \\
&= \bar{\beta}^3\sigma_F^2 + (1-w_i)^2\sigma_*^2\bar{\beta} + \bar{\sigma}^2w_i\beta_i \\
&> 0\,.
\end{aligned}$$

A.8. Direct Effect of Systematic Risk on MES

From equation (9) we recall $\mathrm{MES}_i = -\beta_i\mu + C_\alpha c_{i,j}/\sigma(R_S)$, where $C_\alpha \equiv \alpha^{-1}\phi\left(\Phi^{-1}(\alpha)\right)$ is a positive constant. The MES is special because of its drift related term, which decreases in β_i, whereas in A.7 the ratio $c_{i,j}/\sigma(R_S)$ has already turned out to increase in β_i. We find the following estimate from below for the partial derivative:

$$
\begin{aligned}
\frac{\partial \mathrm{MES}_i}{\partial \beta_i} &= -\mu + \left\{ C_\alpha \frac{\sigma_F \bar{\beta}}{\sigma(R_S)} \left[1 + w_i (1-w_i) \frac{(1-w_i)\sigma_*^2 \beta_i - \beta_* w_i \sigma_i^2}{\bar{\beta}\sigma^2(R_S)} \right] \right\} \times \sigma_F \\
&\geq -\mu + \left\{ C_\alpha \left(1 + \frac{\bar{\sigma}^2}{\bar{\beta}^2 \sigma_F^2} \right)^{-1/2} \left[1 - w_i^2 (1-w_i) \frac{\beta_* \sigma_i^2}{\bar{\beta}\sigma^2(R_S)} \right] \right\} \times \sigma_F \,. \qquad \text{(A-7)}
\end{aligned}
$$

The term in curly braces cannot reasonably become smaller than 1/5, for which we otherwise would have to make extreme assumptions[17]. The question is now how μ and σ_F relate to each other, which depends on the risk horizon. While μ is proportional to the risk horizon, $\sigma(R_S)$ is so to its square root. As the risk horizon is 1 day throughout this paper, the first term is by magnitudes smaller than the second. The 1-day MES will therefore be an increasing function of systematic risk under all plausible conditions. Even on an annual basis, $-\mu$ would usually be dominated by the positive part on the r.h.s. part of inequality (A-7).

A.9. Direct Effect of Size on BETA

As the partial derivative of BETA_i is complicated, we make the simplifying assumption that only bank i has a non-negligible weight in the system, whereas all other banks are infinitesimally small. In the limit, idiosyncratic risks of these banks are diversified away ($\sigma_* = 0$), so that we obtain:

$$
\begin{aligned}
\frac{\partial\,\mathrm{BETA}_i}{\partial w_i} &= \frac{\partial}{\partial w_i}\left[\frac{c_{i,S}}{\sigma^2(R_S)}\right] \propto \sigma^2(R_S)\frac{\partial c_{i,S}}{\partial w_i} - c_{i,S}\frac{\partial \sigma^2(R_S)}{\partial w_i} = \sigma^2(R_S)\frac{\partial c_{i,S}}{\partial w_i} - c_{i,S}\frac{\partial \sigma^2(R_S)}{\partial w_i} \\
&= \left(\bar{\beta}^2\sigma_F^2 + w_i^2\sigma_i^2\right)\left(\beta_i\,\Delta_\beta\,\sigma_F^2 + \sigma_i^2\right) - \left(\beta_i\bar{\beta}\sigma_F^2 + w_i\sigma_i^2\right) 2\left(\bar{\beta}\,\Delta_\beta\sigma_F^2 + w_i\sigma_i^2\right) ,
\end{aligned}
$$

where $\Delta_\beta \equiv \beta_i - \beta_*$. As the conditions under which this expression is positive are still difficult to identify, we focus on the case where w_i is also small enough to set it zero. We obtain:

$$
\left.\frac{\partial\,\mathrm{BETA}_i}{\partial w_i}\right|_{w_i=0} \propto \sigma_i^2 - \beta_i\Delta_\beta\sigma_F^2 \,. \qquad \text{(A-8)}
$$

This derivative is negative if the bank's exposure to systematic risk is above the average and the idiosyncratic risk is comparably small. Note that the growth of such a bank would increase the variance of the system return, which can be seen in equation (A-5), where we have found:

$$
\frac{\partial \sigma^2(R_S)}{\partial w_i} \propto \bar{\beta}\,\Delta_\beta\,\sigma_F^2 + w_i\sigma_i^2 - (1-w_i)\,\sigma_*^2 \,.
$$

Under the limiting assumption $\sigma_* = 0$, the variance of R_S grows in w_i if $\beta_i > \beta_*$.

A.10. Direct Effect of Size on Exposure ΔCoVaR and MES

We first consider the main part of the exposure ΔCoVaR, $c_{i,S}/\sigma(R_S)$. Using the property

$$
\partial\sigma(R_S)/\partial w_i = (2\sigma(R_S))^{-1}\,\partial\sigma^2(R_S)/\partial w_i,
$$

[17] An example where the term just falls short of 1/5 would be $w_i = 0.5$, $\sigma_* = 0$, $\sigma_i = 0.5$ p.a., $\sigma_F = 0.1$ p.a., $\beta_i = 0.25$, $\beta_* = 1.75$, $\alpha = 0.05$. Any ceteris-paribus variation towards less extreme values lets the term rise above 1/5.

we find:

$$
\begin{aligned}
\frac{\partial}{\partial w_i}\left[\frac{c_{i,S}}{\sigma\left(R_S\right)}\right] &\propto \sigma^2\left(R_S\right)\frac{\partial c_{i,S}}{\partial w_i}-\frac{1}{2}c_{i,S}\frac{\partial \sigma^2\left(R_S\right)}{\partial w_i}\\
&= \left(\beta_i\,\Delta_\beta\,\sigma_F^2+\sigma_i^2\right)\left(\bar{\beta}^2\sigma_F^2+w_i^2\sigma_i^2+\left(1-w_i\right)^2\sigma_*^2\right)\\
&\quad -\left(\beta_i\bar{\beta}\sigma_F^2+w_i\sigma_i^2\right)\left(\bar{\beta}\,\Delta_\beta\sigma_F^2+w_i\sigma_i^2-\left(1-w_i\right)\sigma_*^2\right)\\
&= \sigma_F^2\sigma_i^2\left\{\beta_i\Delta_\beta\,w_i^2-\beta_i\bar{\beta}w_i-\bar{\beta}\,\Delta_\beta\,w_i+\bar{\beta}^2\right\}+\sigma_i^2\sigma_*^2\left(1-w_i\right)\\
&\quad +\sigma_F^2\sigma_*^2\beta_i\left(1-w_i\right)\left[\Delta_\beta\left(1-w_i\right)+\bar{\beta}\right]\\
&= \ldots=\sigma_F^2\sigma_i^2\left(1-w_i\right)\beta_*^2+\sigma_i^2\sigma_*^2\left(1-w_i\right)+\sigma_F^2\sigma_*^2\beta_i\left(1-w_i\right)\left(1-w_i\right)\beta_i>0\,.
\end{aligned}
$$

This means that the exposure ΔCoVaR is always an increasing function of w_i. The same holds for the MES since the additive term $-\mu\beta_i$ in equation (7) is independent of w_i.

As already discussed in Section III.B of the article, a positive dependency on size is not necessarily appropriate because size has an ambiguous effect on the system risk $\sigma\left(R_S\right)$.

A.11. Relative Effect of Idiosyncratic Risk on ΔCoVaR

The parameter σ_i has neither an impact on another bank's return volatility $\sigma\left(R_j\right)$ nor on its covariance $c_{j,S}$ with the system return. According to $\Delta\text{CoVaR}_\alpha^{S,j}=c_{j,S}/\sigma\left(R_j\right)\Phi^{-1}(1-\alpha)$, the ΔCoVaR of bank j is constant such that the relative and the direct effect of σ_i fall together, apart from a constant factor.

A.12. Relative Effect of Idiosyncratic Risk on Exposure ΔCoVaR, MES, and BETA

We consider the ratio of two banks' SRMs, such as $\text{BETA}_i/\text{BETA}_j$. For exposure ΔCoVaR and BETA, the ratios are equal to $c_{i,S}/c_{j,S}$. As already stated, $c_{j,S}$ is invariant to σ_i so that only the effect on the covariance $c_{i,S}$ remains to be analyzed. It is obviously positive because $\partial c_{i,S}/\partial\sigma_i=2w_i\sigma_i>0$.

The MES has drift-related addends above and below the fraction line. We neglect them in this section as they are small, based on the arguments provided in Section A.8. We therefore consider the MES to be covered by the analysis of $c_{i,S}/c_{j,S}$.

A.13. Relative Effect of Systematic Risk on Exposure ΔCoVaR, MES, and BETA

For the partial derivative of the ratio of covariances to β_i, we obtain:

$$
\begin{aligned}
\text{(A-9)}\qquad \frac{\partial}{\partial \beta_i}\left[\frac{c_{i,S}}{c_{j,S}}\right] &\propto c_{j,S}\frac{\partial c_{i,S}}{\partial \beta_i}-c_{i,S}\frac{\partial c_{j,S}}{\partial \beta_i}\\
&= \left(\beta_j\bar{\beta}\sigma_F^2+w_j\sigma_j^2\right)\sigma_F^2\left(w_i\beta_i+\bar{\beta}\right)-\left(\beta_i\bar{\beta}\sigma_F^2+w_i\sigma_i^2\right)\sigma_F^2\beta_j w_i\\
&\propto 1+w_j\frac{\sigma_j^2}{\beta_j\bar{\beta}^2\sigma_F^2}\left(w_i\beta_i+\bar{\beta}\right)-w_i^2\frac{\sigma_i^2}{\bar{\beta}^2\sigma_F^2}\,.
\end{aligned}
$$

In absence of a dominating bank, the only negative part of the expression is considerably smaller than 1 under most conditions because of the factor w_i^2. The relative effect of β_i is then positive. However, it may become negative if bank i is really large. Assume for simplicity that bank j is very small so that the middle term vanishes. Then, the ratio $c_{i,S}/c_{j,S}$ will negatively depend on β_i if $w_i\sigma_i>\bar{\beta}\sigma_F$.

A.14. Relative Effect of Systematic Risk on ΔCoVaR

We start with a calculation of the partial derivative:

$$\begin{aligned}
\frac{\partial}{\partial \beta_i}\left[\frac{\Delta\mathrm{CoVaR}_\alpha^{S,i}}{\Delta\mathrm{CoVaR}_\alpha^{S,j}}\right] &= \sigma\left(R_j\right)\frac{\partial}{\partial \beta_i}\left[\frac{c_{i,S}}{c_{j,S}\sigma\left(R_i\right)}\right] \\
&\propto \sigma\left(R_i\right)\frac{\partial}{\partial \beta_i}\left[\frac{c_{i,S}}{c_{j,S}}\right]-\frac{c_{i,S}}{c_{j,S}}\frac{\partial \sigma\left(R_i\right)}{\partial \beta_i} \\
&= \sigma\left(R_i\right)\frac{\sigma_F^2}{c_{j,S}}\left\{\beta_j\bar{\beta}^2\sigma_F^2+w_j\sigma_j^2\bar{\beta}+\left(w_j\sigma_j^2\beta_i-w_i\beta_j\sigma_i^2\right)w_i\right\}-\frac{c_{i,S}}{c_{j,S}}\frac{\sigma_F^2}{\sigma\left(R_i\right)}\beta_i \\
&\propto \sigma^2\left(R_i\right)\left\{\beta_j\bar{\beta}^2\sigma_F^2+w_j\sigma_j^2\bar{\beta}+\left(w_j\sigma_j^2\beta_i-w_i\beta_j\sigma_i^2\right)w_i\right\}-c_{i,S}c_{j,S}\beta_i\,.
\end{aligned}$$

It is difficult to determine under which conditions this expression becomes negative. We therefore use an approximation where we assume that bank i may dominate the system whereas the weight of the benchmark bank j can be neglected.[18] Eliminating all terms containing w_j gives:

$$\begin{aligned}
&\frac{\partial}{\partial \beta_i}\left[\frac{\Delta\mathrm{CoVaR}_\alpha^{S,i}}{\Delta\mathrm{CoVaR}_\alpha^{S,j}}\right] \\
&\propto \ \ldots\approx \sigma^2\left(R_i\right)\left\{\beta_j\bar{\beta}^2\sigma_F^2-w_i^2\beta_j\sigma_i^2\right\}-c_{i,S}c_{j,S}\beta_i \\
&= \ \left(\beta_i^2\sigma_F^2+\sigma_i^2\right)\left\{\beta_j\bar{\beta}^2\sigma_F^2-w_i^2\beta_j\sigma_i^2\right\}-\beta_i\left(\beta_i\bar{\beta}\sigma_F^2+w_i\sigma_i^2\right)\left(\beta_j\bar{\beta}\sigma_F^2+w_j\sigma_j^2\right) \\
&\approx \ \left(\beta_i^2\sigma_F^2+\sigma_i^2\right)\left\{\beta_j\bar{\beta}^2\sigma_F^2-w_i^2\beta_j\sigma_i^2\right\}-\left(\beta_i\bar{\beta}\sigma_F^2+w_i\sigma_i^2\right)\beta_i\beta_j\bar{\beta}\sigma_F^2\,.
\end{aligned}$$

Further consolidation plus introduction of $\kappa\equiv w_i/\left(1-w_i\right)$ and $\rho\equiv\beta_i/\beta_*$ lead to:

$$\begin{aligned}
&\frac{\partial}{\partial \beta_i}\left[\frac{\Delta\mathrm{CoVaR}_\alpha^{S,i}}{\Delta\mathrm{CoVaR}_\alpha^{S,j}}\right]\approx\ldots\propto\sigma_F^2\left(\beta_i^2\bar{\beta}^2\sigma_F^2-\beta_i^2w_i^2\sigma_i^2+\bar{\beta}^2\sigma_i^2-w_i^2\sigma_i^4-\beta_i^2\bar{\beta}^2\beta_j\sigma_F^2-\beta_iw_i\sigma_i^2\bar{\beta}\right) \\
&\propto \ \bar{\beta}^2\sigma_F^2-w_i^2\left(\sigma_i^2+\beta_i^2\sigma_F^2\right)-\beta_iw_i\bar{\beta}\sigma_F^2=\sigma_F^2\bar{\beta}\left[\bar{\beta}-\beta_iw_i\right]-w_i^2\left(\sigma_i^2+\beta_i^2\sigma_F^2\right) \\
&= \ \left(1-w_i\right)\sigma_F^2\bar{\beta}\beta_*-w_i^2\left(\sigma_i^2+\beta_i^2\sigma_F^2\right)=\left(1-w_i\right)\left[w_i\beta_i+\left(1-w_i\right)\beta_*\right]\beta_*-w_i^2\left(\frac{\sigma_i^2}{\sigma_F^2}+\beta_i^2\right) \\
&\propto \ \left[\kappa\beta_i+\beta_*\right]\beta_*-\kappa^2\left(\frac{\sigma_i^2}{\sigma_F^2}+\beta_i^2\right)=\beta_*^2-\kappa\left(\kappa\frac{\sigma_i^2}{\sigma_F^2}+\beta_i\left(\kappa\beta_i-\beta_*\right)\right) \\
&\propto \ 1-\kappa\left(\kappa\frac{\sigma_i^2}{\sigma_F^2\beta_*^2}+\rho\left(\kappa\rho-1\right)\right).
\end{aligned}$$

This expression can be positive or negative; it is discussed in the main text above equation (11).

A.15. Relative Effect of Size on MES, Exposure ΔCoVaR, and BETA

The partial derivative of the ratio of covariances can be simplified to:

$$\begin{aligned}
\text{(A-10)}\quad \frac{\partial}{\partial w_i}\left[\frac{c_{i,S}}{c_{j,S}}\right] &\propto c_{j,S}\frac{\partial c_{i,S}}{\partial w_i}-c_{i,S}\frac{\partial c_{j,S}}{\partial w_i} \\
&= \left(\beta_j\bar{\beta}\sigma_F^2+w_j\sigma_j^2\right)\left(\sigma_i^2+\beta_i\,\Delta_\beta\,\sigma_F^2\right)-\left(\beta_i\bar{\beta}\sigma_F^2+w_i\sigma_i^2\right)\beta_j\Delta_\beta\sigma_F^2 \\
&= \left(\beta_j\beta_*\sigma_F^2+w_j\sigma_j^2\right)\sigma_i^2+w_j\sigma_j^2\beta_i\,\Delta_\beta\,\sigma_F^2 \\
&= \beta_j\beta_*\sigma_F^2\sigma_i^2+w_j\sigma_j^2\left(\sigma_i^2+\beta_i\,\Delta_\beta\,\sigma_F^2\right).
\end{aligned}$$

[18]The relative effect in the reversed case (where bank i is small) is likely to be very similar to the direct effect since the small bank has only weak impact on the index, and so is its effect on the large bank's ΔCoVaR. Hence, the small bank's relative effect is basically the effect on its own ΔCoVaR, divided by a constant.

The derivative can actually become negative but only if the system volatility is a falling function of w_i. In fact, inspecting the last line, the derivative can only be negative if $\sigma_i^2+\beta_i\,\Delta_\beta\,\sigma_F^2<0$; this condition is sufficient for $\partial\sigma^2\left(R_S\right)/\partial w_i<0$, as shown by inequality (A-6). The case is similar to the direct effect of size on the ΔCoVaR (Section A.3). However, the conditions under which the partial derivative in equation (A-10) can become negative are considerably more exotic, as may be illustrated by the following estimate (details omitted):

$$\frac{\partial}{\partial w_i}\left[\frac{c_{i,S}}{c_{j,S}}\right] \quad\propto\quad \ldots \geq 1-\frac{1}{4}w_j\frac{\sigma_j^2\beta_*}{\sigma_i^2\beta_j}.$$

Nevertheless, this positive sensitivity can combine with a negative sensitivity of $\sigma\left(R_S\right)$, as explained below equation (13) in Section III.B of the article.

A.16. Relative Effect of Size on ΔCoVaR

The effect can be traced back to equation (A-10) since $\sigma\left(R_i\right)$ and $\sigma\left(R_j\right)$ are invariant to w_i:

$$\frac{\partial}{\partial w_i}\left[\frac{\Delta\text{CoVaR}_\alpha^{S,i}}{\Delta\text{CoVaR}_\alpha^{S,j}}\right] \quad=\quad \frac{\sigma\left(R_j\right)}{\sigma\left(R_i\right)}\frac{\partial}{\partial w_i}\left[\frac{c_{i,S}}{c_{j,S}}\right]\propto\frac{\partial}{\partial w_i}\left[\frac{c_{i,S}}{c_{j,S}}\right].$$

The relative effect is then the same as for the other measures.

B. The Structural Model for Asset and Equity Returns

B.1. Modeling Assumptions and Simulation

In this section we present the structural model used in the robustness test of Section III.C for the linear case. We extend (and simplify) the model of Collin-Dufresne and Goldstein (2001) which has been selected, first, since it is one of the few models that generate stationary returns[19] both for assets and equity and, second, as it generates a dynamic equity volatility that leads to heavier tails and tail dependence. This stochastic volatility makes the model similar to the empirical approach to the analysis of systemic risk measures taken by Brownlees and Engle (2012) and Acharya, Engle, and Richardson (2012), but there are also essential differences.[20]

We define asset returns as in the linear normal model of Section III in the article, with the modification that returns over finite time intervals are now lognormal. The SDE system for the latent systematic factor F_t and asset values $V_{i,t}$ reads:

$$\frac{\mathrm{d}F_t}{F_t} \quad=\quad \mu\,\mathrm{d}t+\sigma_F\,\mathrm{d}B_t,\quad \frac{\mathrm{d}V_{i,t}}{V_{i,t}}=\beta_i\frac{\mathrm{d}F_t}{F_t}+\sigma_i\,\mathrm{d}B_{i,t}\,,$$

with independent Brownian motions B_t and $B_{i,t}$. The asset returns are stationary by construction. It is convenient to replace the independent Brownian motions by the N-dimensional Gaussian process

$$Z_t \quad\equiv\quad \left(\beta_i\sigma_F B_{F,t}+\sigma_i B_{i,t}\right)_{i=1}^N\,,$$

[19] The models of Leland (1994) and Leland and Toft (1996) might appear as natural alternatives since they include stationary debt pricing. However, neither the equity returns nor those of the market value of assets are stationary in these models.

[20] While their model is richer in that it includes dynamic correlations between systematic and idiosyncratic shocks to equity returns, asset returns are not explicitly modeled. By contrast, we put weight on the consistency of asset and equity returns, which are both stationary, and an explicit modeling of default events.

which has zero drift and the covariance function

$$\Omega_{ij}(t,s) \equiv \operatorname{cov}(Z_{i,t}, Z_{j,s}) = \left(\beta_i\beta_j\sigma_F^2 + \sigma_i^2 I_{\{i=j\}}\right)\min(t,s) \,. \tag{B-1}$$

Most calculations are done in logarithmic terms, which we denote by small characters. The log assets process $v_{i,t} \equiv \log(V_{i,t})$ of bank i is an arithmetic Brownian motion following

$$\mathrm{d}v_{i,t} = \eta_i\,\mathrm{d}t + \mathrm{d}Z_{i,t} \quad \text{with} \quad \eta_i \equiv \beta_i\mu - 0.5\left(\beta_i^2\sigma_F^2 + \sigma_i^2\right) \,. \tag{B-2}$$

Each bank steers its debt by corporate action in order to achieve a certain target leverage.[21] The model approximates this behavior by a controlled dynamic default threshold $K_{i,t}$, which we interpret as the balance-sheet value of debt. It is time-differentiable and assumed to follow, in its logarithmic form, the ODE

$$\mathrm{d}k_{i,t} = \left[\lambda_i\left(v_{i,t} - k_{i,t} + \bar{l}_i\right) + \beta_i\mu\right]\,\mathrm{d}t\,,$$

where target leverage $\bar{l}_i$ and adjustment speed λ_i are strategic parameters.[22] Logarithmic "leverage" is defined as the distance $l_{i,t} = k_{i,t} - v_{i,t}$ between the log default threshold and log assets. As long as the bank is alive, $l_{i,t}$ is an Ornstein-Uhlenbeck (OU) process:

$$\mathrm{d}l_{i,t} = \lambda_i\left(\bar{l}_i - l_{i,t}\right)\mathrm{d}t - \mathrm{d}Z_{i,t}\,.$$

Normally, default would occur at the first time when $K_{i,t} = V_{i,t}$ holds or, equivalently, $l_{i,t} = 0$. As we are interested in observable time series for banks, and for technical reasons, we assume that the supervisor would take a bank into conservatorship and remove it from stock markets when its equity, relative to assets, falls short of a small but positive amount. Formally, we define the "default" time as $\tau_i \equiv \inf\{t : l_{i,t} = l_{\max}\}$ and set $l_{\max} = \log(97\%)$ in the simulations.

Equity is defined as the difference between the market values of assets and debt. For simplicity, we assume the accounting and market value of debt to be identical, so that equity is just $E_{i,t} = V_{i,t} - K_{i,t}$.[23] Normally, a shock to $V_{i,t}$ would partly carry over to the market value of $K_{i,t}$, and especially so over short-term horizons where the adaptation of the smooth process $K_{i,t}$ is of second order, compared to the diffusion shocks to $V_{i,t}$. In our simplified model, however, the short-term variation of assets *completely* carries over to the value of equity, which makes it more volatile especially in moments of high leverage, compared to a model with precise debt pricing. As we mainly test whether our results are robust to the presence of heavier tails in return distributions, we find it acceptable that these tails are a bit heavier than those arising from a fully-fledged debt pricing model.

Using

$$\mathrm{d}E_{i,t} = \mathrm{d}V_{i,t} - \mathrm{d}K_{i,t} = \left[\beta_i\mu E_{i,t} + \lambda_i K_{i,t}\left(l_{i,t} - \bar{l}_i\right)\right]\mathrm{d}t + V_{i,t}\mathrm{d}Z_{i,t}\,,$$

[21] Such corporate action can have various forms but is most conveniently thought of as purely liabilities-related transactions, such as debt/equity swaps or debt-financed stock repurchases.

[22] By adding $\beta_i\mu$ to the drift of $\kappa_{i,t}$, we differ from Collin-Dufresne and Goldstein (2001) in that our parameter $\bar{l}_i$ actually equals the expectation of $l_{i,t}$ under the stationary measure; in the original work there is a gap between them. The difference in the parameters is only a matter of notation.

[23] This assumption is not found in the work of Collin-Dufresne and Goldstein (2001). Focusing on bond pricing, they do not need to model the value of debt and equity explicitly. The only link between their structural model and bond pricing is the distribution of the default time, which is already defined by the Ornstein-Uhlenbeck process.

Itô's lemma gives the following SDE for log equity:

$$\text{(B-3)} \qquad \mathrm{d}\log E_{i,t} = \left[\beta_i \mu + \lambda_i \frac{l_{i,t} - \bar{l}_i}{e^{-l_{i,t}} - 1} - \frac{1}{2}\frac{\beta_i^2 \sigma_F^2 + \sigma_i^2}{\left(1 - e^{l_{i,t}}\right)^2}\right] \mathrm{d}t + \frac{1}{1 - e^{l_{i,t}}}\,\mathrm{d}Z_{i,t}\,.$$

The formula shows two things. First, as long as the log leverage $l_{i,t}$ is stationary, equity returns are stationary, too. Second, the dynamic diffusion generates heteroskedasticity in the equity returns.[24]

We now write target leverages and adjustment speeds in vector form $\bar{l}$ and λ and specify the stationary distribution of $l_t = (l_{i,t})_{i=1}^N$. If we could ignore that processes are stopped at τ_i, the stationary distribution would be $N\left(\bar{l}, \Sigma\right)$, where $\Sigma_{ij} = \left(\beta_i \beta_j \sigma_F^2 + \sigma_i^2 I_{\{i=j\}}\right) / \left(\lambda_i + \lambda_j\right)$. Of course, stopping cannot be ignored since, otherwise, some of the processes would have to start in the default state. We therefore select a distribution of l_t that is stationary *conditional on survival*, meaning that it fulfills

$$\mathbf{P}\left(l_s \in B | \tau_i > s,\ i = 1, ..., N\right) = \mathbf{P}\left(l_t \in B | \tau_i > t,\ i = 1, ..., N\right)$$

for arbitrary times t, s and measurable sets $B \subset \mathbb{R}^N$. This distribution is not analytically available; we approximate it by simulation as described below.

For simulation purposes, we replace the SDE (B-3) by an equation where drift and volatility are kept constant in a small time interval, in our case one day. The simulation of one-day asset and equity returns consists of the following steps. As explained below, multiple independent simulations must be performed in parallel.

1. Seed sample: draw M independent instances from a truncation of the multivariate $N\left(\bar{l}, \Sigma\right)$ distribution to the set $\left(-\infty, l_{\max}\right)^N$, where $l_{\max}$ is the uniform stopping threshold for $l_{i,t}$.

2. Draw M independent instances of the one-day diffusion term from a normal distribution: $\varepsilon \sim N\left(0, \Omega\left(T, T\right)\right)$, where $T \equiv 1/260$ is one trading day and Ω is defined in equation (B-1). Log leverage of the next day is obtained from[25] $l_1 = l_0 + diag(\lambda)\left(\bar{l} - l_0\right) - \varepsilon$ which, however, can also end up with some values larger than $l_{\max}$. As we censor stopping events, in such a case the l_0 is replaced by a randomly selected instance of l_0 from the sample, and a new ε is drawn. If necessary, the replacement is repeated until l_1 is smaller than $l_{\max}$ in all components.[26]

3. Having obtained l_1 from step 2, set $l_0 \equiv l_1$ and go back to step 2. Repeat this loop until the distribution converges to survival-conditional stationarity.[27] After convergence, go to the next step.

4. Draw $\varepsilon \sim N\left(0, \Omega\left(T, T\right)\right)$. Apply equation (B-2) to calculate daily asset returns as

$$R_{i,V} \equiv \exp\left\{\eta_i T + \varepsilon_i\right\} - 1\,.$$

[24]The solution of the SDE could explode if we allowed $l_{i,t}$ to reach zero. To prevent technical problems, we stop the process at τ_i, which bounds the diffusion differential from above. In our simulations, the instantaneous equity volatility can, at max, be about 3.3 times larger than the average.

[25]This AR(1) process is an approximation of the Ornstein-Uhlenbeck process. We could also set mean reversion and variance of the AR process such that it has exactly the same distribution as the Ornstein-Uhlenbeck process observed at discrete times; however, these parameters are almost exactly the same as λ and ε.

[26]If we knew the stationary distribution in advance, resetting l_0 would not be necessary. It is necessary to achieve convergence to the survival-conditional stationary distribution.

[27]We test for survival-conditional stationarity by the convergence of the sample characteristics mean, variance, skewness and kurtosis.

5. Randomly pick one of the M instances of l_0. Calculate one-day equity returns, according to equation (B-3), but keeping coefficients constant for one day, as

$$R_{i,eq} \equiv \left[\beta_i \mu + \lambda_i \frac{l_{i,0} - \bar{l}_i}{e^{-l_{i,0}} - 1} - \frac{1}{2} \frac{\beta_i^2 \sigma_F^2 + \sigma_i^2}{\left(1 - e^{l_{i,0}}\right)^2}\right] T + \frac{1}{1 - e^{l_{i,0}}} \varepsilon_i \,.$$

6. Add R_V and R_{eq} to the sample and go back to step 4.[28]

As in the base case, the index return is defined as a weighted average. For the simulations we set the following base-case parameters. $N = 50$ homogeneous banks of equal size; $\sigma_F = 0.05$, $\mu = 0.03$, $\sigma_i = 0.04$ (all annualized); $\beta_i = 1$, $\bar{l}_i = -0.1$; $\lambda_i = 2.38$. Note that the values relate to bank asset returns, which are typically much less volatile than those of corporates. The risk parameters are broadly consistent with bank asset volatilities from Moody's KMV (own calculations) and estimates of volatilities and leverage dynamics taken from Memmel and Raupach (2010).[29]

B.2. Simulation Results

In the simulations, we generate $M = 50,000$ samples for the initial leverage vector l_0, based on a sequence of 100 days to achieve survival-conditional stationarity. Estimates of SRMs are based on 10 million return vectors, using the estimation methods described at the beginning of Appendix D.

Table B-1 summarizes the results from three tests that closely correspond to the scenarios in Figures 2–4. The remainder of this subsection provides details of the findings discussed in Section III.C.

In the first test we replicate the study of Figure 2 by varying the idiosyncratic risk of asset returns and checking the direct and relative effect on the systemic risk measures. Both for asset and equity returns we observe that an increase in idiosyncratic risk lowers ΔCoVaR, as in the normal model. Importantly, bank i's equity returns get heavier distribution tails when idiosyncratic risk rises. For instance, when σ_i varies from 0.01 to 0.1, the kurtosis of the equity return goes up from 3.10 to 6.66 (assuming $\beta_i = 0.5$) or from 5.47 to 8.38 (assuming $\beta_i = 2$). Returns of the equity-based index have a fairly stable kurtosis around 3.4. This means that the decrease in ΔCoVaR goes along with both increased individual variance and heavier tails.

In the first test, we find negative sensitivities where they did not exist in the normal model: for high β_i, there is both a direct and relative negative effect on MES. Interestingly, no such effect is observed for asset returns, which suggests that tail thickness, as the outstanding difference in the return distributions, plays a role here.[30]

[28] We could also calculate l_1 and use it as the initial leverage vector for the next round. However, drawing l_0 in step 4 independently from the pre-produced sample speeds convergence up as it avoids the otherwise strong autocorrelation of volatilities.

[29] From univariate time-series estimates of capital ratios they obtain a median monthly mean reversions of $\lambda_{\text{monthly}} = 0.18$, which transforms into an Ornstein-Uhlenbeck mean reversion of $\lambda = -12 \times \log(1 - \lambda_{\text{monthly}}) = 2.38$, for which the time unit is one year.

[30] To exclude that the difference in the effect on the systemic risk measures is simply due to differences in the general volatility level, we do the following exercise: equity returns are rescaled after simulation such that they have the same daily standard deviation as their corresponding asset returns. After rescaling, the exposure ΔCoVaR still exhibits the same negative sensitivity to size (which does not exist for asset returns). As the correlation matrices of equity and asset returns are also very similar, tail thickness is the only plausible remaining explanation for the fact that the negative effect is observed with equity returns only.

Second, similar to Figure 3, we test the sensitivity to β_i, using different weights of bank i in the system. To examine whether we obtain similar effects as in the normal setup, other risk factors are given values derived from equation (11) to provoke negative sensitivities.[31] Both for asset and equity returns, an increasing β_i has a negative relative effect on all four systemic risk measures if the weight of a bank is very high (0.45). This finding conforms to the results of the normal model. Again, the increase in β_i is accompanied by a (moderate) increase in tail thickness.[32]

In a third test, similar to Figure 4, we vary the weight w_i of bank i in the system. Using different values for β_i, the results of the normal model are confirmed insofar as an increase of the weight can lower the BETA if the factor sensitivity β_i is high, which holds for asset and equity returns. Further negative sensitivities appear that did not exist in the normal model: for high β_i, there is a negative relative size effect on ΔCoVaR and both a direct and relative negative effect on exposure ΔCoVaR. As in the first test, no such effect is observed for asset returns, which seems to confirm that tail thickness matters.

What we observe when w_i grows while $\beta_i = 2$ is that a highly leptokurtic return (with a constant kurtosis of 8.5) increasingly shapes the index return, the kurtosis of which grows from 3.4 to 5.9. Hence, the riskiness of the index does not only rise for its increased volatility but also for a heavier loss tail. Both effects go along with a negative relative effect on either ΔCoVaR version.

[31] The equations (11) and (A-9), which indicate when negative sensitivities should appear in the normal model, do not actually apply here but may give an indication. In the test, σ_i is doubled from 0.04 to 0.08 (p.a.), while σ_F is halved from 0.05 to 0.025.

[32] When β_i varies from 0.5 to 2 (assuming the highest weight for bank i), the kurtosis of the equity return grows from 5.54 to 6.15. The equity index return exhibits kurtosis values between 4.82 and 5.08.

Table B-1: Effects of Risk Parameters on Systemic Risk Measures in a Non-normal Setting

We use simulations to analyze sensitivities of systemic risk measures to risk parameters in a multivariate extension of Collin-Dufresne and Goldstein (2001). Lognormal asset returns are drawn from correlated geometric Brownian motions. Equity returns are derived from a stochastic differential equation with continuously adapted debt. Both returns are stationary, conditional on banks' survival. A *direct effect* is understood as the change in a bank's own systemic risk measure owing to a change in one of the bank's parameters. The *relative effect* describes changes in the ratio between the systemic risk measures of two banks, e.g., $\mathrm{MES}_i/\mathrm{MES}_j$, while the risk parameter of bank i changes. A plus (minus) sign indicates that the systemic risk measure / the ratio of two measures grows (falls) with the parameter. If both signs appear, the sign can be positive or negative, depending on other parameters. If framed by exclamation marks, the outcome is different from its counterpart in Panel A of Table 1 (normal linear setup). The base-case parameters (p.a. for drift, volatility and mean reversion) are set to $\mathrm{E}(F) = 0.03$, $\sigma_F = 0.05$, $\sigma_j = 0.04$, $\beta_j = 1$ for asset returns; target log debt ratio $\bar{l}_j = -0.1$, mean reversion $\lambda_j = 2.38$, $w_j = 1/50$ for all j, $N = 50$; quantile level $\alpha = 0.01$ for the CoVaR measures and 0.05 for the MES. For the effect of idiosyncratic risk, σ_i varies from 0.01 to 0.1. Monotonicity in σ_i is checked for β_i between 0.5 and 2. For the effect of systematic risk, β_i varies from 0.5 to 2. Monotonicity in β_i is checked for a weight w_i between 0.02 (equal share) and 0.45. In this exercise we set $\sigma_i = 0.08$ and $\sigma_F = 0.025$ to provoke the effect found in the normal model. For the size effect, w_i varies from 0.02 (equal share) to 0.45 while β_i is fixed at values between 0.5 and 3.

Parameter	Effect	Return type	ΔCoVaR	Exp. ΔCoVaR	MES	BETA
Idiosyncratic Risk σ_i	Direct	Assets	−	+	+	+
		Equity	−	+	!!±!!	+
	Relative	Assets	−	+	+	+
		Equity	−	+	!!±!!	+
Systematic Risk β_i	Direct	Assets	+	+	+	+
		Equity	+	+	+	+
	Relative	Assets	±	±	±	±
		Equity	±	±	±	±
Size w_i	Direct	Assets	+	+	+	±
		Equity	+	!!±!!	+	±
	Relative	Assets	+	+	+	+
		Equity	!!±!!	!!±!!	+	+

C. Sensitivities to Risk Parameters if Bank i Is Removed from the System Index

In this appendix we analyze sensitivities as introduced in Section III.B of the article and carried out in Appendix A, with the modification that bank i is excluded from the system index used in the calculation of bank i's SRM.

In the next subsections we collect results for a counterpart to Table 1 in the main text. The results are summarized in Table C-1. Panel A presents the general results, which are possible signs of partial derivatives, while Panel B shows signs for a system that consists of two banks only.

Table C-1: Effect of Risk Parameters on Modified Systemic Risk Measures, Excluding the Bank of Interest from the Index

We analyze sensitivities of SRMs to certain risk parameters in a linear setting. Returns are described through $R_i = \beta_i F + \varepsilon_i$ and $R^*_{S,i} = \sum_{j \neq i} w^*_j R_j$ with independent $F \sim N\left(\mu, \sigma_F^2\right)$ and $\varepsilon_i \sim N\left(0, \sigma_i^2\right)$. Note that R_i is excluded from the index return $R^*_{S,i}$, which relies on weights $w^*_j = w_j / \left(1 - w_i\right)$. A *direct effect* of a parameter is understood as the partial derivative of an SRM. A *relative effect* refers to the ratio between the SRMs of two banks. It is the ratio's partial derivative to a parameter of the bank in the numerator, e.g., $\partial \left(\mathrm{MES}_i / \mathrm{MES}_j\right) / \partial \sigma_i$. Panel A presents possible signs of the derivatives. A superscript ^{n} marks cases where the sign applies under normal conditions. Only very implausible parameter combinations would generate the opposite sign. Panel B reports the partial derivatives' signs for a system consisting of two banks only.

Parameter	Effect type	ΔCoVaR	Exp. ΔCoVaR	MES	BETA
Panel A. Range of the Sign of Partial Derivative					
Idiosyncratic Risk σ_i	Direct	$-$	0	0	0
	Relative	$-$	$+$	$+^n$	$+$
Systematic Risk β_i	Direct	$+$	$+$	$+^n$	$+$
	Relative	$\pm$	$+$	$+^n$	$+$
Size w_i	Direct	0	0	0	0
	Relative	$\pm$	$\pm$	$\pm$	$+$
Panel B. Two Banks					
Idiosyncratic Risk σ_i	Direct	$-$	0	0	0
	Relative	$-$	$+$	$+^n$	$+$
Systematic Risk β_i	Direct	$+$	$+$	$+^n$	$+$
	Relative	$-$	$+$	$+^n$	$+$
Size w_i	Direct	0	0	0	0
	Relative	0	0	0	0

C.1. Notation and Basic Properties

The modified index that leaves bank i out is given by

$$R^*_{S,i} \equiv \sum_{k \neq i} w^*_k R_k,$$

with $w^*_k \equiv w_k / \left(1 - w_i\right)$, $k \neq i$, as introduced in Appendix A. The representations (8) and (9) of SRMs in the linear model remain correct, but they now refer to the modified index:

$$\Delta\mathrm{CoVaR}_\alpha^{S|i} = \frac{\mathrm{cov}\left(R^*_{S,i}, R_i\right)}{\sigma\left(R_i\right)} \Phi^{-1}(1-\alpha), \qquad \Delta\mathrm{CoVaR}_\alpha^{i|S} = \frac{\mathrm{cov}\left(R^*_{S,i}, R_i\right)}{\sigma\left(R^*_{S,i}\right)} \Phi^{-1}(1-\alpha),$$

$$\mathrm{MES}_i = -\beta_i \mu + \frac{\mathrm{cov}\left(R^*_{S,i}, R_i\right)}{\sigma\left(R^*_{S,i}\right)} \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{\alpha}, \qquad \mathrm{BETA}_i = \frac{\mathrm{cov}\left(R^*_{S,i}, R_i\right)}{\sigma^2\left(R^*_{S,i}\right)}.$$

We need to be more specific in the notation of average factor loadings:

$$\beta_{*,i} \equiv \sum_{k \neq i} w^*_k \beta_k.$$

Note that $w_k^* = w_k / (1 - w_i) = A_k / \left(\sum_{l \neq i} A_l\right)$ (where A_k is bank size in dollars) remains constant when w_i changes. As a consequence, $\beta_{*,i}$ is invariant to the risk parameters of bank i. The same holds true for the aggregate idiosyncratic risk of the index

$$\sigma_{*,i}^2 \equiv \sum_{k \neq i} \left(w_k^*\right)^2 \sigma_k^2,$$

which means that the variance of the index return

$$\sigma^2\left(R_{S,i}^*\right) = \beta_{*,i}^2 \sigma_F^2 + \sigma_{*,i}^2 \tag{C-1}$$

is also completely invariant to the risk parameters of bank i. By contrast, the distribution of the index used in the SRM of another bank j does depend on bank i's parameters, as becomes obvious in the form

$$\beta_{*,j} = \frac{w_i}{1 - w_j} \beta_i + H_{i,j}, \quad \text{where } H_{i,j} \equiv \frac{1}{1 - w_j} \sum_{k \notin \{i,j\}} w_k \beta_k . \tag{C-2}$$

Similar representations hold for $\sigma_{*,j}^2$. Finally, the return covariance between bank i and its index is given by

$$\operatorname{cov}\left(R_{S,i}^*, R_i\right) = \beta_i \beta_{*,i} \sigma_F^2 . \tag{C-3}$$

C.2. Direct Effects

Taking equation (C-3) into account, the invariance of $\beta_{*,i}$ to w_i and σ_i immediately implies:

$$\frac{\partial}{\partial \sigma_i}\left[\operatorname{cov}\left(R_{S,i}^*, R_i\right)\right] = \frac{\partial}{\partial w_i}\left[\operatorname{cov}\left(R_{S,i}^*, R_i\right)\right] = 0.$$

As $\sigma\left(R_{S,i}^*\right)$ is invariant to these parameters as well (see equation (C-1) and below), w_i and σ_i have no direct effect on the SRMs, with the only exception of ΔCoVaR. This measure has $\sigma\left(R_i\right)$ in the denominator, resulting in the same negative direct impact of σ_i as in Section III of the article.

The loading β_i to systematic risk has a positive direct effect on all SRMs, which is trivial for BETA and the exposure ΔCoVaR, where only the covariance is sensitive to β_i. In the case of ΔCoVaR, we obtain:

$$\frac{\partial \Delta \mathrm{CoVaR}_\alpha^{S|i}}{\partial \beta_i} \propto \frac{\partial}{\partial \beta_i}\left[\frac{\operatorname{cov}\left(R_{S,i}^*, R_i\right)}{\sigma\left(R_i\right)}\right] \propto \frac{\partial}{\partial \beta_i}\left[\left(\text{const} + \beta_i^{-2}\right)^{-1/2}\right] > 0.$$

The sensitivity of the MES to β_i is simpler than in the base case:

$$\frac{\partial \mathrm{MES}_i}{\partial \beta_i} = -\mu + \left\{\frac{\phi\left(\Phi^{-1}(\alpha)\right)}{\alpha}\left(1 + \frac{\sigma_{*,i}^2}{\beta_{*,i}^2 \sigma_F^2}\right)^{-1/2}\right\} \times \sigma_F .$$

A comparison with the estimate (A-7) in Appendix A.8 shows that the same arguments apply: the one-day MES is positively dependent on β_i under virtually all conditions, whereas the dependency can become negative at a one-year horizon for extreme parameter combinations.

C.3. Relative Effects on ΔCoVaR

As for the original SRMs, σ_i has no side effect on the ΔCoVaR of another bank so that the relative effect has the same negative sign as the direct effect.

To identify the relative size effect on ΔCoVaR, we observe that the standard deviations of individual returns in the denominators are invariant to w_i, just as the index return $R^*_{S,i}$. This gives, recalling the notation $w^*_k = w_k/\,(1-w_i)$ and the fact that w^*_k is constant while w_i changes,

$$\begin{aligned}\frac{\Delta\mathrm{CoVaR}_\alpha^{S|i}}{\Delta\mathrm{CoVaR}_\alpha^{S|j}} &= \frac{\mathrm{cov}\left(R^*_{S,i}, R_i\right)\sigma\left(R_j\right)}{\mathrm{cov}\left(R^*_{S,j}, R_j\right)\sigma\left(R_i\right)} \propto \left(\mathrm{cov}\left(R^*_{S,j}, R_j\right)\right)^{-1} = \beta_j^{-1}\beta_{*,j}^{-1}\sigma_F^{-2} \\ &\propto \beta_{*,j}^{-1}\,.\end{aligned}$$

At this point it is simpler to represent this expression by dollar bank sizes rather than weights:

$$\beta_{*,j} = \frac{\sum_{k\neq j} A_k\beta_k}{\sum_{k\neq j} A_k} = \frac{A_i\beta_i + \sum_{k\notin\{i,j\}} A_k\beta_k}{A_i + \sum_{k\notin\{i,j\}} A_k}\,.$$

Its sensitivity to dollar size is:

$$\begin{aligned}\frac{\partial}{\partial A_i}\left[\beta_{*,j}\right] &= \frac{\partial}{\partial A_i}\left[\frac{A_i\beta_i + \sum_{k\notin\{i,j\}} A_k\beta_k}{A_i + \sum_{k\notin\{i,j\}} A_k}\right] \\ &\propto \left(\sum_{k\neq j} A_k\right)\frac{\partial}{\partial A_i}\left[A_i\beta_i + \sum_{k\notin\{i,j\}} A_k\beta_k\right] - \left(\sum_{k\neq j} A_k\beta_k\right)\frac{\partial}{\partial A_i}\left[A_i + \sum_{k\notin\{i,j\}} A_k\right] \\ &= \left(\sum_{k\neq j} A_k\right)\beta_i - \sum_{k\neq j} A_k\beta_k = \sum_{k\notin\{i,j\}} A_k\beta_i - \sum_{k\notin\{i,j\}} A_k\beta_k \\ &\propto \beta_i - \sum_{k\notin\{i,j\}} w^{**}_k\beta_k\,, \quad \text{where} \quad w^{**}_k \equiv \frac{A_k}{\sum_{l\notin\{i,j\}} A_l}.\end{aligned}$$

Note that this expression makes sense only if there are more than two banks.[33] As the ΔCoVaR ratio is *inversely* proportional to $\beta_{*,i}$ in its sensitivity to size, the ratio shrinks when bank i grows under the condition that the bank has an *above*-average exposure β_i to systematic risk. While this has no impact on the index return $R^*_{S,i}$, growth does then make the system as a whole more volatile.

To assess the relative effect of β_i on ΔCoVaR, eliminating all parts that are insensitive to β_i gives:

$$\begin{aligned}\frac{\Delta\mathrm{CoVaR}_\alpha^{S|i}}{\Delta\mathrm{CoVaR}_\alpha^{S|j}} &= \frac{\mathrm{cov}\left(R^*_{S,i}, R_i\right)\sigma\left(R_j\right)}{\mathrm{cov}\left(R^*_{S,j}, R_j\right)\sigma\left(R_i\right)} \propto \frac{\mathrm{cov}\left(R^*_{S,i}, R_i\right)}{\mathrm{cov}\left(R^*_{S,j}, R_j\right)\sigma\left(R_i\right)} = \frac{\beta_i\beta_{*,i}\sigma_F^2}{\beta_j\beta_{*,j}\sigma_F^2\sqrt{\beta_i^2\sigma_F^2+\sigma_i^2}} \\ &\propto \frac{\beta_i}{\beta_{*,j}\sqrt{\beta_i^2\sigma_F^2+\sigma_i^2}}\,.\end{aligned}$$

[33] The special two-bank case is trivial because the system "index" consists of one bank only. There is no sensitivity to size in that case; see also Panel B of Table C-1.

Expanding $\beta_{*,j}$ according to equation (C-2), the derivative of the latter expression is:

$$\begin{aligned}
&\frac{\partial}{\partial \beta_i}\left[\frac{\beta_i}{\left(\frac{w_i}{1-w_j}\beta_i + H_{i,j}\right)\sqrt{\beta_i^2\sigma_F^2+\sigma_i^2}}\right] \\
\propto\ & \left(\frac{w_i}{1-w_j}\beta_i + H_{i,j}\right)\sqrt{\beta_i^2\sigma_F^2+\sigma_i^2} - \beta_i\frac{\partial}{\partial \beta_i}\left[\left(\frac{w_i}{1-w_j}\beta_i + H_{i,j}\right)\sqrt{\beta_i^2\sigma_F^2+\sigma_i^2}\right] \\
=\ & \left(\frac{w_i}{1-w_j}\beta_i + H_{i,j}\right)\sqrt{\beta_i^2\sigma_F^2+\sigma_i^2} - \beta_i\left(\frac{w_i}{1-w_j}\sqrt{\beta_i^2\sigma_F^2+\sigma_i^2} + \beta_i\sigma_F^2\frac{\left(\frac{w_i}{1-w_j}\beta_i + H_{i,j}\right)}{\sqrt{\beta_i^2\sigma_F^2+\sigma_i^2}}\right) \\
\propto\ & \left(\frac{w_i}{1-w_j}\beta_i + H_{i,j}\right)\sigma^2\left(R_i\right) - \beta_i\left(\frac{w_i}{1-w_j}\sigma^2\left(R_i\right) + \beta_i\sigma_F^2\left(\frac{w_i}{1-w_j}\beta_i + H_{i,j}\right)\right) \\
=\ & H_{i,j}\left(\sigma^2\left(R_i\right) - \beta_i^2\sigma_F^2\right) - \beta_i^2\sigma_F^2\frac{w_i}{1-w_j} = H_{i,j}\sigma_i^2 - \beta_i^2\sigma_F^2\frac{w_i}{1-w_j} \\
\propto\ & \left(\sum_{k\notin\{i,j\}}\frac{w_k}{1-w_j}\beta_k\right)\sigma_i^2 - \beta_i^2\sigma_F^2\frac{w_i}{1-w_j}.
\end{aligned}$$

The derivative can become negative if exceptional size combines with a loading to systematic risk above the average and a low level of idiosyncratic risk. As a consequence, β_i has an ambiguous relative effect on ΔCoVaR.

C.4. Relative Effects on BETA

The effects on SRMs of other banks $j \neq i$ are more complicated than the direct ones because R_i is included in the banking index which the SRM of bank j refers to. We cannot restrict ourselves to the quotient of covariances because the measures' denominators, which now refer to different indices, do not cancel out anymore:

$$\frac{\text{BETA}_i}{\text{BETA}_j} = \frac{\operatorname{cov}\left(R_{S,i}^*, R_i\right)\sigma^2\left(R_{S,j}^*\right)}{\operatorname{cov}\left(R_{S,j}^*, R_j\right)\sigma^2\left(R_{S,i}^*\right)} = \frac{\beta_i\beta_{*,i}}{\beta_j\beta_{*,j}} \times \frac{\beta_{*,j}^2\sigma_F^2+\sigma_{*,j}^2}{\sigma^2\left(R_{S,i}^*\right)}.$$

Eliminating all parts that are insensitive to the risk parameters of bank i, we obtain:

$$\frac{\text{BETA}_i}{\text{BETA}_j} \propto \frac{\beta_i\left(\beta_{*,j}^2\sigma_F^2+\sigma_{*,j}^2\right)}{\beta_{*,j}} = \beta_i\left(\beta_{*,j}\sigma_F^2 + \frac{\sigma_{*,j}^2}{\beta_{*,j}}\right).$$

This ratio has a (weak) positive dependency on σ_i through $\sigma_{*,j}^2$. The same holds for the relative effect on the exposure ΔCoVaR (and approximately the MES), where the root of $\left(\beta_{*,j}^2\sigma_F^2+\sigma_{*,j}^2\right)$ stands in the numerator.

The average $\beta_{*,j}$ is influenced by β_i and w_i in the following way:

$$\beta_{*,j} = \frac{w_i}{1-w_j}\beta_i + \frac{1}{1-w_j}\sum_{k\notin\{j,i\}}w_k\beta_k = \frac{w_i}{1-w_j}\beta_i + H_{i,j},$$

which yields:

$$\frac{\text{BETA}_i}{\text{BETA}_j} \propto \beta_i \left(\left[\frac{w_i}{1-w_j} \beta_i + H_{i,j} \right] \sigma_F^2 + \frac{\sigma_{*,j}^2}{\frac{w_i}{1-w_j}\beta_i + H_{i,j}} \right)$$
$$= \beta_i \left[\frac{w_i}{1-w_j} \beta_i + H_{i,j} \right] \sigma_F^2 + \frac{\sigma_{*,j}^2}{\frac{w_i}{1-w_j} + H_{i,j}\beta_i^{-1}}.$$

As $H_{i,j}$ is invariant to β_i and w_i, the ratio positively depends on β_i.

The size effect is more complicated because w_i also affects the degree to which idiosyncratic risks are diversified.

To get a feeling for the range of possible outcomes, we leave it at two limiting cases. In the first case, there are two banks only. We set $i = 1$ and note that the "index" to which the SRM of one bank refers consists of the other bank only. Correspondingly, $H_{1,2} = 0$ and $\sigma_{*,2}^2 = \sigma_1^2$. The ratio boils down to:

$$\frac{\text{BETA}_1}{\text{BETA}_2} \propto \left[\frac{w_1}{w_1} \beta_1 + H_{1,2} \right] \sigma_F^2 + \frac{\sigma_{*,j}^2}{\frac{w_1}{w_1}\beta_1 + H_{1,2}} = \beta_1 \sigma_F^2 + \frac{\sigma_1^2}{\beta_1},$$

which is insensitive to w_1.

The other limiting case is given by a system consisting of one large bank i and an infinitely granular remainder. The maximum diversification among these banks reduces $\sigma_{*,j}^2$ to $w_i^2 \sigma_i^2$ and yields $H_{i,j} = (1 - w_i)\, \beta_{*,i}$ in the limit such that we obtain:

$$\frac{\text{BETA}_i}{\text{BETA}_j} \propto \bar{\beta}\sigma_F^2 + \frac{\sigma_i^2}{\bar{\beta}} w_i^2 .$$

This gives for the derivative:

$$\frac{\partial}{\partial w_i} \left[\frac{\text{BETA}_i}{\text{BETA}_j} \right] \propto \Delta_\beta \left(\bar{\beta}\sigma_F^2 - w_i^2 \frac{\sigma_i^2}{\bar{\beta}} \right) + 2\sigma_i^2 w_i,$$

which again stresses the role of Δ_β and shows that the derivative can easily become negative. We now relate this observation to the ambiguous effect of size on total system risk. From the representation (A-5) of $\partial\sigma^2(R_S)/\partial w_i$ we obtain:

$$\frac{\partial}{\partial w_i} \left[\frac{\text{BETA}_i}{\text{BETA}_j} \right] \propto C \times \frac{\partial \sigma^2(R_S)}{\partial w_i} + \sigma_i^2 w_i \frac{\beta_{*,i}}{\bar{\beta}},$$

in which C is a constant. The positive second addend shows that the ratio of BETAs can shrink in w_i only if this decreases the system's volatility. However, this does not necessarily mean appropriateness: if the relative size effect is positive and the effect on $\sigma^2(R_S)$ negative, a growing bank i would be "punished" (relative to its competitors) for making the system safer.

C.5. Relative Effects on Exposure ΔCoVaR and MES

For brevity we set the drift term in the MES to zero, which makes the ratios for the exposure ΔCoVaR and the MES coincide. The only difference to the BETA ratios is that square roots are taken of variances:

$$\frac{\Delta\text{CoVaR}_\alpha^{i|S}}{\Delta\text{CoVaR}_\alpha^{j|S}} = \frac{\text{cov}\left(R_{S,i}^*, R_i\right) \sigma\left(R_{S,j}^*\right)}{\text{cov}\left(R_{S,j}^*, R_j\right) \sigma\left(R_{S,i}^*\right)} = \frac{\beta_i \beta_{*,i}}{\beta_j \beta_{*,j}} \times \frac{\sqrt{\beta_{*,j}^2 \sigma_F^2 + \sigma_{*,j}^2}}{\sigma\left(R_{S,i}^*\right)}$$
$$\propto \frac{\beta_i \sqrt{\beta_{*,j}^2 \sigma_F^2 + \sigma_{*,j}^2}}{\beta_{*,j}}.$$

The last expression has been purified from all factors that are insensitive to σ_i, β_i, and w_i. Let us first analyze the sensitivity to β_i. The ratio:

$$\begin{aligned}\frac{\Delta\text{CoVaR}_\alpha^{i|S}}{\Delta\text{CoVaR}_\alpha^{j|S}} &\propto \sqrt{\beta_i^2\sigma_F^2 + \frac{\beta_i^2}{\beta_{*,j}^2}\sigma_{*,j}^2} = \sqrt{\beta_i^2\sigma_F^2 + \left(\frac{\beta_i}{\frac{w_i}{1-w_j}\beta_i + H_{i,j}}\right)^2 \sigma_{*,j}^2} \\ &= \sqrt{\beta_i^2\sigma_F^2 + \left(\frac{w_i}{1-w_j} + \frac{H_{i,j}}{\beta_i}\right)^{-2}\sigma_{*,j}^2}\end{aligned}$$

is obviously monotonic in β_i.

Reducing the SRM ratio to the variables affected by w_i gives:

$$\frac{\Delta\text{CoVaR}_\alpha^{i|S}}{\Delta\text{CoVaR}_\alpha^{j|S}} \propto \frac{\beta_i\sqrt{\beta_{*,j}^2\sigma_F^2 + \sigma_{*,j}^2}}{\beta_{*,j}} \propto \sqrt{\sigma_F^2 + \left(\frac{\sigma_{*,j}}{\beta_{*,j}}\right)^2},$$

so that it suffices to analyze the ratio $\sigma_{*,j}/\beta_{*,j}$. We define $w_{k|j} \equiv w_k/(1-w_j)$ and expand the ratio to:

$$\frac{\sigma_{*,j}}{\beta_{*,j}} = \frac{\sqrt{w_{i|j}^2\sigma_i^2 + \sum_{k\notin\{j,i\}} w_{k|j}^2\sigma_k^2}}{w_{i|j}\beta_i + H_{i,j}}.$$

As $w_{i|j}$ is monotonic in w_i, we can also take the derivative to the former, which gives:

$$\begin{aligned}\frac{\partial}{\partial w_i}\left[\frac{\sigma_{*,j}}{\beta_{*,j}}\right] &\propto \frac{\partial}{\partial w_{i|j}}\left[\frac{\sigma_{*,j}}{\beta_{*,j}}\right] \propto \beta_{*,j}\frac{\partial}{\partial w_{i|j}}[\sigma_{*,j}] - \sigma_{*,j}\frac{\partial}{\partial w_{i|j}}[\beta_{*,j}] \\ &= \beta_{*,j}\frac{\partial}{\partial w_{i|j}}\left[\sqrt{w_{i|j}^2\sigma_i^2 + \sum_{k\notin\{j,i\}} w_{k|j}^2\sigma_k^2}\right] - \sigma_{*,j}\frac{\partial}{\partial w_{i|j}}\left[w_{i|j}\beta_i + H_{i,j}\right] \\ &= \beta_{*,j}\frac{w_{i|j}\sigma_i^2}{\sigma_{*,j}} - \sigma_{*,j}\beta_i \propto \beta_{*,j}\frac{\sigma_i^2}{\sigma_{*,j}^2}w_{i|j} - \beta_i\,.\end{aligned}$$

This can become negative if β_i is above the average (without bank j), given by $\beta_{*,j}$, if bank i is not too large and the other banks' idiosyncratic risks are not too diversified. The relative effect on exposure ΔCoVaR and MES can thus be positive or negative. As before, there are realistic parameter constellations under which these effects combine with opposite effects on $\sigma(R_S)$.

D. Sensitivities in the Contagion Case

This appendix complements Section IV, explaining in detail how the contagion intensity γ_1 affects SRMs. The numerical calculations are based on 100 million independent simulations and estimation procedures for the SRMs as described in Footnote 13 of the article.

D.1. ΔCoVaR

To understand how ΔCoVaR is affected by the contagion intensity γ_1, we express the system return as a function of the return of an individual bank. For ease of exposition, we will incorporate the choice of uniform unit β_j, $1 \leq j \leq N$, and equally-weighted banks that we made for the simulation. When applied to bank 1, the left-hand part of equation (14) then implies $F = R_1 - \varepsilon_1$, which can be plugged into the representation of R_S to eliminate the factor return F:

$$R_S \;=\; R_1 - \frac{N-1}{N}\left[1 - I_{\{\varepsilon_1<\kappa\}}\gamma_1\right]\varepsilon_1 + \frac{1}{N}\sum_{j=2}^{N}\varepsilon_j\,. \tag{D-1}$$

When we use equation (D-1) to study the system return conditional on a quantile of R_1, it is not enough to replace R_1 by its quantile. The conditional distribution of ε_1 also differs from the unconditional one. However, it is not affected by the contagion intensity γ_1, which facilitates the analysis.

Changing γ_1 can influence the ΔCoVaR through effects on both the 50% quantile and the 1% quantile of R_S. For the 50% quantile, effects will be relatively small because the probability that contagion occurs if R_1 is at its median is very small. This probability, which again is independent of γ_1, can be determined by exploiting the fact that the conditional distribution of ε_1 is normal (see Section D.5). For the parameter combination used here, contagion occurs with a probability of 1.00% if R_1 is at its median, and with a probability of 50.02% if R_1 is at its 1% quantile (see Equation (D-8)). For the purpose of understanding the patterns of Graph A in Figure 5, it is therefore sufficient to focus on the 1% CoVaR.

Graph A shows that changes in the contagion intensity do not affect the ΔCoVaR until γ_1 reaches a value of around 0.75. This may seem surprising, given that contagion happens with a probability of over 50% once R_1 is at its 1% quantile. However, it does not necessarily follow that contagion events are crucial for the CoVaR, which is the 1% quantile of R_S conditional on R_1 taking some value. Equation (D-1) shows that one way of arriving at a low conditional realization of R_S is to have a very positive realization of ε_1. If ε_1 is positive, however, there is no contagion.

The extent to which low realizations of R_S are associated with contagion has already been illustrated in Figure 6 and explained in Section IV.A of the article.

For an infected bank – here we take it to be bank 2 – we can derive:

$$R_S \;=\; R_2 - \frac{N-1}{N}\varepsilon_2 + \frac{1}{N}\left(1 - I_{\{\varepsilon_1<\kappa\}}\gamma_1\right)\varepsilon_1 + \frac{1}{N}\sum_{j=3}^{N}\varepsilon_j\,. \tag{D-2}$$

The direct effects of γ_1 and ε_1 that we discussed above now play a smaller role because they enter the equation with a factor of $1/N$ rather than $(N-1)/N$. However, there is an additional effect because the quantiles of R_2 also depend on γ_1. Increasing γ_1 lowers both the median and the 1% quantile of R_2, with the effect on the latter being more pronounced. This is the key factor behind the pattern shown in Graph A of Figure 5: contagion increases the risk of the infected bank as well as the entire system.

Comparing the infectious and the infected bank, Graph A of Figure 5 shows that contagion drives a wedge between the ΔCoVaR of the two banks, which increases with the strength of the spillovers. In the presence of contagion, ΔCoVaR assigns a larger systemic risk to the infected bank.

While this pattern is consistent with the discussion from above, we would like to provide another way of understanding it. In Figure D-1, we visualize the conditional distribution of the system return by plotting it against the return of the infectious bank and the return of an infected bank, respectively. We choose the contagion intensity γ_1 to be 0.75 and show three scatterplots.

Graph A contains the full sample, Graph B only contagion cases ($\varepsilon_i < \kappa$), and Graph C only cases without contagion. Conditional on the bank returns being at their 1% quantiles, the system return has a lower mean in the case of the infectious bank.

While this effect tends to make the ΔCoVaR of the infected bank more extreme, there is a stronger effect working in the opposite direction. For the infectious bank, the conditional variance of the system return is relatively low. In the contagion case, the system return is highly correlated with the infectious bank because that bank's idiosyncratic risk has spread through the system. If there is no contagion, the conditional volatility is relatively low because it is then relatively likely that the system return has been brought about by a low factor return.[34]

The low conditional variance of the system return means that – according to ΔCoVaR – the system appears to have a relatively low risk, conditional on the infectious bank being at its 1% quantile.

D.2. Exposure ΔCoVaR

Comparing the four measures in Figure 5, the exposure ΔCoVaR of the infectious bank in Graph B exhibits the most complex pattern. At first, it becomes larger. Then it shrinks, but this new tendency is again reversed. Rearranging equation (D-1), we can examine how the return of the infectious bank depends on the system return:

$$R_1 \quad = \quad R_S + \frac{N-1}{N}\left(1 - I_{\{\varepsilon_1<\kappa\}}\gamma_1\right)\varepsilon_1 - \frac{1}{N}\sum_{j=2}^{N}\varepsilon_j \,.$$

Using this equation is now considerably more involved than above. A change in γ_1 affects the quantiles of R_S as well as the conditional distribution of ε_1. An inspection of the simulated CoVaR figures reveals which forces drive the observed patterns: the initial upward move goes back to a decrease in the $\text{CoVaR}_{50\%}$; this decrease is then overcompensated by changes in the $\text{CoVaR}_{1\%}$, which exhibits a trough-shaped behavior, with the bottom of the trough being close to $\gamma_1 = 0.5$. We explain those in turn.

An increase in γ_1 makes it less likely that R_S is at its median once contagion has occurred because contagion shifts R_S away from its median and this shift is stronger, the larger γ_1.[35] In consequence, a higher γ_1 means that there will be fewer realizations with ε_1 below the contagion threshold if R_S is at its median. There is another effect having the same impact on $\text{CoVaR}_{50\%}$: an increasing γ_1 pulls $\left(1 - I_{\{\varepsilon_1<\kappa\}}\gamma_1\right)\varepsilon_1$ towards zero if there is contagion. Together, this makes the $\text{CoVaR}_{50\%}$ decrease, as it is defined as the negative of the 1% quantile of R_1 conditional on R_S being at its median. An opposite effect – an increase of γ_1 lowers the median of R_S – is relatively small.

What happens if we condition on the 1% quantile of R_S in order to determine the $\text{CoVaR}_{1\%}$? An increase in γ_1 now makes it *more* likely that it was contagion that led to the

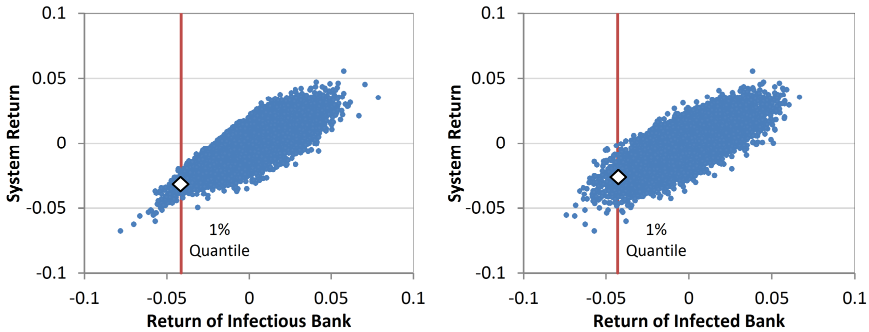
[34]If there is no contagion, the system return can be low because of a low factor realization or a low average realization of the infected banks' idiosyncratic risk. With 49 banks, however, the variance of the average idiosyncratic shock is very low, making it less likely to be the reason for a low system return.

[35]More precisely, contagion shifts the system return *down* so that some realizations of R_S will also be shifted (from above) towards the median. However, even with $\gamma_1 = 0$ the majority of "contagion" events has realizations of R_S below the median. If γ_1 rises, more returns with contagion are therefore pushed away from the median than are pushed towards it. This lowers the presence of contagion events among those having a return at the median, in total. There is also a slight compensating effect since the median of R_S is decreased, too; but the effect has lower order because the median is mainly driven by F and aggregate idiosyncratic risk, which both do not depend on γ_1.

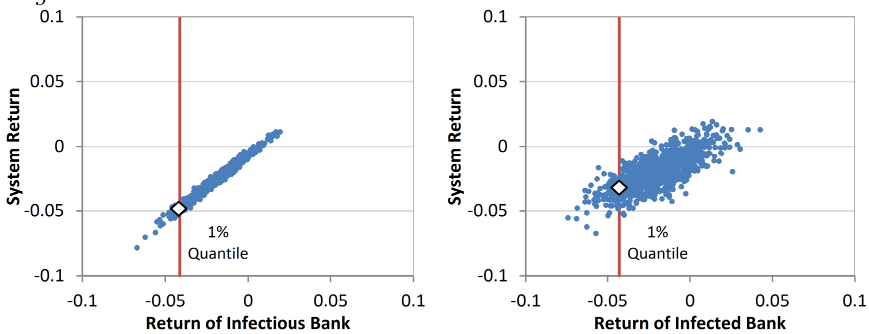
Figure D-1: Simulated System Returns Versus Returns of Infectious and Infected Banks

We simulate returns for the same system as in Figure 5 in the article. The contagion intensity of bank 1 is set to $\gamma_1 = 0.75$. We plot simulated system returns R_S against individual returns R_1 (infectious bank, on the left) and R_j (an infected bank, on the right). Graph A plots the full sample. Graph B contains only cases of contagion, where $\varepsilon_1 < \kappa$. Graph C contains cases of no contagion. The vertical line marks the event that the individual bank return is at its 1% quantile, the event which $\text{CoVaR}_{1\%}$ conditions on. The diamond marks the corresponding conditional mean of the system return.

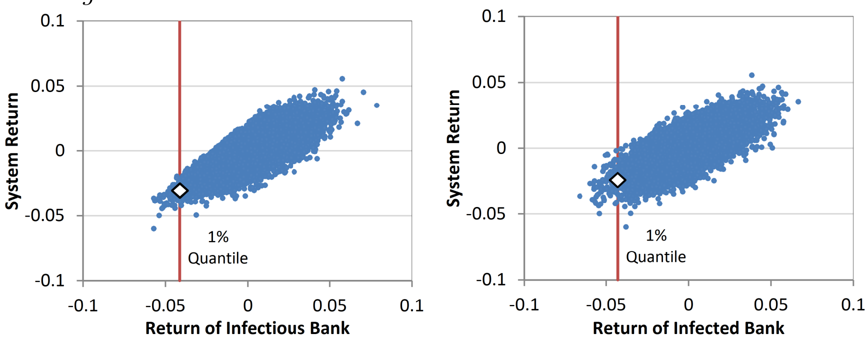
Graph A. Full Sample



Graph B. Contagion



Graph C. No Contagion



extreme realization of R_S. More contagion implies more extremely negative realizations of ε_1, but what matters for the CoVaR$_{1\%}$ is $\left(1 - I_{\{\varepsilon_1<\kappa\}}\gamma_1\right)\varepsilon_1$, which is pulled towards zero as γ_1 increases.

To get an intuition why the second effect is stronger, consider the extreme case in which γ_1 equals 1. Then, the smallest value that $\left(1 - I_{\{\varepsilon_1<\kappa\}}\gamma_1\right)\varepsilon_1$ can take is κ, even though there will be many realizations with an ε_1 smaller than κ. The overall trough-shaped pattern in the exposure ΔCoVaR arises because this effect is at some point outsized by another effect of γ_1 which works in the opposite direction: increasing γ_1 lowers the 1% quantile of R_S which we condition on.

For the infected banks the pattern is less complex: the ΔCoVaR grows in γ_1. We can rearrange equation (D-2):

$$R_2 = R_S + \frac{N-1}{N}\varepsilon_2 + \frac{1}{N}\left(I_{\{\varepsilon_1<\kappa\}}\gamma_1 - 1\right)\varepsilon_1 - \frac{1}{N}\sum_{j=3}^{N}\varepsilon_j\,.$$

The effects of changes in the conditional distribution of $\left(1 - I_{\{\varepsilon_1<\kappa\}}\gamma_1\right)\varepsilon_1$ are now less important than in the case of the infectious bank studied above because the factor $1/N$ is much smaller than $(N-1)/N$. Changes in the conditional quantiles of R_2 are therefore driven by changes in the quantiles of R_S. Since the 1% quantile of R_S is more sensitive to changes in γ_1 than the 50% quantile, the exposure ΔCoVaR of bank 2 becomes larger.

D.3. MES

For the infectious bank, we again inspect:

$$R_1 = R_S + \frac{N-1}{N}\left(1 - I_{\{\varepsilon_1<\kappa\}}\gamma_1\right)\varepsilon_1 - \frac{1}{N}\sum_{j=2}^{N}\varepsilon_j\,.$$

MES and exposure ΔCoVaR are similar in that we condition on a tail event of the system return. Increasing γ_1 makes it more likely that contagion has occurred in the conditioning event that R_S is below its 5% quantile. A higher probability of contagion means that the conditional means of both R_S and ε_1 are more negative. But the direct effect of γ_1 works in the opposite direction. Through $\left(1 - I_{\{\varepsilon_1<\kappa\}}\gamma_1\right)\varepsilon_1$, an increase in γ_1 makes the MES less extreme. The concave shape of the MES arises because the first effect dominates for small γ_1, while the second effect gains weight when γ_1 gets larger.

For an infected bank, the reasoning is very close to the one for the exposure ΔCoVaR:

$$R_2 = R_S + \frac{N-1}{N}\varepsilon_2 + \frac{1}{N}\left(I_{\{\varepsilon_1<\kappa\}}\gamma_1 - 1\right)\varepsilon_1 - \frac{1}{N}\sum_{j=3}^{N}\varepsilon_j\,.$$

What matters most are changes in the quantile of R_S, which is driven down by increases in γ_1. The second term does not contribute much because it contains the factor $1/N$.

For γ_1 very close to 1, the MES indicates that the infected bank is riskier than the infectious bank. This observation is easiest understood if infection occurred and $\gamma_1 = 1$. In this case, the equations for R_1 and R_2 differ only in ε_2, which adds to R_2 by $+\varepsilon_2 (N-1)/N$ while it adds to R_1 by $-\varepsilon_2/N$. Conditional on $\{R_S < Q_S^\alpha\}$, the expected value of ε_2 is negative because it also contributes to the system return.

D.4. BETA

We now explain the pattern observed in Graph D of Figure 5 that the BETA of the infectious bank is hump-shaped in the contagion intensity γ_1, while the BETA curve of the infected bank is nearly flat.

To this end, we trace the unconditional second moments in the formula for BETA back to conditional moments. Setting $\chi \equiv I_{\{\varepsilon_1<\kappa\}}$ for the contagion dummy, the numerator of BETA can be decomposed into:

$$\mathrm{cov}\left(R_1, R_S\right) \quad = \quad \mathrm{E}\left(\mathrm{cov}\left(R_1, R_S|\chi\right)\right) + \mathrm{cov}\left(\mathrm{E}\left(R_1|\chi\right), \mathrm{E}\left(R_S|\chi\right)\right),$$

where the χ-conditional moments are random variables with two possible realizations, one for $\chi = 1$ and one for $\chi = 0$. It is crucial to observe that the values for the non-contagion case $\chi = 0$ do not depend on γ_1 because we are then in the standard one-factor case, albeit under a special distribution for ε_1; that distribution is not affected by γ_1 either. Putting everything invariant to γ_1 into constants gives:

$$\text{(D-3)} \qquad \begin{aligned} \mathrm{cov}\left(R_1, R_S\right) \quad = \quad & c_1 \mathrm{cov}\left(R_1, R_S|\varepsilon_1 < \kappa\right) + c_2 \\ & + c_3\left(\mathrm{E}\left(R_1|\varepsilon_1 < \kappa\right) - c_4\right)\left(\mathrm{E}\left(R_S|\varepsilon_1 < \kappa\right) - c_5\right) . \end{aligned}$$

We now study the parts that are sensitive to γ_1. In equation (D-1) we can replace R_1 by $F + \varepsilon_1$ to obtain:

$$R_S \quad = \quad F + \left[\frac{N-1}{N} I_{\{\varepsilon_1<\kappa\}}\gamma_1 + \frac{1}{N}\right]\varepsilon_1 + \frac{1}{N}\sum_{j=2}^{N}\varepsilon_j .$$

Introducing $\sigma^2_{1,\mathrm{cont}}$ as the contagion-conditional variance of ε_1, we find:

$$\begin{aligned} \mathrm{cov}\left(R_1, R_S|\varepsilon_1 < \kappa\right) \quad &= \quad \mathrm{cov}\left(F + \varepsilon_1, F + \left[\frac{N-1}{N}\gamma_1 + \frac{1}{N}\right]\varepsilon_1 + \frac{1}{N}\sum_{j=2}^{N}\varepsilon_j \,\middle|\, \varepsilon_1 < \kappa\right) \\ &= \quad \sigma_F^2 + \left[\frac{N-1}{N}\gamma_1 + \frac{1}{N}\right]\sigma^2_{1,\mathrm{cont}}, \end{aligned}$$

which is linear in γ_1. The mean of R_1 conditional on contagion is

$$\mathrm{E}\left(R_1|\varepsilon_1 < \kappa\right) \quad = \quad \mathrm{E}\left(F\right) + \mathrm{E}\left(\varepsilon_1|\varepsilon_1 < \kappa\right)$$

and does not depend on γ_1, while there is a linear dependency in the conditional mean of R_S:

$$\text{(D-4)} \qquad \mathrm{E}\left(R_S|\varepsilon_1 < \kappa\right) \quad = \quad \mathrm{E}\left(F\right) + \left(\frac{N-1}{N}\gamma_1 + \frac{1}{N}\right)\mathrm{E}\left(\varepsilon_1|\varepsilon_1 < \kappa\right).$$

Introducing new constants, equation (D-3) is reduced to:

$$\mathrm{cov}\left(R_1, R_S\right) \quad = \quad c_1 \mathrm{cov}\left(R_1, R_S|\varepsilon_1 < \kappa\right) + c_6 + c_7 \mathrm{E}\left(R_S|\varepsilon_1 < \kappa\right),$$

which shows that the unconditional covariance is linear in γ_1.

Turning to the denominator of BETA, a similar decomposition gives:

$$\sigma^2\left(R_S\right) \quad = \quad c_1\sigma^2\left(R_S|\varepsilon_1 < \kappa\right) + c_2 + c_3\left(\mathrm{E}\left(R_S|\varepsilon_1 < \kappa\right) - c_4\right)^2,$$

of which we already know that the conditional mean on the right is linear in γ_1, resulting in quadratic dependency of the entire second term. Also the first term is quadratic in γ_1:

$$\sigma^2\left(R_S|\varepsilon_1<\kappa\right) \quad = \quad \sigma_F^2+\left[\frac{N-1}{N}\gamma_1+\frac{1}{N}\right]^2\sigma_{1,\text{cont}}^2+\frac{1}{N^2}\sigma_j^2\,,$$

and thus the whole $\sigma^2\left(R_S\right)$.

Altogether, the observation that the covariance of R_S and R_1 is linear in γ_1 while the variance is quadratic in γ_1 explains the hump-shaped relationship between the contagion intensity γ_1 and $\text{cov}\left(R_1,R_S\right)/\sigma^2\left(R_S\right)$, which is the BETA of the infectious bank.

For the infected banks, we need to analyze the components of:

(D-5) $$\begin{aligned}\text{cov}\left(R_2,R_S\right) \quad &= \quad d_1\text{cov}\left(R_2,R_S|\varepsilon_1<\kappa\right)+d_2\\ &\quad +d_3\left(\text{E}\left(R_2|\varepsilon_1<\kappa\right)-d_4\right)\left(\text{E}\left(R_S|\varepsilon_1<\kappa\right)-d_5\right)\,.\end{aligned}$$

We observe that the conditional covariance is now quadratic in γ_1:

$$\begin{aligned}\text{cov}\left(R_2,R_S|\varepsilon_1<\kappa\right) \quad &= \quad \text{cov}\left(F+\varepsilon_2+\gamma\varepsilon_1,F+\left[\frac{N-1}{N}\gamma_1+\frac{1}{N}\right]\varepsilon_1+\frac{1}{N}\sum_{j=2}^{N}\varepsilon_j\,\middle|\,\varepsilon_1<\kappa\right)\\ &= \quad \sigma_F^2+\left[\frac{N-1}{N}\gamma_1^2+\frac{1}{N}\gamma_1\right]\sigma_{1,\text{cont}}^2+\frac{1}{N}\sigma_2^2\,.\end{aligned}$$

In contrast to the infectious bank, the conditional mean $\text{E}\left(R_2|\varepsilon_1<\kappa\right)$ linearly depends on γ_1. In equation (D-5) it is multiplied with a further linear expression, as equation (D-4) has shown, such that the whole product is quadratic in γ_1. In the definition of BETA, both the numerator and the denominator are thus quadratic in γ_1, which gives an intuition for the almost flat relationship between γ_1 and the BETA of an infected bank.

D.5. Conditional Distribution of the Noise Term ε_1

Given the return $R_1=\beta_1F+\varepsilon_1$ of the infectious bank, we calculate the distribution of ε_1 under the condition that R_1 equals its quantile value at level α. This is done by an orthogonal linear representation, in the same way as in Section III of the article.

Starting from the assumptions $F\sim N\left(\mu,\sigma_F^2\right)$, $\varepsilon_1\sim N\left(0,\sigma_1^2\right)$ and independence of F and ε_1, we calibrate an equation $\varepsilon_1=a+bR_1+\eta$ such that R_1 and η are independent and $\text{E}\left(\varepsilon_1\right)=\text{E}\left(\eta\right)=0$. This requires $a=-b\beta_1\mu$ and

(D-6) $$\varepsilon_1 \quad = \quad b\left(R_1-\beta_1\mu\right)+\eta,$$

with $b=\text{cov}\left(R_1,\varepsilon_1\right)/\sigma^2\left(R_1\right)=\sigma_1^2/\sigma^2\left(R_1\right)$. The noise orthogonal to R_1 has variance

$$\sigma_\eta^2 \quad = \quad \sigma_1^2-b^2\sigma^2\left(R_1\right)=\sigma_1^2\beta_1^2\sigma_F^2/\sigma^2\left(R_1\right)\,,$$

which completes the conditional moments. Putting $Q_\alpha\left(R_1\right)=\beta_1\mu+\sigma\left(R_1\right)\Phi^{-1}\left(\alpha\right)$ into (D-6), we obtain:

(D-7) $$\begin{aligned}\varepsilon_1\left|\left\{R_1=Q_\alpha\left(R_1\right)\right\}\right. \quad &\sim \quad N\left(b\left(Q_\alpha\left(R_1\right)-\beta_1\mu\right),\sigma_\eta^2\right)\\ &= \quad N\left(\frac{\sigma_1^2}{\sigma\left(R_1\right)}\Phi^{-1}\left(\alpha\right),\frac{\sigma_1^2\beta_1^2\sigma_F^2}{\sigma^2\left(R_1\right)}\right)\,.\end{aligned}$$

We are also interested in the conditional probability of the contagion event, which is $\{\varepsilon_1 < \kappa\}$. Let us assume that the unconditional probability of contagion is χ. The R_1-conditional probability is calculated by standardization of ε_1 based on the moments given in formula (D-7):

$$\text{(D-8)} \qquad \begin{aligned} \mathbf{P}\left(\varepsilon_1 < \kappa | R_1 = Q_\alpha\left(R_1\right)\right) &= \mathbf{P}\left(\varepsilon_1 < \sigma_1 \Phi^{-1}\left(\chi\right) | R_1 = Q_\alpha\left(R_1\right)\right) \\ &= \Phi\left(\frac{\sigma_1 \Phi^{-1}\left(\chi\right) - \mathrm{E}\left(\varepsilon_1 | R_1 = Q_\alpha\left(R_1\right)\right)}{\sigma\left(\varepsilon_1 | R_1 = Q_\alpha\left(R_1\right)\right)}\right) \\ &= \Phi\left(\frac{\sigma\left(R_1\right)\Phi^{-1}\left(\chi\right) - \sigma_1 \Phi^{-1}\left(\alpha\right)}{\beta_1 \sigma_F}\right). \end{aligned}$$

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