Internet Appendix to

"Tips from TIPS: the Informational Content of Treasury Inflation-Protected Security Prices"

- Not intended for publication -

1 All Parameter estimates

Table 1 reports parameter estimates for all five models mentioned in the paper.

2 Additional Model Results

Figure 1 shows the results for Model LI, while Figure 2 shows the results of Model LII estimated over the pre-crisis period.

Figure 3 plots the yield fitting errors from Model LII. We see that the fit is generally worse for shorter-term nominal yields and during the crisis period.

3 Robustness Checks

3.1 Gaussian Assumption for Expected Inflation

It's now well known in the literature that, by allowing flexible correlations between the factors, the affine-Gaussian bond pricing model outperforms affine models with stochastic volatilities in matching term premium dynamics.[1] A similar argument can be made for using Gaussian models to study inflation risk premiums, which derive from the correlation between the real pricing kernel and inflation, even though such models by construction cannot capture *time-varying* inflation uncertainty and cannot decompose the inflation risk premium further into time-varying inflation risks and time-varying prices of inflation risk.

In addition to time-varying inflation uncertainties, recent studies of inflation caps and floors by **?** and **?** find that investors appear to attach significantly more weight to extreme inflation outcomes (either deflation or high inflation) than a normal distribution would suggest. These observations raise some doubt on the appropriateness of modeling inflation as a conditional Gaussian process. Nevertheless, due to the short history of inflation caps and floors, both papers focus on a short sample dominated by the financial crisis and the zero lower bound period; it therefore remains to be seen whether the Gaussian assumption for inflation, both under the physical and the risk-neutral measures, works better over a longer time span as the one used in the current study.

We first examine the inflation distribution under the *physical* measure. **?** analyze the probabilistic forecasts for inflation in the SPF over a long quarterly sample of 1969-2001. They find that for most years, the histograms are bell shaped, reasonably symmetric with most of the probability mass concentrated in

[1] See **?** and **?**, among others.

Table 1: Parameter Estimates

	Model NL-noIE		Model NL		Model LI		Model LII		Model LII-PC	
State Variables Dynamics										
$dx_t = \mathcal{K}(\mu - x_t)dt + \Sigma dB_t$										
$\mathcal{K}_{11}$	0.8550	(0.3533)	0.6849	(0.4589)	0.8302	(0.6993)	0.4317	(0.1622)	0.7358	(0.3542)
$\mathcal{K}_{22}$	0.1343	(0.0562)	0.1309	(0.0471)	0.1004	(0.0425)	0.0961	(0.0499)	0.0316	(0.0357)
$\mathcal{K}_{33}$	1.4504	(0.3633)	1.4259	(0.7216)	1.2353	(0.9516)	1.8425	(0.4757)	1.3386	(0.6951)
$100 \times \Sigma_{21}$	-0.7526	(0.5524)	-1.8236	(1.1939)	-1.1547	(0.8448)	-1.6133	(0.9020)	-0.6414	(0.2435)
$100 \times \Sigma_{31}$	-4.4450	(4.8007)	-4.8415	(8.4964)	-7.1258	(30.8741)	-1.7824	(0.9339)	-4.8511	(8.3985)
$100 \times \Sigma_{32}$	-0.9597	(0.2356)	-1.0313	(0.2948)	-1.0456	(0.4755)	-0.7864	(0.1713)	-0.5316	(0.2111)
Nominal Pricing Kernel										
$dM_t^N/M_t^N = -r^N(x_t)dt - \lambda(x_t)'dB_t$,										
$r^N(x_t) = \rho_0^N + \rho_1^{N'} x_t, \;\; \lambda(x_t) = \lambda_0^N + \Lambda^N x_t$										
ρ_0^N	0.0474	(0.0048)	0.0468	(0.0046)	0.0467	(0.0048)	0.0480	(0.0062)	0.0480	(0.0086)
$\rho_{1,1}^N$	3.6695	(3.0529)	4.9405	(5.2771)	6.2746	(22.3760)	2.4285	(0.7752)	3.2396	(5.1459)
$\rho_{1,2}^N$	0.8844	(0.1321)	0.9109	(0.1387)	0.8850	(0.2141)	0.7810	(0.0968)	0.4424	(0.0892)
$\rho_{1,3}^N$	0.7169	(0.0355)	0.7173	(0.0175)	0.7419	(0.0226)	0.7031	(0.0195)	0.6333	(0.0256)
$\lambda_{0,1}^N$	0.3241	(0.1606)	0.3270	(0.1484)	-0.0097	(0.2087)	0.1557	(0.1697)	0.2216	(0.2936)
$\lambda_{0,2}^N$	-0.4335	(0.1819)	-0.4019	(0.1533)	-0.3696	(0.2725)	-0.5355	(0.2110)	-0.4906	(0.4491)
$\lambda_{0,3}^N$	-1.2754	(0.3726)	-1.2417	(0.3888)	-1.1435	(0.3903)	-1.3591	(0.4438)	-1.5077	(2.1659)
$[\Sigma\Lambda^N]_{11}$	-0.6953	(0.9033)	-0.6529	(1.5192)	0.6238	(2.4162)	-0.0138	(0.1295)	-0.3677	(1.1468)
$[\Sigma\Lambda^N]_{21}$	2.1331	(2.6939)	2.4644	(4.7106)	0.5454	(3.2112)	0.2964	(0.5159)	1.1200	(2.8701)
$[\Sigma\Lambda^N]_{31}$	3.0734	(6.4541)	3.8734	(13.5898)	-1.0467	(22.6787)	0.1262	(0.5179)	3.6061	(12.9735)
$[\Sigma\Lambda^N]_{12}$	0.0339	(0.0409)	0.0650	(0.0583)	-0.0732	(0.0474)	-0.0223	(0.0650)	-0.0827	(0.1901)
$[\Sigma\Lambda^N]_{22}$	-0.1447	(0.0233)	-0.2128	(0.0531)	-0.0458	(0.0418)	-0.1151	(0.0731)	-0.1613	(0.1064)
$[\Sigma\Lambda^N]_{32}$	-0.3576	(0.3013)	-0.6065	(0.9297)	0.3068	(1.8944)	-0.1980	(0.1120)	-0.4788	(0.5115)
$[\Sigma\Lambda^N]_{13}$	-0.0809	(0.1141)	-0.1135	(0.1026)	0.1866	(0.4329)	0.0790	(0.1603)	-0.0369	(0.2512)
$[\Sigma\Lambda^N]_{23}$	0.6000	(0.1980)	0.7232	(0.3218)	0.0875	(0.1439)	0.5394	(0.2512)	0.4512	(0.2111)
$[\Sigma\Lambda^N]_{33}$	0.1553	(0.8626)	0.3736	(1.8229)	-1.0059	(2.1457)	-0.4945	(0.4766)	0.7245	(1.5411)
Log Price Level										
$d\log Q_t = \pi(x_t)dt + \sigma_q' dB_t + \sigma_q^{\perp} dB_t^{\perp}, \; \pi(x_t) = \rho_0^{\pi} + \rho_1^{\pi'} x_t$										
ρ_0^{π}	0.0262	(0.0016)	0.0285	(0.0015)	0.0294	(0.0021)	0.0288	(0.0026)	0.0278	(0.0079)
$\rho_{1,1}^{\pi}$	-0.0326	(0.5805)	-0.4711	(1.7446)	-0.5261	(4.3530)	0.1582	(0.3076)	0.3895	(0.5149)
$\rho_{1,2}^{\pi}$	0.0867	(0.0578)	0.2378	(0.0400)	0.3515	(0.0849)	0.2684	(0.0300)	0.3883	(0.0376)
$\rho_{1,3}^{\pi}$	-0.2213	(0.1859)	-0.2804	(0.1584)	-0.1999	(0.2596)	-0.1356	(0.1442)	0.0485	(0.0845)
$100 \times \sigma_{q,1}$	-0.0796	(0.0445)	0.0038	(0.0734)	0.0000	(0.1009)	-0.1495	(0.0409)	-0.0815	(0.0585)
$100 \times \sigma_{q,2}$	0.0066	(0.0673)	0.0869	(0.0739)	0.1625	(0.0620)	0.0763	(0.0581)	0.0575	(0.0581)
$100 \times \sigma_{q,3}$	-0.0278	(0.0589)	-0.2586	(0.0459)	-0.1526	(0.0674)	0.0224	(0.0619)	0.0154	(0.0533)
$100 \times \sigma_q^{\perp}$	0.9229	(0.0268)	0.9461	(0.0300)	0.9508	(0.0346)	0.8975	(0.0264)	0.7018	(0.0213)

Table 1 Continued

	Model NL-noIE		Model NL		Model LI		Model LII		Model NL-PreCrisis	
TIPS Liquidity Premium										
$l_t = \tilde{\gamma}\tilde{x}_t + \gamma' x_t,\ d\tilde{x}_t = \tilde{\kappa}(\tilde{\mu} - \tilde{x}_t)dt + \tilde{\sigma}dW_t,\ \tilde{\lambda}_t = \tilde{\lambda}_0 + \tilde{\lambda}_1\tilde{x}_t.$										
$\tilde{\gamma}$					0.8376	(0.0224)	0.8393	(0.0225)	0.5427	(0.0344)
$\tilde{\kappa}$					0.5097	(0.2113)	0.4900	(0.2051)	0.1936	(0.2416)
$\tilde{\mu}$					0.0067	(0.0049)	0.0077	(0.0050)	0.0167	(0.0122)
$\tilde{\lambda}_0$					0.3754	(0.3571)	0.4136	(0.3413)	0.2847	(0.5339)
$\tilde{\sigma}\tilde{\lambda}_1$					-0.3981	(0.2114)	-0.3770	(0.2052)	-0.1041	(0.2412)
γ_1							-0.8403	(0.2826)	-0.3915	(0.5743)
γ_2							-0.0527	(0.1024)	0.1032	(0.0802)
γ_3							0.0121	(0.2293)	-0.0000	(0.1607)
Measurement Errors: Nominal Yields										
$100 \times \delta_{N,3m}$	0.1314	(0.0020)	0.1314	(0.0020)	0.1311	(0.0021)	0.1312	(0.0021)	0.1028	(0.0027)
$100 \times \delta_{N,6m}$	0.0188	(0.0015)	0.0192	(0.0015)	0.0211	(0.0015)	-0.0212	(0.0014)	-0.0215	(0.0017)
$100 \times \delta_{N,1y}$	0.0655	(0.0022)	0.0655	(0.0022)	0.0653	(0.0022)	0.0653	(0.0022)	0.0529	(0.0018)
$100 \times \delta_{N,2y}$	0.0000	(51.7227)	0.0000	(9.0140)	0.0000	(3995.5010)	0.0000	(4062.7066)	-0.0000	(104.5475)
$100 \times \delta_{N,4y}$	0.0397	(0.0016)	0.0396	(0.0016)	0.0396	(0.0016)	0.0396	(0.0016)	0.0292	(0.0012)
$100 \times \delta_{N,7y}$	0.0000	(150.5043)	-0.0000	(100.1489)	0.0000	(4423.6406)	0.0000	(5024.8333)	-0.0000	(148.9753)
$100 \times \delta_{N,10y}$	0.0530	(0.0015)	0.0529	(0.0015)	0.0533	(0.0015)	0.0530	(0.0015)	0.0487	(0.0018)
Measurement Errors: TIPS Yields										
$100 \times \delta_{\mathcal{T},5y}$	0.5374	(0.0801)	0.5400	(0.0785)	0.0806	(0.0033)	0.0812	(0.0033)	0.0642	(0.0047)
$100 \times \delta_{\mathcal{T},7y}$	0.4217	(0.0849)	0.4231	(0.0843)	-0.0000	(6302.1210)	-0.0000	(6307.8897)	0.0000	(26.1627)
$100 \times \delta_{\mathcal{T},10y}$	0.3879	(0.0632)	0.3874	(0.0605)	0.0653	(0.0033)	-0.0644	(0.0033)	-0.0610	(0.0050)
Measurement Errors: Survey Forecasts of Nominal Short Rate										
$100 \times \delta_{f,6m}$	0.1890	(0.0146)	0.1893	(0.0146)	0.1872	(0.0141)	0.1891	(0.0146)	0.1654	(0.0137)
$100 \times \delta_{f,12m}$	0.2965	(0.0222)	0.2945	(0.0218)	0.2944	(0.0219)	0.2968	(0.0224)	0.2225	(0.0203)

This table reports parameter estimates and standard errors for all five models we estimate. Standard errors are calculated using the BHHH formula and are reported in parentheses.

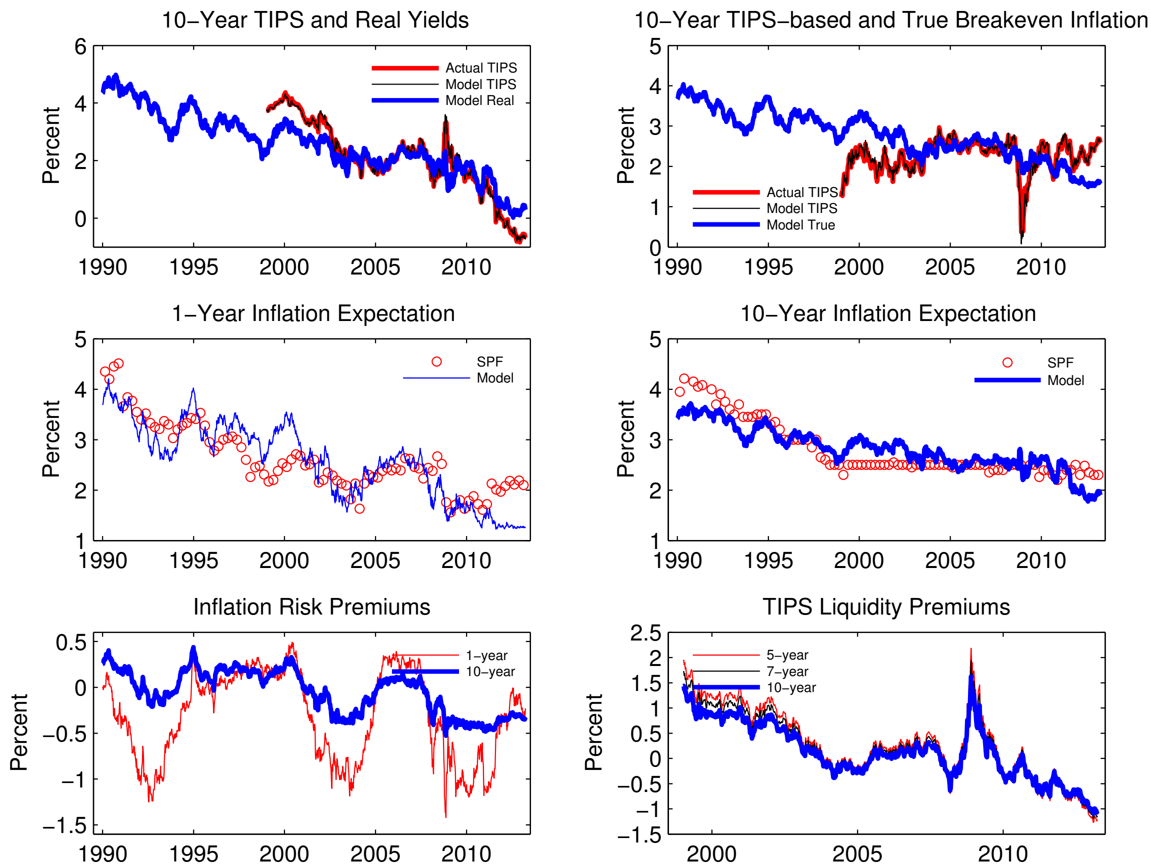


Figure 1: Model LI

The top left panel plot the 10-year actual TIPS yields (red), the 10-year model-implied TIPS yields (black) and the 10-year model-implied real yields (blue). The top right panel plots the 10-year actual TIPS breakevens (red), the 10-year model-implied TIPS breakevens (black) and the 10-year model-implied true breakevens (blue). The middle panels plot the 1- and 10-year model-implied inflation expectation, respectively, together with their survey counterparts from the SPF. The bottom left panel plot the 1- and 10-year model-implied inflation risk premiums. The bottom right panel plot the 5-, 7-, and 10-year model-implied TIPS-indexed bond yield differences.

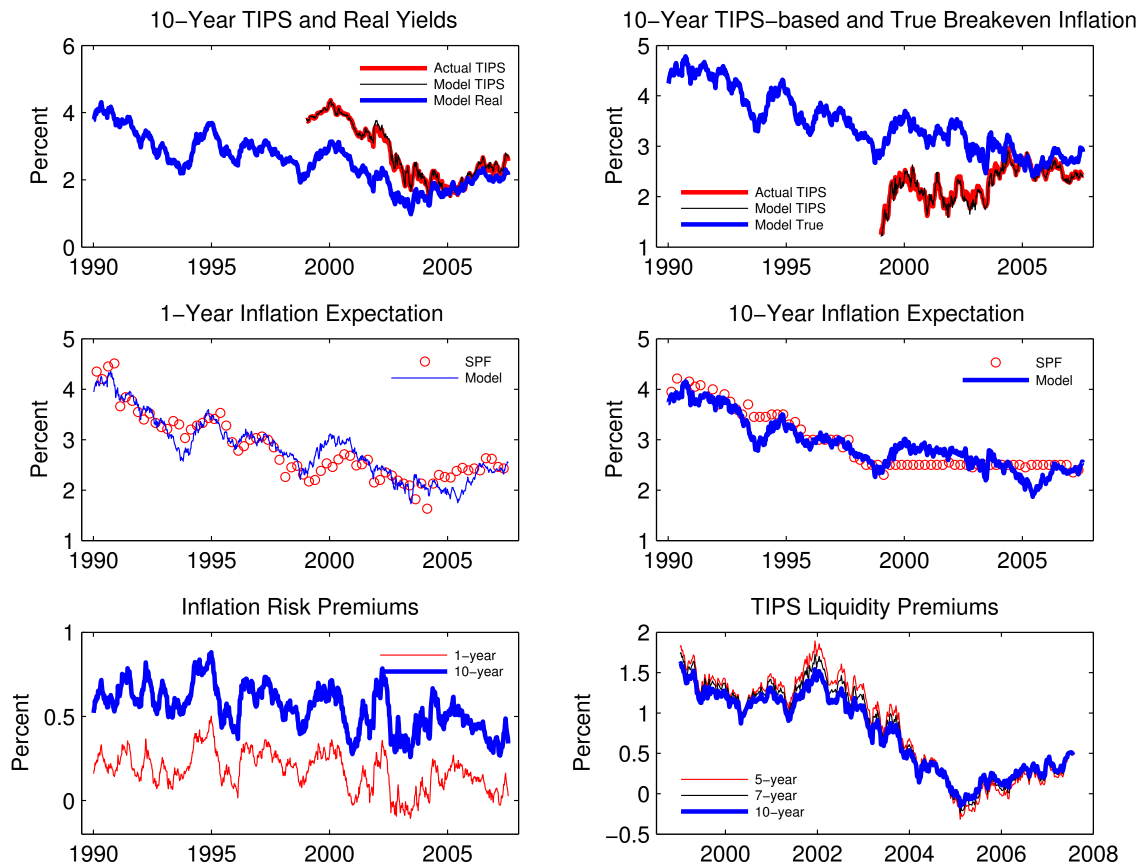


Figure 2: Model LII Estimated over the Pre-Crisis Period

The top left panel plot the 10-year actual TIPS yields (red), the 10-year model-implied TIPS yields (black) and the 10-year model-implied real yields (blue). The top right panel plots the 10-year actual TIPS breakevens (red), the 10-year model-implied TIPS breakevens (black) and the 10-year model-implied true breakevens (blue). The middle panels plot the 1- and 10-year model-implied inflation expectation, respectively, together with their survey counterparts from the SPF. The bottom left panel plot the 1- and 10-year model-implied inflation risk premiums. The bottom right panel plot the 5-, 7-, and 10-year model-implied TIPS-indexed bond yield differences.

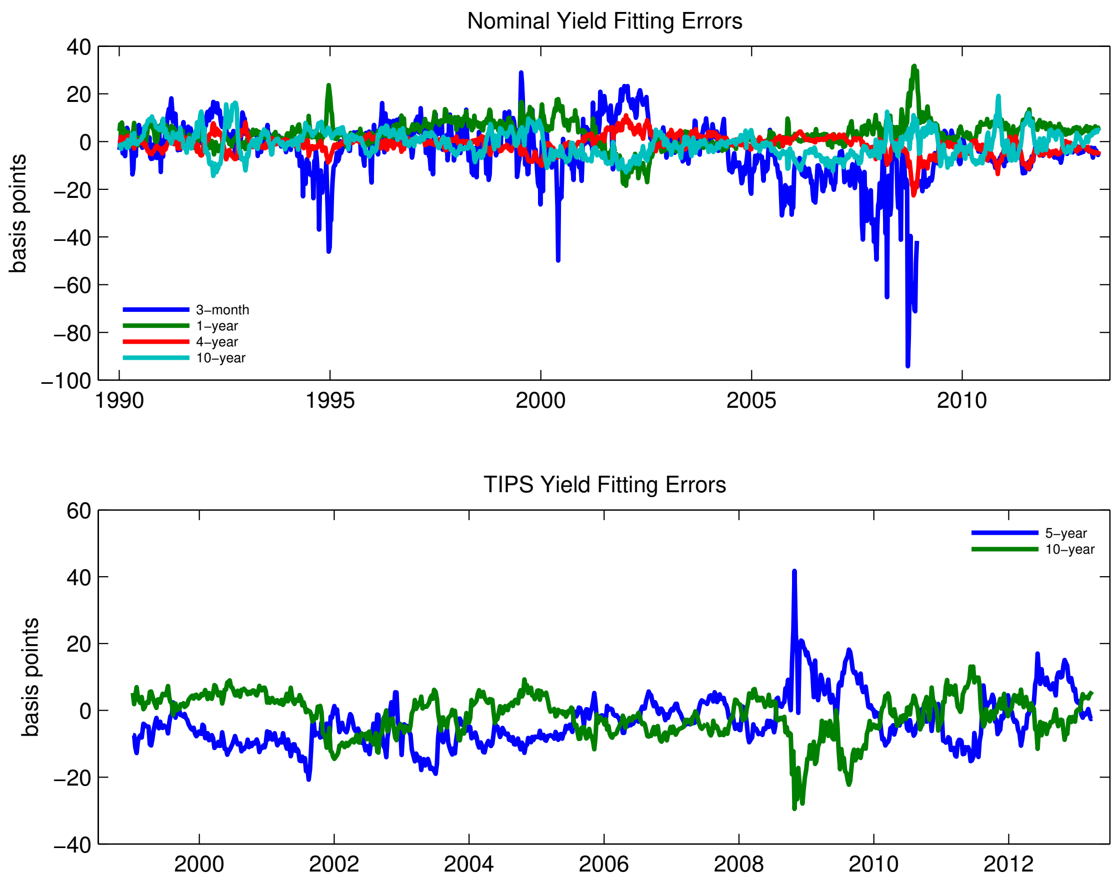


Figure 3: Time Series of Yield Fitting Errors from Model LII

This chart plots the time series of fitting errors on nominal yields (top panel) and TIPS yields (bottom panel) based on Model LII.

interior intervals, suggesting that the normal distribution provides a good approximation to the physical distribution of inflation in those years. An update of their results using the same methodology for each year from 1992 to 2013, shown in Figures 4 and 5, demonstrates that the normal distribution continues to provide a reasonable approximation to the physical distribution of inflation forecasts over recent years. This is consistent with the findings in **?**: Figures 8 and 9 of their paper show that even during the years of 2010-2012, a period that was dominated by deflation scares, the physical distribution of expected inflation remains reasonably symmetric and assign much lower odds to tail outcomes than the corresponding options-implied PDFs.[2]

To formally test the normality of each distribution shown in Figure 4 and 5, we use the χ^2 statistic described in **?**. The values of this statistic for one- and two-year ahead forecasts are reported in the third and fifth columns of Table 2, respectively. The associated levels of significance indicate that we reject the normality assumption for 13 out of 22 distributions (60% of the time) at the one-year horizon and for 9 out of 22 distributions (40% of time) at the two-year horizon. We interpret the results as suggesting that, despite the crude approximation of the true distribution using a few bins and the sensitivity of the test to the treatment of the open intervals, the normality distribution can be thought of as a reasonable approximation about half of the time over this period and more so for longer forecast horizons.

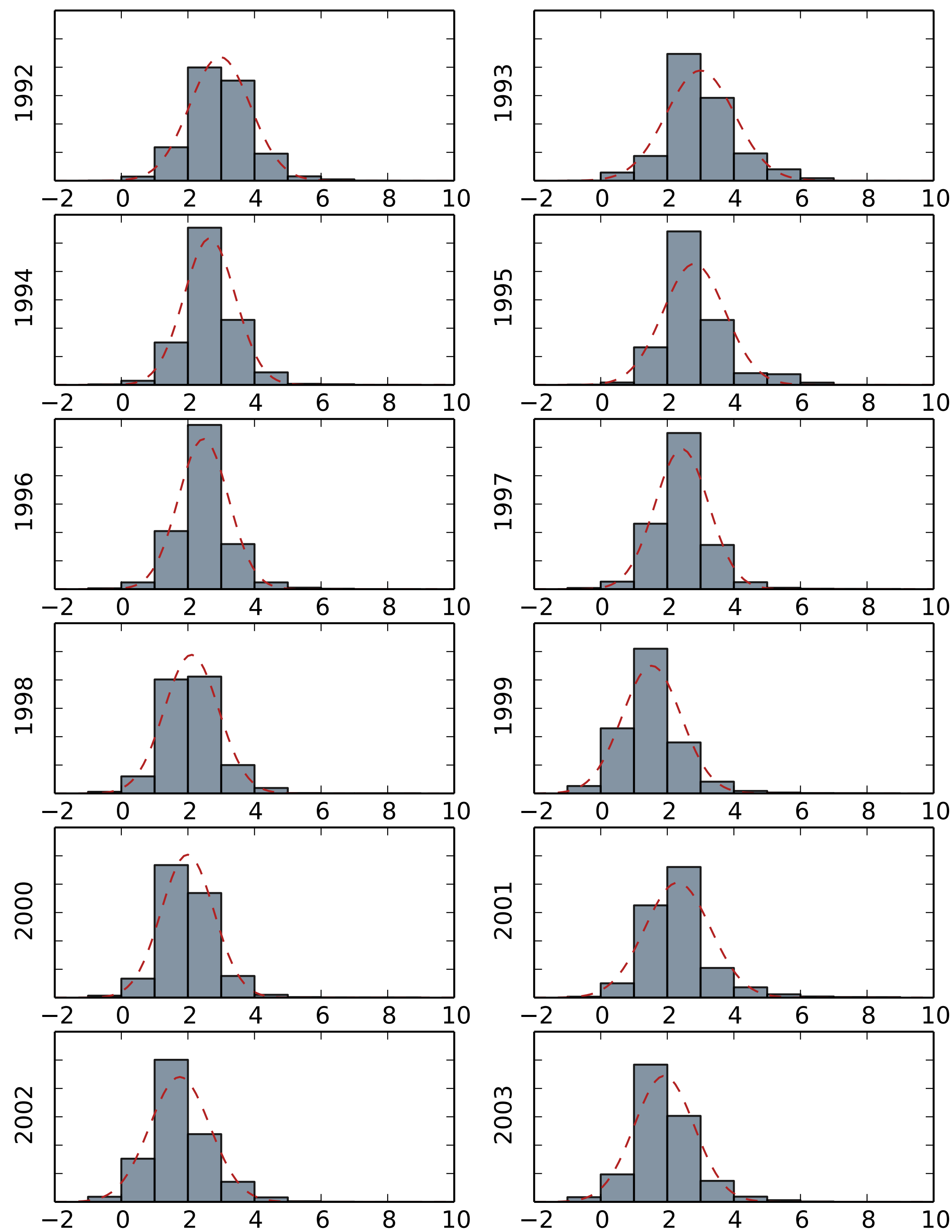
Turning to *risk-neutral* distributions, Figure 6 plot the skewness and excess kurtosis of risk-neutral distributions of inflation over the next one, five, and ten years, constructed from zero-coupon inflation caps using a similar model-free methodology as in **?**. The caps-implied skewness was notably negative at 5- and 10-year horizons in the immediate aftermath of the crisis, but have hovered around zero since late 2010 despite lingering worries about deflation. Similarly, the excess kurtosis was significantly positive between late 2009 and late 2010, suggesting investors perceived higher risks of tail inflation outcomes than implied by a normal distribution. The excess kurtosis had also largely dissipated by late 2010, although more recently it has drifted up again for the 5-year horizon.

Overall, the Gaussian model seems to be a more reasonable approximation of inflation dynamics over a long sample period like ours, although its inability to capture time-varying volatilities, asymmetric distributions, or heavy tails can be more problematic for periods with heightened deflation concerns such as 2009-2010, which nonetheless constitutes only a small part of our sample period. We therefore view the general affine-Gaussian model as an important benchmark to investigate before exploring more sophisticated models.

3.2 Parameter Stability

The literature has documented significant market dislocations in the nominal Treasury/TIPS market during the 2008 financial crisis (see **?** and **?**, among others). We therefore re-estimate Model LII over a pre-crisis sample ending on July 25, 2007. As can be seen from Table 1, the parameter estimates are very similar to those from Model LII estimated over the full sample. A comparison between Figure 2 in this appendix and the bottom panels of Figures **??** and **??** in the paper shows that the model-implied real yields, inflation

[2] Those PDFs are constructed using two different models: the unobserved component stochastic volatility model of **?** and the time-varying-parameter VAR model with stochastic volatility of **?**.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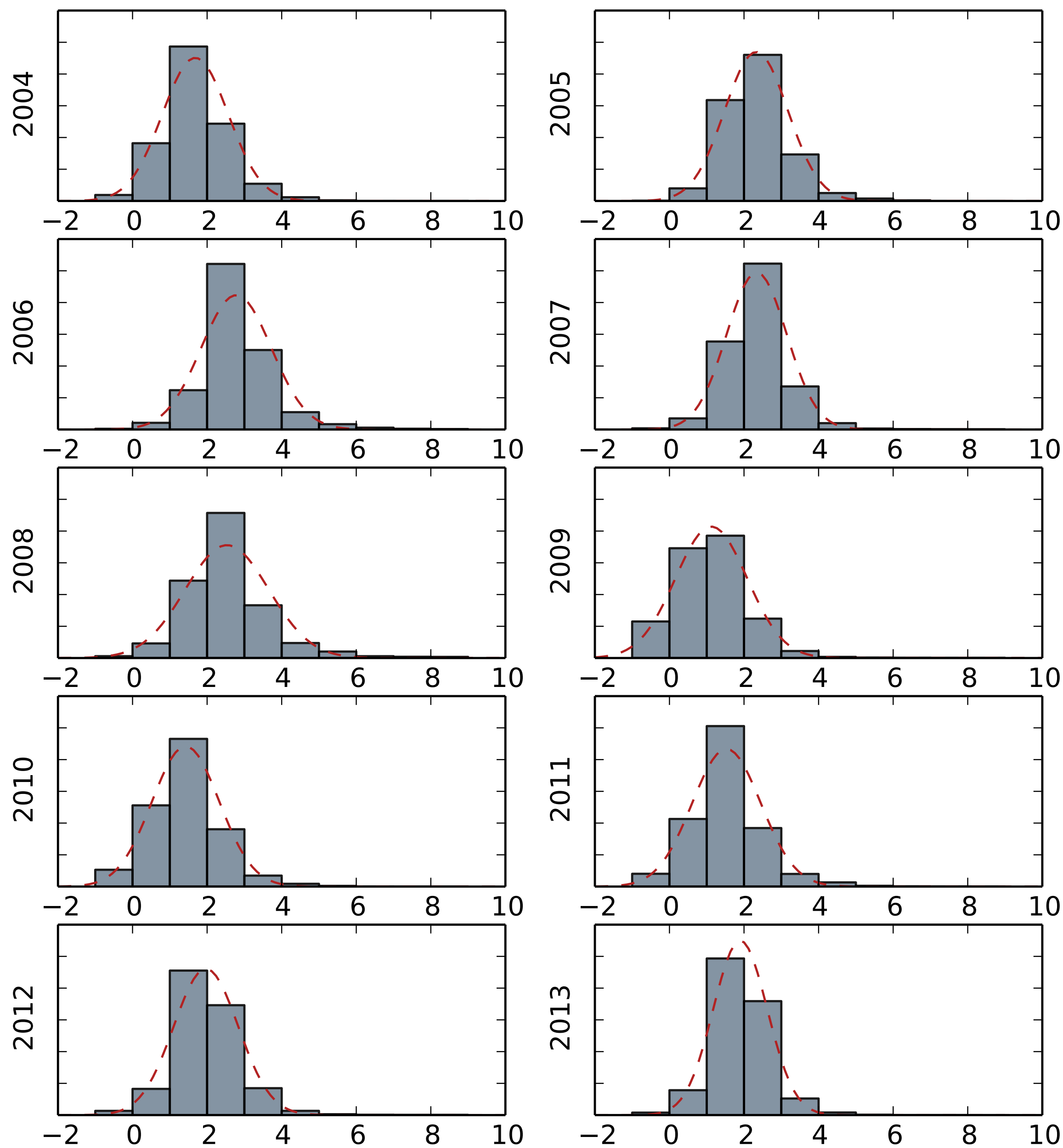
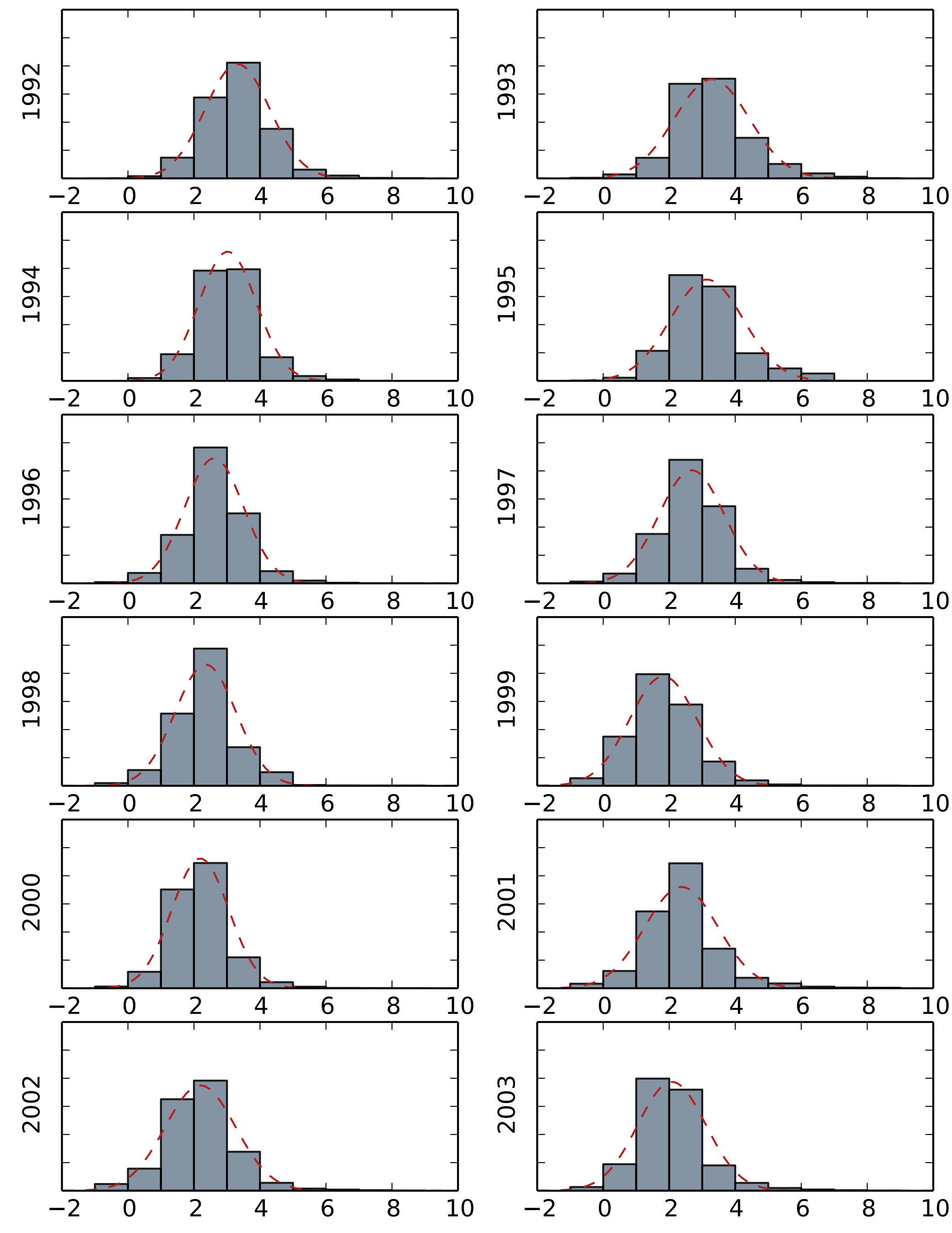


Figure 4: Distribution of 1-Year Ahead Expected Inflation

Histograms of 1-year ahead inflation forecasts from the SPF and the fitted distributions.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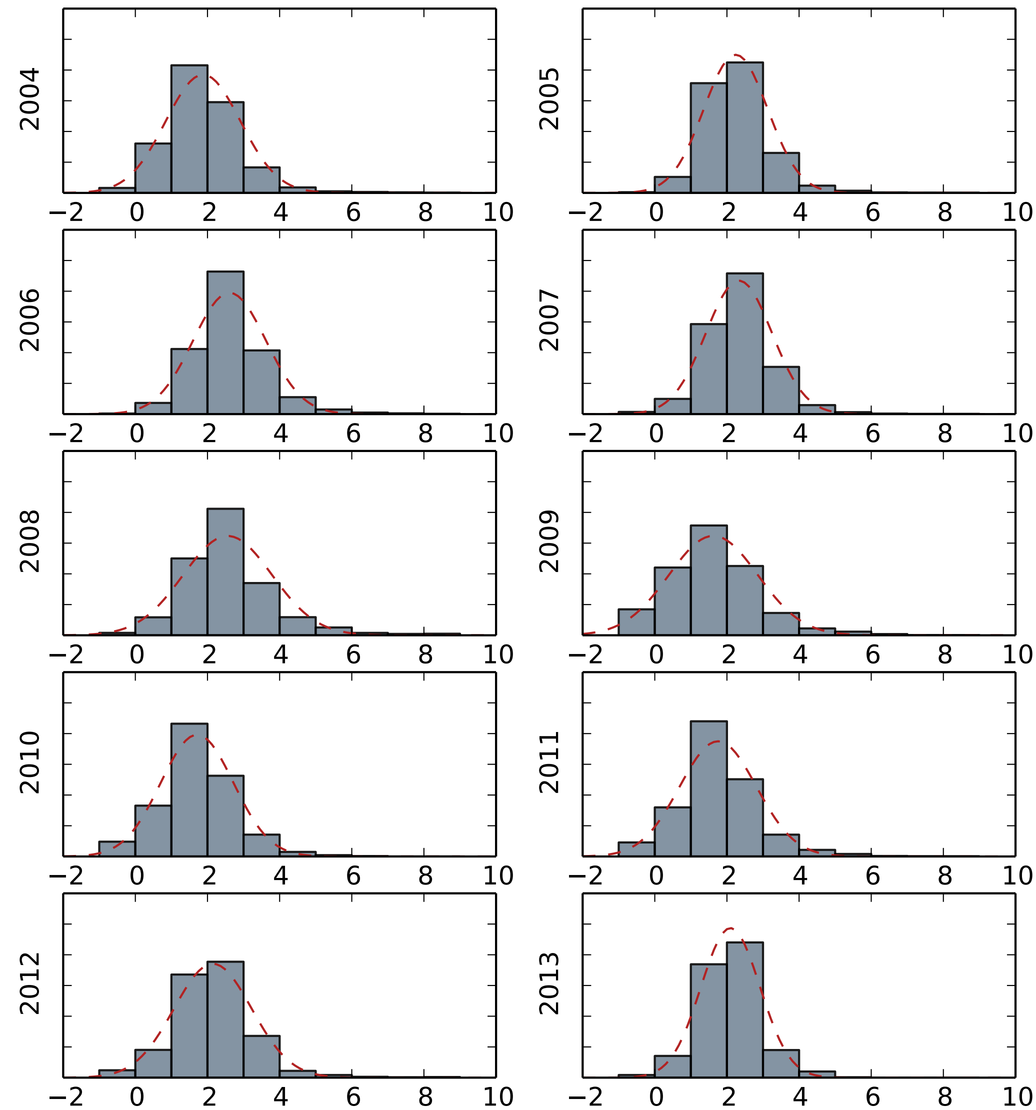

Figure 5: Distribution of 2-Year Ahead Expected Inflation

Histograms of 2-year ahead inflation forecasts from the SPF and the fitted distributions.

Table 2: Normality Test of SPF Forecasts

Year	No. of Forecasts	1-year test stat	1-year p-value	2-year test stat	2-year p-value
1992	34	0.04	1.00	0.08	1.00
1993	31	0.11	1.00	0.10	1.00
1994	26	0.26	1.00	0.08	1.00
1995	26	0.33	1.00	0.14	1.00
1996	36	0.45	1.00	0.06	1.00
1997	35	8419.37	0.00	10.46	0.16
1998	29	22185.89	0.00	43.15	0.00
1999	28	57274.68	0.00	38.78	0.00
2000	33	194851.32	0.00	1006.51	0.00
2001	29	194.97	0.00	7.96	0.34
2002	30	4377.78	0.00	7.60	0.37
2003	33	0.35	1.00	27.12	0.00
2004	27	26149.43	0.00	216.25	0.00
2005	31	7.80	0.35	400.02	0.00
2006	50	33.64	0.00	7.94	0.34
2007	42	21526.65	0.00	273.61	0.00
2008	41	11.61	0.11	3.19	0.87
2009	38	66105.14	0.00	1.33	0.99
2010	40	0.18	1.00	0.13	1.00
2011	41	15830.11	0.00	21.89	0.00
2012	41	16414.07	0.00	53.26	0.00
2013	41	20.79	0.00	0.13	1.00

This table reports the χ^2 test statistics and the associated p-values for one- and two-year ahead forecasts. The p-values are calculated as the probability of a χ^2-distributed variable with 7 degrees of freedom exceeding the actual test statistic under the null hypothesis of normality.

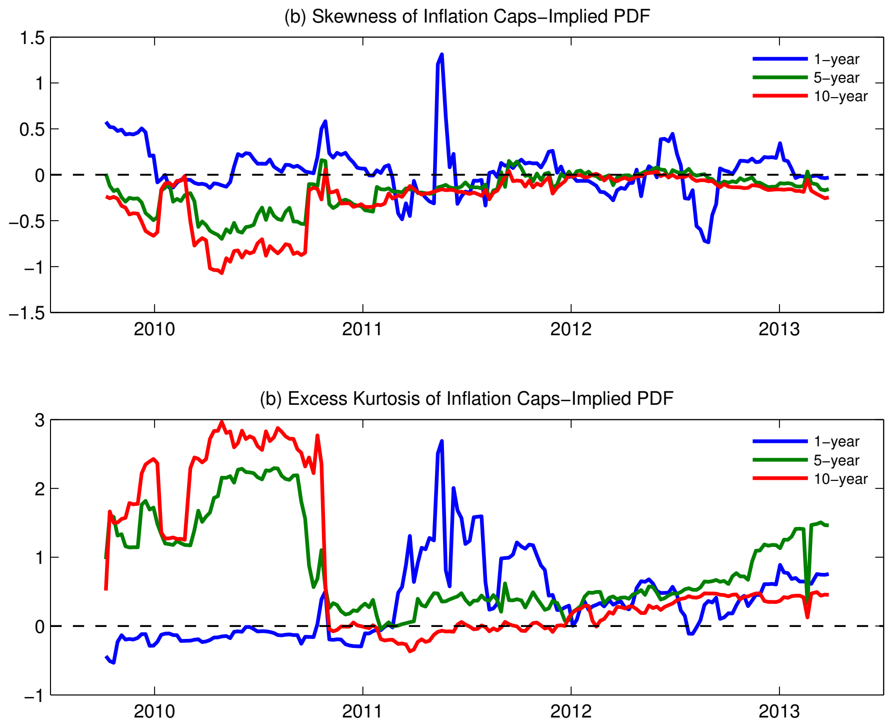

Figure 6: Skewness and Kurtosis from Inflation Caps

Panels (a) and (b) plot the skewness and excess kurtosis, respectively, of the 1-, 5-, and 10-year inflation probability distributions constructed from inflation caps.

expectations, inflation risk premiums, and the difference between TIPS yields and indexed bond yields are almost identical to what the full-sample Model LII predicts for the same period.

4 Decomposing Nominal Yields

Although it is not the focus of the current paper, our models can also be used to separate nominal yields into real yields, expected inflation and inflation risk premiums:

$$y^N_{t,\tau} = y^R_{t,\tau} + I_{t,\tau} + \wp_{t,\tau}. \tag{1}$$

Figure 7 plots 1- and 10-year nominal yields and their constituents, whereas Table 3 reports the variance decomposition results.

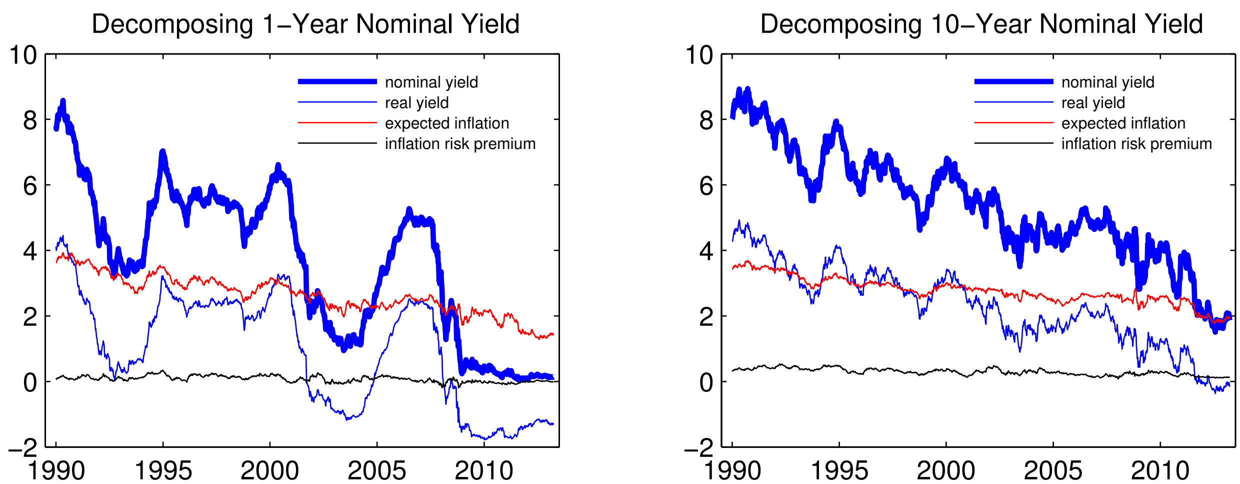

Figure 7: Decomposing Nominal Yields

The two panels decompose 1- and 10-year nominal yields into real yields, expected inflation and inflation risk premiums according to Equation (1).

These results indicate that, at least during our sample period, real yield changes explain more than three quarters of the variations in nominal yields at all maturities. Inflation expectation explains about 20% (10%) of the variations in the 1-quarter (10-year) nominal yield. Inflation risk premiums account for the remaining 2-10% of the nominal yield changes. This stands in contrast to previous studies using a longer sample period but not using TIPS yields, which typically find relatively smooth real yields but volatile inflation expectation or inflation risk premiums.[3] The limited evidence we have so far from TIPS seems to suggest that real yields may also vary considerably over time.

[3]See **?**, Figure 2 and **?**, Figure 7 for example.

Table 3: In-Sample Variance decomposition of Nominal Yields

Maturity	real yield	inf exp	inf risk prem
1-quarter	0.7639	0.2214	0.0147
	(0.1078)	(0.1039)	(0.0193)
1-year	0.7743	0.2032	0.0224
	(0.1101)	(0.0987)	(0.0246)
5-year	0.7852	0.1716	0.0433
	(0.1262)	(0.0943)	(0.0579)
10-year	0.7850	0.1488	0.0663
	(0.1326)	(0.0884)	(0.0720)

Note: This table reports the in-sample variance decompositions of nominal yields into real yields, expected inflation, the inflation risk premiums, all based on Model LII estimates. The variance decomposition is calculated according to

$$1 = \frac{cov\left(y_{t,\tau}^{N}, y_{t,\tau}^{R}\right)}{var\left(y_{t,\tau}^{N}\right)} + \frac{cov\left(y_{t,\tau}^{N}, I_{t,\tau}\right)}{var\left(y_{t,\tau}^{N}\right)} + \frac{cov\left(y_{t,\tau}^{N}, \wp_{t,\tau}^{I}\right)}{var\left(y_{t,\tau}^{N}\right)}.$$

Standard errors calculated using the delta method are reported in parentheses.

5 Davies (1987) Likelihood Ration Test Statistic

This section describes the details in constructing the **?** Likelihood Ration Test Statistic mentioned in Section **??**. Denote by θ the vector of nuisance parameters of size s, and define the likelihood ratio statistic as a function of θ:

$$LR(\theta) = 2\left[\log L_1(\theta) - \log L_0\right],$$

where $L_1(\theta)$ is the likelihood value of the alternative model for any admissible values of the nuisance parameters $\theta \in \Omega$, and L_0 is the maximized likelihood value of the null model. For an estimated LR value of M, **?** derives an upper bound for its significance as

$$\Pr\left[\sup_{\theta\in\Omega} LR(\theta) > M\right] < \Pr\left[LR(\theta) > M\right] + VM^{\frac{1}{2}(s-1)} \exp^{-(M/2)} \frac{2^{-s/2}}{\Gamma(s/2)}$$

where $\Gamma(.)$ represents the Gamma function and V is defined as

$$V = \int_{\Omega} \left|\frac{\partial LR(\theta)}{\partial\theta}\right| d\theta.$$

? further assumes that the likelihood ratio statistic has a single peak at $\widehat{\theta}$, which reduces V to $2M^{\frac{1}{2}}$.