Internet Appendices for

Dynamic Portfolio Choice with Linear Rebalancing Rules

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I. Generalized Transaction Costs, Risk Measures, and Factor Dynamics

In this section, we provide examples of dynamic portfolio choice problems involving non-quadratic transaction costs, generalized risk penalties, and complicated return dynamics that fall into our general framework.

Example 5 (Non-quadratic transaction costs). In practice, many trading costs, such as the bid–ask spread, broker commissions, and exchange fees, are intrinsically proportional to the trade size. Letting χ_i be the proportional transaction cost rate (e.g., an aggregate sum of bid–ask costs and commission fees) for trading security i, the investor will incur a total cost of

$$\mathsf{TC}(\mathbf{u}) \triangleq \sum_{t=1}^{T} \sum_{i=1}^{N} \chi_i |u_{i,t}|.$$

The proportional transaction costs are a classical cost structure that is well studied in the literature (e.g., Constantinides (1986)).

Furthermore, other trading costs occur due to disadvantageous transaction prices caused by the price impact of the trade. The management of trading costs due to price impacts has recently attracted considerable interest (e.g., Almgren and Chriss (2000), Obizhaeva and Wang (2013)). Many models of transaction costs due to price impacts imply a nonlinear relation between trade size and the resulting transaction cost, for example

$$\mathsf{TC}(\mathbf{u}) \triangleq \sum_{t=1}^{T} \sum_{i=1}^{N} \chi_i |u_{i,t}|^{\beta},$$

where $\beta \geq 1$ and χ_i is a security-specific proportionality constant.[10]

Generally, when the trade size is small relative to the total traded volume, proportional costs will dominate. On the other hand, when the trade size is large, costs due to price impact will dominate. Hence, both of these types of trades are important. However, the LQC framework of Example 2 only allows for quadratic transaction costs (i.e., $\beta = 2$).

[10] Gatheral (2010) notes that $\beta = \frac{3}{2}$ is a typical assumption in practice.

Example 6 (Terminal wealth risk). The objective function of Example 2 includes a term to penalize excessive risk. In particular, the per-period quadratic penalty, $x_t^\top \Sigma x_t$, is used, to satisfy the requirements of the LQC model. However, penalizing risk additively in a per-period fashion is not standard. Such a risk penalty does not correspond to traditional forms of investor risk preferences, such as maximizing the expected utility of terminal wealth, and the economic meaning of such a penalty is unclear. An investor is typically more interested in the risk associated with terminal wealth rather than a sum of per-period penalties.

To account for terminal wealth risk, let $\rho\colon \mathbb{R} \to \mathbb{R}$ be a real-valued convex function meant to penalize the excessive risk of terminal wealth (e.g., $\rho(w) = \frac{1}{2}w^2$ for a quadratic penalty) and consider the optimization problem

$$\text{(A-1)} \qquad \underset{\pi \in \mathcal{U}_{\mathbb{F}}}{\text{maximize}} \ \mathsf{E}_\pi \left[W(\mathbf{x}, \mathbf{r}) - \mathsf{TC}(\mathbf{u}) - \gamma \rho\Big(W(\mathbf{x}, \mathbf{r}) \Big) \right],$$

where $\gamma > 0$ is a risk proportionality constant.

It is not difficult to see that the objective in (A-1) satisfies Assumption 3 and hence fits into our model. However, even when the risk penalty function $\rho(\cdot)$ is quadratic, (A-1) does not allow for a tractable LQC solution, since the quadratic objective does not decompose across time.

Example 7 (Maximum drawdown risk). In addition to the terminal measures of risk described in Example 6, an investor could also be interested in controlling intertemporal measures of risk defined over the entire time trajectory. For example, a fund manager could be sensitive to a string of successive losses that may lead to the withdrawal of assets under management. One way to limit such losses is to control the maximum drawdown, defined as the worst loss of the portfolio between any two points of time during the investment horizon.[11]. Formally,

$$\mathsf{MD}(\mathbf{x}, \mathbf{r}) \triangleq \max_{1 \le t_1 \le t_2 \le T} \left(- \sum_{t=t_1}^{t_2} x_t^\top r_{t+1}, 0 \right).$$

It is easy to see that the maximum drawdown is a convex function of $\mathbf{x}$. Hence, the portfolio optimization problem

$$\text{(A-2)} \qquad \underset{\pi \in \mathcal{U}_{\mathbb{F}}}{\text{maximize}} \ \mathsf{E}_\pi \Big[W(\mathbf{x}, \mathbf{r}) - \mathsf{TC}(\mathbf{u}) - \gamma \mathsf{MD}(\mathbf{x}, \mathbf{r}) \Big],$$

where $\gamma \ge 0$ is a constant controlling trade-off between wealth and the maximum drawdown penalty, satisfies Assumption 3. Moreover, standard convex optimization theory shows that the problem (A-2) is equivalent to solving the constrained problem

$$\text{(A-3)} \qquad \begin{array}{ll} \underset{\pi \in \mathcal{U}_{\mathbb{F}}}{\text{maximize}} & \mathsf{E}_\pi \Big[W(\mathbf{x}, \mathbf{r}) - \mathsf{TC}(\mathbf{u}) \Big], \\ \text{subject to} & \mathsf{E}_\pi \left[\mathsf{MD}(\mathbf{x}, \mathbf{r}) \right] \le C, \end{array}$$

[11]For example, see Grossman and Zhou (1993) for an earlier example.

where C (which depends on the choice of γ) is a limit on the allowed expected maximum drawdown.

Note that other mechanisms for risk aversion, such as penalties based on convex or coherent risk measures, can easily be incorporated into our framework in a manner analogous to Examples 4 and 6.

Example 8 (Complex dynamics). We can also generalize the dynamics of Example 2. Consider factor and return dynamics given by

$$f_{t+1} = (I - \Phi)\, f_t + \epsilon_{t+1}^{(1)}, \qquad r_{t+1} = \mu_t + (B + \xi_{t+1}) f_t + \epsilon_{t+1}^{(2)}$$

for each time $t \geq 0$. In this case, each $\xi_{t+1} \in \mathbb{R}^{N \times K}$ is an extra noise term that captures model uncertainty regarding factor loadings. We assume that

$$\mathsf{E}\left[(B + \xi_{t+1})\, f_t \mid \mathcal{F}_t\right] = B f_t, \qquad \operatorname{var}\left[(B + \xi_{t+1})\, f_t \mid \mathcal{F}_t\right] = \overline{f}_t^\top \Upsilon \overline{f}_t,$$

where $\mathcal{F}_t$ is the sigma-algebra incorporating all random variables realized by time t and $\overline{f}_t \in \mathbb{R}^{K \times N}$ is a matrix given by $\overline{f}_t \triangleq \begin{bmatrix} f_t & f_t & \dots & f_t \end{bmatrix}$.

With this model, the conditional variance of returns becomes dependent on the factor structure and is time varying, that is, $\operatorname{var}[r_{t+1} | \mathcal{F}_t] = \overline{f}_t^\top \Upsilon \overline{f}_t + \Sigma$. This result is consistent with the empirical work of Fama and French (1996), for example. In this setting, a per-period conditional variance risk penalty analogous to that in (7) becomes $\mathsf{RA}_t(\mathbf{x}, \mathbf{f}) = x_t^\top \left(\overline{f}_t^\top \Upsilon \overline{f}_t + \Sigma\right) x_t$. The resulting optimal control problem no longer falls within the LQC framework.

II. Efficient Exact Formulations

In this section, we will provide efficient exact formulations of dynamic portfolio choice problems using the class of linear policies for our feasible set of policies. In particular, we will consider a number of the examples of dynamic portfolio choice problems discussed in Section II.B. These examples include features such as constraints on portfolio holdings, transaction costs, and risk measures. In each case, we will demonstrate how the optimization problem (6) can be transformed into a deterministic convex program by explicit analytical evaluation of the objective function $\mathsf{E}[\mathrm{p}(\cdot, \mathbf{f})]$ and the constraint set $\mathcal{C}$.

Exact formulations require the evaluation of expectations taken over the sample path of factor realizations $\mathbf{f}$. In order to do this, we will make the following assumption for the rest of this section:

Assumption 4 (Gaussian factors). Assume that the sample path $\mathbf{f}$ of factor realizations is jointly Gaussian. In particular, denote by $F_t \triangleq (f_1, \dots, f_t)^\top \in \mathbb{R}^{Kt}$ the vector of all factors observed by time t. We assume that $F_t \sim N(\theta_t, \Omega_t)$, where $\theta_t \in \mathbb{R}^{Kt}$ is the mean vector and $\Omega_t \in \mathbb{R}^{Kt \times Kt}$ is the covariance matrix.

Note that this assumption is not necessary for the practical application of our method. Indeed, as discussed in Section II.A, the SAA or stochastic approximation

methods can be used to determine the optimal policy in the absence of such an assumption. However, under Assumption 4, the optimal linear policy can be solved via a deterministic convex optimization program that does not involve any sampling. This is often more computationally efficient as well as more accurate (since there are no errors introduced by sampling).

With this assumption, the trades of any linear policy will also be jointly normally distributed, as each such policy is affine transformations of the factors. Formally, let

$$M_t \triangleq \begin{bmatrix} E_{1,t} & E_{2,t} & \dots & E_{t,t} \end{bmatrix} \in \mathbb{R}^{N \times Kt} \tag{A-4}$$

be the matrix of time t policy coefficients, so that the trade vector is given by $u_t = c_t + M_t F_t$. With this representation, it is easy to see that $u_t \sim N(\bar{u}_t, V_t)$, where the mean vector and covariance matrix are given by

$$\bar{u}_t \triangleq \mathsf{E}[u_t] = c_t + M_t \theta_t, \qquad V_t \triangleq \operatorname{var}(u_t) = M_t \Omega_t M_t^\top. \tag{A-5}$$

Similarly, the portfolio x_t at time t is normally distributed. we have that

$$x_t = x_0 + \sum_{i=1}^{t} u_i = x_0 + \sum_{i=1}^{t} \left(c_i + \sum_{s=1}^{i} E_{s,i} f_s \right) = d_t + \sum_{s=1}^{t} J_{s,t} f_s, \tag{A-6}$$

where $d_t \triangleq x_0 + \sum_{i=1}^{t} c_i$ and $J_{s,t} \triangleq \sum_{\ell=s}^{t} E_{s,\ell}$. With this representation, it is easy see that $x_t \sim N(\kappa_t, Y_t)$, where

$$\kappa_t \triangleq \mathsf{E}[x_t] = d_t + P_t \theta_t, \qquad Y_t \triangleq \operatorname{var}(x_t) = P_t \Omega_t P_t^\top, \tag{A-7}$$

$$P_t \triangleq \begin{bmatrix} J_{1,t} & J_{2,t} & \dots & J_{t,t} \end{bmatrix}. \tag{A-8}$$

A. Linear Constraints

We will provide formulations for linear equality or inequality constraints on trades or positions, in the context of linear rebalancing policies. These types of constraints appear frequently in portfolio choice due to regulatory reasons such as short-sale restrictions, liquidation purposes, or diversification needs (e.g., keeping a specific industry exposure under a certain limit).

1. Equality Constraints

Equality constraints appear often in portfolio choice, particularly in portfolio execution problems when the investor needs to liquidate a certain portfolio (i.e., $x_T = \mathbf{0}$) or constructs a certain target portfolio by the end of the time horizon (i.e., $x_T = \bar{x}$).

Suppose that for some time t, we have a linear equality constraint on the trade vector u_t, of the form $Au_t = b$. Here, $A \in \mathbb{R}^{M \times N}$ and $b \in \mathbb{R}^N$. This constraint can be written as

$$Ac_t + AM_t F_t = b. \tag{A-9}$$

Under Assumption 4, the left side of (A-9) is normally distributed. Therefore, for (A-9) to hold almost surely, we must have that the left side has mean b and zero covariance. Thus, we require that

$$Ac_t = b, \qquad AM_t = \mathbf{0}. \tag{A-10}$$

Thus, the linear equality constraint (A-9) on the trade vector u_t is equivalent to the linear equality constraint (A-10) on the policy coefficients (c_t, M_t). Linear equality constraints on the portfolio position x_t can be handled similarly.

2. Inequality Constraints

Inequality constraints on trades or positions are common as well. One example is a short-sale constraint, which would require that $x_t \geq \mathbf{0}$ for all times t. When the factor realizations do not have bounded support, inequality constraints cannot be enforced almost surely. This is true in the Gaussian case: there is a chance, however small, that factors may take extreme values, and if the policy is a linear function of the factors, this may cause an inequality constraint to be violated. In other words, under Assumption 4 with linear constraints, any non-deterministic linear rebalancing policy will typically be infeasible. Note that while this is immediate in the exact formulation, this difficulty remains if we employ other solution methods such as SAA. In the SAA approach, for example, any non-deterministic linear rebalance policy will become infeasible given enough samples.

In order to account for such constraints in a linear rebalancing policy, instead of enforcing inequality constraints almost surely, we will enforce them at a given level of confidence. For example, given a vector $a \in \mathbb{R}^N$ and a scalar b, instead of enforcing the linear constraint $a^\top u_t \leq b$, almost surely, we can consider a relaxation where we seek to guarantee that it is violated with small probability. In other words, we can impose the chance constraint $\mathsf{P}(a^\top u_t > b) \leq \eta$, for a small value of the parameter η. The following lemma illustrates that this can be accomplished explicitly:

Lemma 1. Given $\eta \in [0, 1/2]$, a non-zero vector $a \in \mathbb{R}^N$, and a scalar b, the chance constraint $\mathsf{P}(a^\top u_t > b) \leq \eta$ is equivalent to the constraint

$$a^\top (c_t + M_t\theta_t) - b + \Phi^{-1}(1-\eta) \left\| \Omega_t^{1/2} M_t^\top a \right\|_2 \leq 0$$

on the policy coefficients (c_t, M_t), where $\Phi^{-1}(\cdot)$ is the inverse cumulative normal distribution.

Proof. This proof follows standard arguments in convex optimization (see, e.g., Boyd and Vandenberghe (2004)). Let $\bar{u}_t$ and V_t be the mean and the variance of u_t as given in (A-5). Then,

$$\mathsf{P}(a^\top u_t > b) = \mathsf{P}(\beta_t + \sigma_t Z > 0),$$

where

$$\beta_t \triangleq a^\top \bar{u}_t - b, \qquad \sigma_t \triangleq \|V_t^{1/2} a\|_2 \neq 0,$$

and Z is a standard normal random variable. Thus,

$$\mathsf{P}(a^\top u_t > b) = 1 - \Phi(-\beta_t/\sigma_t).$$

Note that the this probability is less than or equal to η if and only if

$$\beta_t + \Phi^{-1}(1-\eta)\sigma_t \leq 0.$$

Substituting (A-5) into definitions for β_t and σ_t, we obtain the desired result. □

A similar approach can be applied to incorporate linear inequality constraints on the portfolio position x_t with high confidence.

In many situations (e.g., short-sale constraints), it may not be sufficient to enforce an inequality constraint only probabilistically. In such cases, when a linear rebalancing policy is applied, the resulting trades can be projected onto the constraint set so as to ensure that the constraints are always satisfied. When the linear policy is designed, however, it is helpful to incorporate the desired constraints probabilistically so as to account for their presence. We will demonstrate this idea in the application in Section III.

B. Transaction Costs

In this section, we will provide efficient exact formulations for the transaction cost functions discussed in Section II.B, in the context of linear rebalancing policies. In general, one might consider a total transaction cost of

$$\mathsf{TC}(\mathbf{u}) \triangleq \sum_{t=1}^{T} \mathsf{TC}_t(u_t)$$

for executing the sample path of trades $\mathbf{u}$, where $\mathsf{TC}_t(u_t)$ is the cost of executing the trade vector u_t at time t. As seen in Section II.B, we typically wish to subtract an expected transaction cost term from investor's objective. Hence efficient exact formulations for transaction costs involve explicit analytical computation of $\mathsf{E}[TC(\mathbf{u})] = \sum_{t=1}^{T} \mathsf{E}[\mathsf{TC}_t(u_t)]$, when each trade vector u_t is specified by a linear policy.

Under a linear policy, $u_t \sim N(\bar{u}_t, V_t)$ is distributed as a normal random variable with mean and covariance, $(\bar{u}_t, V_t)$, specified with the policy coefficients, $(\mathbf{E}, \mathbf{c})$, through (A-5). Then, the evaluation of expected transaction costs reduces to the evaluation of the expected value of the per-period transaction cost function $\mathsf{TC}_t(\cdot)$ for a Gaussian random variable. This can be handled on a case-by-case basis as follows:

- Quadratic transaction costs. In the case of quadratic transaction costs, as seen in Example 2, the per period transaction cost function is given by $\mathsf{TC}_t(u_t) \triangleq \frac{1}{2} u_t^\top \Lambda u_t$, where $\Lambda \in \mathbb{R}^{N\times N}$ is a positive definite matrix. In this case, $\mathsf{E}[\mathsf{TC}_t(u_t)] = \frac{1}{2}\left(\bar{u}_t \Lambda \bar{u}_t + \mathrm{tr}(\Lambda V_t)\right).$

- Proportional transaction costs. In the case of proportional transaction costs, as

discussed in Example 5, the per period transaction cost function is given by

$$\mathsf{TC}_t(u_t) \triangleq \sum_{i=1}^{N} \chi_i |u_{t,i}|,$$

where $\chi_i > 0$ is a proportionality constant specific to security i. Using the properties of the folded normal distribution, we obtain

$$\mathsf{E}[\mathsf{TC}_t(u_t)] = \sum_{i=1}^{N} \chi_i \left(\sqrt{\frac{2V_{t,i}}{\pi}} \exp\left\{ -\frac{\bar{u}_{t,i}^2}{2V_{t,i}} \right\} + \bar{u}_{t,i} \left\{ 1 - 2\Phi\left(-\frac{\bar{u}_{t,i}}{\sqrt{V_{t,i}}} \right) \right\} \right),$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable.

- Nonlinear transaction costs. In the case of nonlinear transaction costs, as discussed in Example 5, the per period transaction cost function is given by

$$\mathsf{TC}_t(u_t) \triangleq \sum_{i=1}^{N} \chi_i |u_{t,i}|^{\beta},$$

where $\chi_i > 0$ is a proportionality constant specific to security i, and $\beta \geq 1$ is an exponent capturing the degree of nonlinearity. As in the proportional case, evaluating the Gaussian expectation explicitly results in

$$\mathsf{E}[\mathsf{TC}_t(u_t)] = \sum_{i=1}^{N} \chi_i \Gamma\left(\frac{1+\beta}{2}\right) \frac{(2V_{t,i})^{\frac{\beta}{2}}}{\sqrt{\pi}} \; {}_1F_1\left(-\frac{\beta}{2}; \frac{1}{2}; -\frac{\bar{u}_{t,i}^2}{2V_{t,i}} \right),$$

where $\Gamma(\cdot)$ is the gamma function and ${}_1F_1(\cdot)$ is the confluent hypergeometric function of the first kind.

C. Terminal Wealth and Risk Aversion

In many of the portfolio choice examples in Section II.B, an investor wishes to maximize expected wealth net of transaction costs, subject to a penalty for risk, that is,

$$\text{(A-11)} \qquad \underset{\pi \in \mathcal{U}_{\mathbb{F}}}{\text{maximize}} \; \mathsf{E}_{\pi}\Big[W(\mathbf{x}, \mathbf{r}) - \mathsf{TC}(\mathbf{u}) - \mathsf{RA}(\mathbf{x}, \mathbf{f}, \mathbf{r}) \Big].$$

Here, $W(\cdot)$ is the terminal wealth associated with a sample path, $\mathsf{TC}(\cdot)$ are the transaction costs, and $\mathsf{RA}(\cdot)$ is a penalty for risk aversions. Exact calculation of expected transaction costs for linear policies were discussed in Internet Appendix II.B. Here, we will discuss exact calculation of the expected terminal wealth and the risk aversion penalty.

To begin, note that the terminal wealth depends on realized returns in addition to factor realizations. Hence, we will make the following assumption:

Assumption 5 (Gaussian returns). As in Example 2, assume that for each time $t \geq 0$,

returns evolve according to

$$r_{t+1} = \mu_t + Bf_t + \epsilon^{(2)}_{t+1}, \tag{A-12}$$

where μ_t is a deterministic vector, B is a matrix of factor loadings, and $\varepsilon^{(2)}_t$ are zero-mean i.i.d. Gaussian disturbances with covariance Σ.

Note that the critical assumption we are making here is that the factor realizations $\mathbf{f}$ and the sample path of security returns $\mathbf{r}$ are jointly Gaussian. The particular form (A-12) is chosen out of convenience but is not necessary.

We can calculate the expected terminal wealth as

$$\begin{aligned}
\mathsf{E}[W(\mathbf{x},\mathbf{r})] &= W_0 + \sum_{t=1}^{T} \mathsf{E}[x_t^\top r_{t+1}] \;=\; W_0 + \sum_{t=1}^{T} \left(\mu_t^\top \kappa_t + \mathsf{E}[x_t^\top B f_t]\right), \\
&= W_0 + \sum_{t=1}^{T} \Bigg(\mu_t^\top \kappa_t + d_t^\top B \delta_t \\
&\qquad + \sum_{s=1}^{t} \left(\delta_s^\top (B(I-\Phi)^{t-s} J_{s,t}^\top)\delta_s + \operatorname{tr}\left(B(I-\Phi)^{t-s} J_{s,t}^\top \omega_s\right)\right)\Bigg),
\end{aligned}$$

where ω_s is the sth $K \times K$ diagonal block matrix of Ω_t.

For the risk aversion penalty, we consider two cases:

- Per period risk penalty. Consider risk aversion penalties that decompose over time as

  $$\mathsf{RA}(\mathbf{x},\mathbf{f},\mathbf{r}) = \sum_{t=1}^{N} \mathsf{RA}_t(x_t),$$

  where $\mathsf{RA}_t(\cdot)$ is a function which penalizes for risk aversion based on the positions held at time t. One such case is the quadratic penalty $\mathsf{RA}_t(x_t) \triangleq \frac{\gamma}{2} x_t^\top \Sigma x_t$ of Example 2, where $\gamma > 0$ is a risk penalty proportionality constant. Here, the investor seeks to penalize in proportion to the conditional per period variance of the portfolio value. So long as the expectation of $\mathsf{RA}_t(\cdot)$ can be calculated for Gaussian arguments, then the overall expected risk aversion penalty can be calculated exactly. This can be accomplished for a variety of functions. For example, quadratic penalties can be handled in a manner analogous to the quadratic transaction costs discussed in Section II.B.

- Terminal wealth risk penalty. Alternatively, as discussed in Example 6, a more natural risk aversion criteria might be to penalize risk as a function of the terminal wealth. Specifically, an investor with mean–variance preferences would consider a risk aversion penalty $\mathsf{RA}(\mathbf{x},\mathbf{f},\mathbf{r}) \triangleq -\frac{\gamma}{2}\operatorname{var}(W(\mathbf{x},\mathbf{r}))$, where $\gamma > 0$ is a risk penalty proportionality constant.

  Following the notation of Section II, we will compute $\operatorname{var}[W(\mathbf{x},\mathbf{r})]$ analytically and demonstrate that the resulting expression is a quadratic convex function of the policy coefficients. Without loss of generality, assume that $W_0 = 0$ and $\mu_t = \mathbf{0}$. Let

$d \in \mathbb{R}^{NT\times 1}$, $e \in \mathbb{R}^{NT\times 1}$, $L \in \mathbb{R}^{NT\times KT}$, and $D \in \mathbb{R}^{NT\times KT}$ with

$$d \triangleq \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{T-1} \\ d_T \end{bmatrix}, \quad e \triangleq \begin{bmatrix} \epsilon_1^{(2)} \\ \epsilon_2^{(2)} \\ \vdots \\ \epsilon_{T-1}^{(2)} \\ \epsilon_T^{(2)} \end{bmatrix}, \quad L \triangleq \begin{bmatrix} B & 0 & \dots & 0 & 0 \\ 0 & B & & \dots & 0 \\ \vdots & & \ddots & \dots & 0 \\ & \dots & & B & 0 \\ 0 & \dots & & 0 & B \end{bmatrix},$$

$$D \triangleq \begin{bmatrix} J_{1,1} & 0 & \dots & 0 & 0 \\ J_{1,2} & J_{2,2} & & \dots & 0 \\ \vdots & & \ddots & \dots & 0 \\ & \dots & & \dots & 0 \\ J_{1,T} & \dots & & J_{T-1,T} & J_{T,T} \end{bmatrix}.$$

Then, observe that $\begin{bmatrix} x_1^\top & \dots & x_T^\top \end{bmatrix}^\top = d + DF_T$, and $W(\mathbf{x}, \mathbf{r}) = (d + DF_T)^\top (LF_T + e)$. Using the independence between e and F_T and $\mathsf{E}[e] = 0$, we obtain

$$\begin{aligned} \text{(A-13)} \qquad \operatorname{var}(W(\mathbf{x}, \mathbf{r})) &= \mathsf{E}\left[(d + DF_T)^\top (LF_T + e)(LF_T + e)^\top (d + DF_T)\right] \\ &\quad - \mathsf{E}\left[(d + DF_T)^\top (LF_T + e)\right]^2 \\ &= \mathsf{E}\left[(d + DF_T)^\top ee^\top (d + DF_T)\right] \\ &\quad + \mathsf{E}\left[(d + DF_T)^\top (LF_T)(LF_T)^\top (d + DF_T)\right] \\ &\quad - \mathsf{E}\left[(d + DF_T)^\top (LF_T)\right]^2 \end{aligned}$$

We need the following fact from multivariate statistics in order to compute the expectations.

Fact 1. If z is a random vector with mean m and variance S, and Q, A, H are constant matrices, and a is a constant vector, then

$$\begin{aligned} &\mathsf{E}\left[z^\top Qz\right] = \operatorname{tr}(QS) + m^\top Qm, \\ &\mathsf{E}\left[(Az + a)^\top Hz(Hz)^\top (Az + a)\right] = \operatorname{tr}\left(AS(H^\top A + A^\top H)SH^\top\right) \\ &\quad + \left((Am + a)^\top H + (Hm)^\top A\right) S \left((Am + a)^\top H + (Hm)^\top A\right)^\top \\ &\quad + \left(\operatorname{tr}\left(ASH^\top\right) + (Am + a)^\top Hm\right)^2. \end{aligned}$$

We will consider each of the three terms in (A-13) separately. The first expectation, $\mathsf{E}\left[(d + DF_T)^\top ee^\top (d + DF_T)\right]$, can be evaluated using the independence of $\epsilon_s^{(2)}$ and

$\epsilon_t^{(2)}$ when $s \neq t$ and Fact 1.

$$\begin{aligned}
\mathsf{E}\left[(d+DF_T)^\top ee^\top (d+DF_T)\right] &= \sum_{t=1}^{T} \mathsf{E}\left[x_t^\top \mathsf{E}\left[\epsilon_{t+1}^{(2)}\left(\epsilon_{t+1}^{(2)}\right)^\top |\mathcal{F}_t\right] x_t\right] \\
&= \sum_{t=1}^{T} \mathsf{E}\left[x_t^\top \Sigma x_t\right] \\
&= \sum_{t=1}^{T} (d_t + P_t\theta_t)^\top \Sigma (d_t + P_t\theta_t) + \operatorname{tr}\left(\Sigma P_t \Omega_t P_t^\top\right).
\end{aligned}$$

Note that this term is equivalent to the sum of per-period risk penalties considered in the previous section.

The remaining expectations in (A-13) can be directly computed using Fact 1 with replacing z with F_T where $F_T \sim N(\Theta_T, \Omega_T)$.

$$\begin{aligned}
\mathsf{E}\left[(d+DF_T)^\top (LF_T)(LF_T)^\top (d+DF_T)\right] &= \operatorname{tr}\left(D\Omega_T(L^\top D + D^\top L)\Omega_T L^\top\right) \\
&+ \left((D\Theta_T + d)^\top L + (L\Theta_T)^\top D\right)\Omega_T\left((D\Theta_T + d)^\top L + (L\Theta_T)^\top D\right)^\top \\
&+ \left(\operatorname{tr}\left(D\Omega_T L^\top\right) + (D\Theta_T + d)^\top L\Theta_T\right)^2.
\end{aligned}$$

Finally,

$$\mathsf{E}\left[(d+DF_T)^\top (LF_T)\right]^2 = \left(\operatorname{tr}\left(D\Omega_T L^\top\right) + (D\Theta_T + d)^\top L\Theta_T\right)^2.$$

Summing all three terms in (A-13), we obtain the exact expression for the variance of terminal wealth:

(A-14)

$$\begin{aligned}
\operatorname{var}(W(\mathbf{x},\mathbf{r})) = &\sum_{t=1}^{T} (d_t + P_t\theta_t)^\top \Sigma (d_t + P_t\theta_t) + \operatorname{tr}\left(\Sigma P_t \Omega_t P_t^\top\right) \\
&+ \operatorname{tr}\left(D\Omega_T(L^\top D + D^\top L)\Omega_T L^\top\right) \\
&+ \left((D\Theta_T + d)^\top L + (L\Theta_T)^\top D\right)\Omega_T\left((D\Theta_T + d)^\top L + (L\Theta_T)^\top D\right)^\top.
\end{aligned}$$

Note that this expression is convex in our decision variables, as expected.

III. Derivation of the LQC Policies

We can derive a closed-form solution for our trading policy when the problem satisfies the LQC framework. We guess a functional form for the value function and show that this functional form is preserved at each time step.

Using dynamic programming principle and $u_t = (x_t - x_{t-1})$, the value function

$V_t(x_{t-1}, f_t)$ satisfies

$$V_{t-1}(x_{t-1}, f_t) = \underset{x_t}{\text{maximize}} \left(x_t^\top (Bf_t) - \frac{1}{2}(x_t - x_{t-1})^\top \Lambda (x_t - x_{t-1}) + \mathsf{E}[V_t(x_t, f_{t+1})] \right).$$

We guess the following quadratic form for our value function:

$$V_t(x_t, f_{t+1}) = -\frac{1}{2} x_t^\top A_{xx,t} x_t + x_t^\top A_{xf,t} f_{t+1} + \frac{1}{2} f_{t+1}^\top A_{ff,t} f_{t+1} + \frac{1}{2} m_t.$$

Then,

$$\begin{aligned} \mathsf{E}[V_t(x_t, f_{t+1})] &= -\frac{1}{2} x_t^\top A_{xx,t} x_t + x_t^\top A_{xf,t} (I - \Phi) f_t \\ &\quad + \frac{1}{2} f_t^\top (I - \Phi)^\top A_{ff,t} (I - \Phi) f_t + \frac{1}{2} \left(\operatorname{tr}(\Psi A_{ff,t}) + m_t \right). \end{aligned}$$

At the last period, we need $x_T = 0$, and our value function equals

$$V_{T-1}(x_{T-1}, f_t) = -\frac{1}{2} x_{T-1}^\top \Lambda x_{T-1}$$

which satisfies our functional form with

$$A_{xx,T-1} = \Lambda \qquad A_{xf,T-1} = \mathbf{0} \qquad A_{ff,T-1} = \mathbf{0} \qquad m_{T-1} = 0.$$

For all $t < T - 1$, we maximize the quadratic objective $-\frac{1}{2} x_t^\top Q_t x_t + x_t^\top q_t + b_t$ where

$$\begin{aligned} Q_t &= \Lambda + A_{xx,t} \\ q_t &= \Lambda x_{t-1} + (B + A_{xf,t} (I - \Phi)) f_t \\ b_t &= -\frac{1}{2} x_{t-1}^\top \Lambda x_{t-1} + \frac{1}{2} f_t^\top (I - \Phi)^\top A_{ff,t} (I - \Phi) f_t + \operatorname{tr}(\Psi A_{ff,t}) + m_t \end{aligned}$$

Then, the optimal x_t is given by $Q_t^{-1} q_t$ and x_t and u_t are given by

$$\begin{aligned} x_t &= (\Lambda + A_{xx,t})^{-1} (\Lambda x_{t-1} + (B + A_{xf,t} (I - \Phi)) f_t) \\ u_t &= (\Lambda + A_{xx,t})^{-1} (\Lambda x_{t-1} + (B + A_{xf,t} (I - \Phi)) f_t) - x_{t-1} \end{aligned}$$

The maximum then occurs at $\frac{1}{2} q_t^\top Q_t^{-1} q_t + b_t$ and we obtain the following recursions:

$$\begin{aligned} A_{xx,t-1} &= -\Lambda (\Lambda + A_{xx,t})^{-1} \Lambda + \Lambda \\ A_{xf,t-1} &= \Lambda (\Lambda + A_{xx,t})^{-1} (B + A_{xf,t} (I - \Phi)) \\ A_{ff,t-1} &= (B + A_{xf,t} (I - \Phi))^\top (\Lambda + A_{xx,t})^{-1} (B + A_{xf,t} (I - \Phi)) + (I - \Phi)^\top A_{ff,t} (I - \Phi) \\ m_{t-1} &= \operatorname{tr}(\Psi A_{ff,t}) + m_t \end{aligned}$$

Using these recursions, we can compute the optimal expected payoff of the dynamic

program. Using $f_0 = N(0, \Omega_0)$,

$$\begin{aligned}
& \mathsf{E}[V_0(x_0, f_1)] = \mathsf{E}\left[\mathsf{E}[V_0(x_0, f_1)|f_0]\right] \\
= \; & \mathsf{E}\Big[-\frac{1}{2}x_0^\top A_{xx,0}x_0 + x_0^\top A_{xf,0}\left(I-\Phi\right)f_0 + \frac{1}{2}f_0^\top\left(I-\Phi\right)^\top A_{ff,0}\left(I-\Phi\right)f_0 \\
& + \frac{1}{2}\left(\mathrm{tr}(\Omega_0 A_{ff,0}) + m_0\right)\Big] \\
= \; & -\frac{1}{2}x_0^\top A_{xx,0}x_0 + \frac{1}{2}\left(\mathrm{tr}\left(\Omega_0(I-\Phi)^\top A_{ff,0}(I-\Phi)\right) + \sum_{t=0}^{T-2}\mathrm{tr}(\Psi A_{ff,0})\right).
\end{aligned}$$

IV. Exact Formulation of Best Linear Execution Policy

We will first compute the expectation in the objective of (18) and write the equivalent deterministic form. We will then replace probabilistic constraints with deterministic constraints using Lemma 1, and finally obtain the deterministic version of the stochastic program in (18).

We start working with the expectation in the objective function. For each t, we have to compute the expectation of the following two terms, $\mathsf{E}\left[x_t^\top(Bf_t)\right]$, and $\mathsf{E}\left[u_t^\top\Lambda u_t\right]$. First, we derive the statistics for f_t, u_t and x_t. We first note that

$$f_t \;=\; (I-\Phi)^T e_0 + \sum_{s=1}^{t}(I-\Phi)^{t-s}\epsilon_s^{(1)}.$$

Letting $F_t \triangleq (f_1, \ldots, f_t)^\top$, Then, in vector form, we have the following representation

$$F_t = \begin{bmatrix} (I-\Phi)f_0 \\ (I-\Phi)^2 f_0 \\ \vdots \\ (I-\Phi)^{t-1}f_0 \\ (I-\Phi)^t f_0 \end{bmatrix} + \underbrace{\begin{bmatrix} I & 0 & \ldots & 0 & 0 \\ (I-\Phi) & I & 0 & \ldots & 0 \\ \vdots & (I-\Phi) & \ddots & \ldots & 0 \\ (I-\Phi)^{t-1} & \ldots & \ddots & I & 0 \\ (I-\Phi)^t & \ldots & & (I-\Phi) & I \end{bmatrix}}_{\triangleq A_t} \begin{bmatrix} \epsilon_1^{(1)} \\ \epsilon_2^{(1)} \\ \vdots \\ \epsilon_{t-1}^{(1)} \\ \epsilon_t^{(1)} \end{bmatrix}.$$

Using this representation, we compute the mean

$$\theta_t \triangleq \mathsf{E}[F_t] = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{t-1} \\ \delta_t \end{bmatrix} \triangleq \begin{bmatrix} (I-\Phi)f_0 \\ (I-\Phi)^2 f_0 \\ \vdots \\ (I-\Phi)^{t-1}f_0 \\ (I-\Phi)^t f_0 \end{bmatrix}, \tag{A-15}$$

and the covariance matrix

$$\Omega_t \triangleq \text{var}[F_t] = A_t \begin{bmatrix} \Psi & 0 & \dots & 0 & 0 \\ 0 & \Psi & & \dots & 0 \\ \vdots & & \ddots & \dots & 0 \\ & \dots & & \Psi & 0 \\ 0 & \dots & & 0 & \Psi \end{bmatrix} A_t^\top. \tag{A-16}$$

Note that Ω_t is a block diagonal matrix with t blocks of size $K \times K$. Recall that in Section II, we defined

$$M_t \triangleq \begin{bmatrix} E_{1,t} & E_{2,t} & \dots & E_{t,t} \end{bmatrix} \tag{A-17}$$

Then, $u_t = c_t + M_t F_t$ and we have the following moments for u_t:

$$\begin{aligned} \mu_t &\triangleq \mathsf{E}[u_t] = c_t + M_t\theta_t \\ V_t &\triangleq \text{var}(u_t) = M_t \Omega_t M_t^\top. \end{aligned} \tag{A-18}$$

Therefore, u_t is normally distributed with mean μ_t and covariance matrix V_t. Similarly, we can obtain the statistics for x_t. Using (A-7),

$$\begin{aligned} \kappa_t &\triangleq \mathsf{E}[x_t] = d_t + P_t\theta_t \\ Y_t &\triangleq \text{var}(x_t) = P_t \Omega_t P_t^\top. \end{aligned}$$

Using Fact 1, we can compute each term in the expectation.

$$\begin{aligned} \mathsf{E}\left[x_t^\top (Bf_t)\right] &= \mathsf{E}\left[d_t^\top B f_t + \sum_{s=1}^{t} f_s^\top J_{s,t}^\top B f_t\right] \\ &= d_t^\top B\delta_t + \sum_{s=1}^{t} \mathsf{E}\left[f_s^\top J_{s,t}^\top B \mathsf{E}\left[f_t | f_s\right]\right] \\ &= d_t^\top B\delta_t + \sum_{s=1}^{t} \mathsf{E}\left[f_s^\top J_{s,t}^\top B (I-\Phi)^{t-s} f_s\right] \\ &= d_t^\top B\delta_t + \sum_{s=1}^{t} \left(\delta_s^\top (B(I-\Phi)^{t-s} J_{s,t}^\top)\delta_s + \text{tr}\left(B(I-\Phi)^{t-s} J_{s,t}^\top \omega_s\right)\right) \end{aligned}$$

where ω_s is the sth diagonal block matrix of Ω_t having a size of $K \times K$. Finally, for the transaction cost terms,

$$\begin{aligned} \mathsf{E}\left[u_t^\top \Lambda u_t\right] &= \mathsf{E}\left[(c_t + M_t F_t)^\top \Lambda (c_t + M_t F_t)\right] \\ &= (c_t + M_t\theta_t)^\top \Lambda (c_t + M_t\theta_t) + \text{tr}\left(\Lambda M_t \Omega_t M_t^\top\right) \end{aligned}$$

Summing up all the terms, our final objective function in deterministic form equals

$$\underset{c_t, E_{s,t}}{\text{maximize}} \sum_{t=1}^{T} \Bigg\{ d_t^\top B \delta_t + \sum_{s=1}^{t} \left(\delta_s^\top (B(I-\Phi)^{t-s} J_{s,t}^\top) \delta_s + \operatorname{tr}\left(B(I-\Phi)^{t-s} E_{s,t}^\top \omega_s \right) \right)$$
$$+ \frac{1}{2} \left((c_t + M_t\theta_t)^\top \Lambda (c_t + M_t\theta_t) + \operatorname{tr}\left(\Lambda M_t \Omega_t M_t^\top \right) \right) \Bigg\},$$

which is a quadratic function of the policy parameters.

We now rewrite the equality constraint, $x_T = 0$ in terms of policy parameters. In order to enforce this equality, we need

$$d_T = 0 \text{ and } J_{s,T} = \mathbf{0} \qquad s = 1, \ldots, T.$$

Lastly, we replace probabilistic constraints with deterministic constraints using Lemma 1. Note that $\mathsf{P}(x_t \leq 0) \leq \eta$ can be written as $\mathsf{P}(-x_t \geq 0) \leq \eta$. Then, using Lemma 1,

$$(-d_t - P_t\theta_t) + \Phi^{-1}(1-\eta) \left\| \left(P_t \Omega_t P_t^\top \right)^{1/2} \right\|_2 \leq 0.$$

Similarly, we obtain that $\mathsf{P}(u_t \geq 0) \leq \eta$ can be replaced by

$$(c_t + M_t\theta_t) + \Phi^{-1}(1-\eta) \left\| \left(M_t \Omega_t M_t^\top \right)^{1/2} \right\|_2 \leq 0.$$

Combining all the results, we obtain the deterministic version of the stochastic program in (18), a second-order cone program:

(A-19)

$$\begin{aligned}
\underset{c_t, E_{s,t}}{\text{maximize}} \quad & \textstyle\sum_{t=1}^{T} \Big\{ d_t^\top B \delta_t + \sum_{s=1}^{t} \left(\delta_s^\top (B(I-\Phi)^{t-s} J_{s,t}^\top) \delta_s + \operatorname{tr}\left(B(I-\Phi)^{t-s} E_{s,t}^\top \omega_s \right) \right), \\
& + \tfrac{1}{2} \left((c_t + M_t\theta_t)^\top \Lambda (c_t + M_t\theta_t) + \operatorname{tr}\left(M_t^\top \Lambda M_t \Omega_t \right) \right) \Big\}, \\
\text{subject to} \quad & d_t = x_0 + \textstyle\sum_{i=1}^{t} c_i \qquad t = 1, \ldots, T, \\
& J_{s,t} = \textstyle\sum_{i=s}^{t} E_{s,i} \qquad 1 \leq s \leq t \leq T, \\
& (-d_t - P_t\theta_t) + \Phi^{-1}(1-\eta) \left\| \left(P_t \Omega_t P_t^\top \right)^{1/2} \right\|_2 \leq 0 \qquad t = 1, \ldots, T, \\
& (c_t + M_t\theta_t) + \Phi^{-1}(1-\eta) \left\| \left(M_t \Omega_t M_t^\top \right)^{1/2} \right\|_2 \leq 0 \qquad t = 1, \ldots, T, \\
& d_T = 0 \text{ and } J_{s,T} = \mathbf{0}.
\end{aligned}$$

Note that the number of decision variables is greater than that of the original execution problem in (13). Total number of decision variables in a problem with N securities, K factors and T periods equals $2NT + NKT(T+1)$ which is on the order of $O(NKT^2)$.